Pre-Class Problems 26 Inverse Functions

Discussion of Inverse Functions

These are the type of problems that you will be working on in class.

You can go to the solution for each problem by clicking on the problem letter.

1. If $f(x) = 4 + 3x - x^2$ and $f(-2) = -6$, then find $(f^{-1})'(-6)$.

2. If $f(x) = 4 + 3x - x^2$ and $f(-2) = -6$, then find $(f^{-1})'(-6)$ by finding the inverse function first.

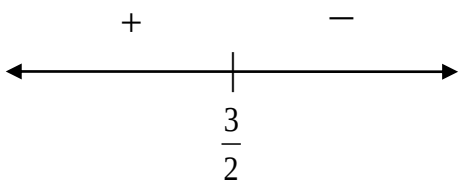
3. If $g(x) = x^3 - 5x^2 - 8x + 12$ and $g\left(\frac{1}{2}\right) = \frac{23}{4}$, then find the equations (in point-slope form) of the tangent line and the normal line to the graph of the function $x = g^{-1}(y)$ at the point for which $y = \frac{23}{4}$.

SOLUTIONS:

1. $f(x) = 4 + 3x - x^2$, $f(-2) = -6$ Find $(f^{-1})'(-6)$

$f'(x) = 3 - 2x$

Sign of $f'(x)$:



Since -2 is in the interval $\left(-\infty, \frac{3}{2}\right)$ and $f'(x) > 0$ for all x in this interval, then the function f is increasing on this interval. Thus, by the theorem below we have that

$$(f^{-1})'(-6) = \frac{1}{f'(f^{-1}(-6))} = \frac{1}{f'(-2)} = \frac{1}{7}$$

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since $f'(-2) = 7$.

Answer: $\frac{1}{7}$ Back to Problem 1

2. If $f(x) = 4 + 3x - x^2$, $f(-2) = -6$ Find $(f^{-1})'(-6)$ by finding the inverse function first.

Since f is a polynomial, then f is continuous for all real numbers. Since -2 is in the interval $\left(-\infty, \frac{3}{2}\right)$ and $f'(x) > 0$ for all x in this interval, then the function f is increasing on this interval. Since the function f is continuous and monotonic on the interval $\left(-\infty, \frac{3}{2}\right)$, then by the theorem below, the function f have an inverse on this interval.

Let $y = f(x)$. Then $y = 4 + 3x - x^2$. Now, solve for x in terms of y. In order to do this, we will use the Quadratic Formula.

$$y = 4 + 3x - x^2 \Rightarrow x^2 - 3x + y - 4 = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = 1$, $b = -3$, and $c = y - 4$. Thus,

$$x = \frac{3 \pm \sqrt{9 - 4(1)(y-4)}}{2} = \frac{3 \pm \sqrt{9 - 4(y-4)}}{2} = \frac{3 \pm \sqrt{9 - 4y + 16}}{2} =$$

$\frac{3 \pm \sqrt{25 - 4y}}{2}$. Since $f(-2) = -6$, then $f^{-1}(-6) = -2$. Thus, in order to have the value of $y = -6$ to map back to the value of $x = -2$, we must have that $x = \frac{3 - \sqrt{25 - 4y}}{2}$. Thus, $f^{-1}(y) = \frac{3 - \sqrt{25 - 4y}}{2}$.

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Since $f^{-1}(y) = \dfrac{3 - \sqrt{25 - 4y}}{2} = \dfrac{1}{2}\left[3 - \sqrt{25 - 4y}\right] = \dfrac{1}{2}\left[3 - (25 - 4y)^{1/2}\right]$,

then $(f^{-1})'(y) = \dfrac{1}{2}\left[0 - \dfrac{1}{2}(25 - 4y)^{-1/2}(-4)\right] = \dfrac{1}{2}\left[2(25 - 4y)^{-1/2}\right] =$

$(25 - 4y)^{-1/2} = \dfrac{1}{\sqrt{25 - 4y}}$. Thus, $(f^{-1})'(y) = \dfrac{1}{\sqrt{25 - 4y}} \Rightarrow$

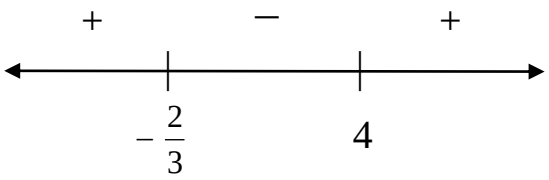
$(f^{-1})'(-6) = \dfrac{1}{\sqrt{25 + 24}} = \dfrac{1}{\sqrt{49}} = \dfrac{1}{7}$

Answer: $\dfrac{1}{7}$ Back to Problem 2

3. $g(x) = x^3 - 5x^2 - 8x + 12$, $g\left(\dfrac{1}{2}\right) = \dfrac{23}{4}$

$g'(x) = 3x^2 - 10x - 8 = (x - 4)(3x + 2)$

Sign of $g'(x)$:



Since $\dfrac{1}{2}$ is in the interval $\left(-\dfrac{2}{3}, 4\right)$ and $g'(x) < 0$ for all x in this interval, then the function g is decreasing on this interval. Thus, by the theorem below we have that

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$$(g^{-1})'\left(\frac{23}{4}\right) = \frac{1}{g'\left(g^{-1}\left(\frac{23}{4}\right)\right)} = \frac{1}{g'\left(\frac{1}{2}\right)} = -\frac{4}{49}$$

since $g'\left(\frac{1}{2}\right) = \frac{3}{4} - 5 - 8 = \frac{3}{4} - \frac{52}{4} = -\frac{49}{4}$.

Thus, $m_{\tan} = (g^{-1})'\left(\frac{23}{4}\right) = -\frac{4}{49}$ and $m_{normal} = -\frac{1}{m_{\tan}} = \frac{49}{4}$

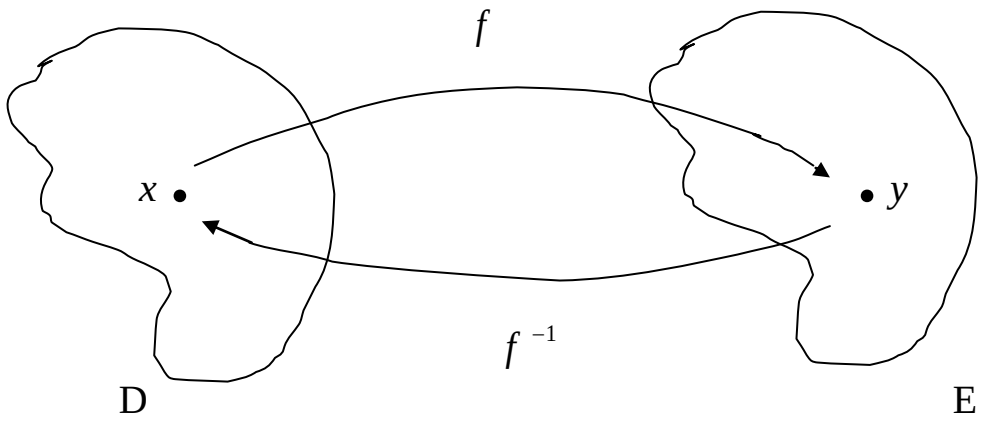
Tangent Point = Normal Point = $\left(\frac{23}{4}, g^{-1}\left(\frac{23}{4}\right)\right) = \left(\frac{23}{4}, \frac{1}{2}\right)$

Answer: Tangent Line: $y - \frac{1}{2} = -\frac{4}{49}\left(x - \frac{23}{4}\right)$

Normal Line: $y - \frac{1}{2} = \frac{49}{4}\left(x - \frac{23}{4}\right)$

Back to Problem 3

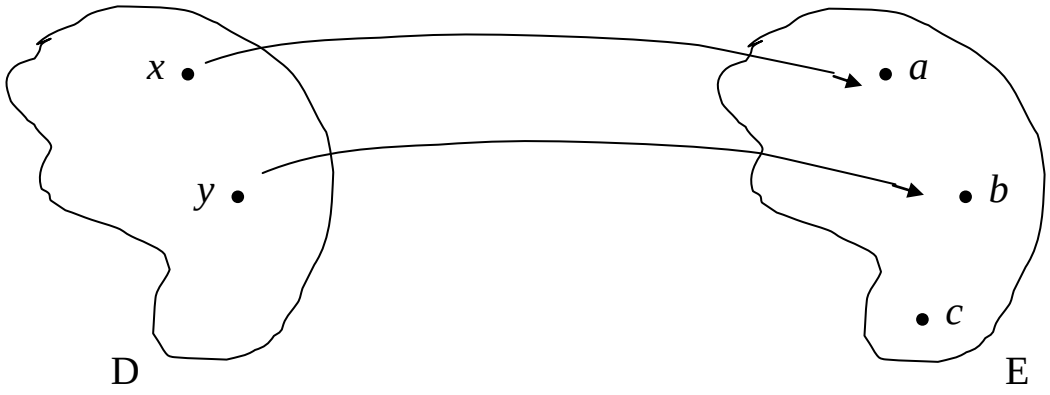
Example Consider the following function f and its inverse f^{-1}.



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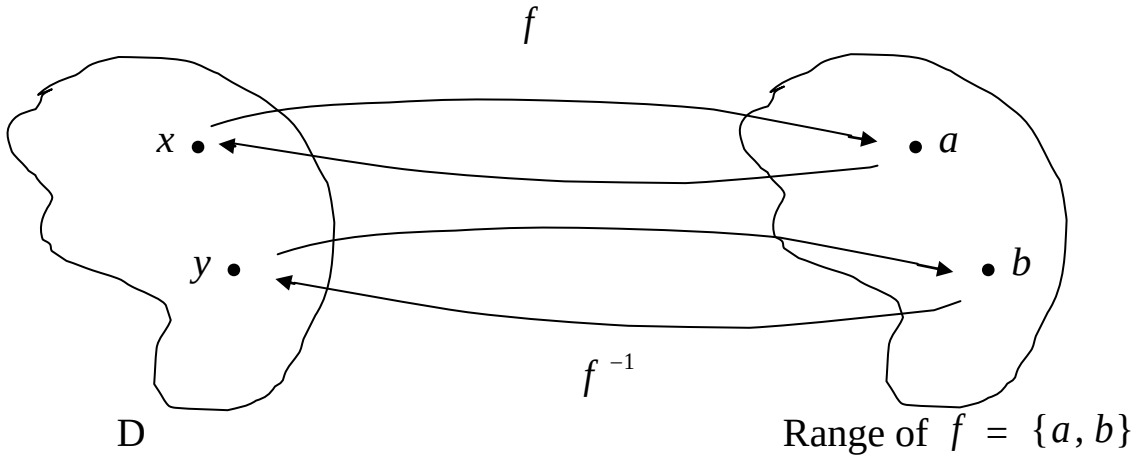
NOTE: The function f maps x in the set D to y in the set E and f^{-1} maps y back to x. Of course, this is what an inverse function is suppose to do.

Example Consider the following function f. Does f have an inverse?



The function f maps x in the set D to a in the set E. So, the inverse function would map a back to x. The function f maps y in the set D to b in the set E. So, the inverse function would map b back to y. However, where does the inverse function map c. Note that the domain of the function f is the set $\{x, y\}$ and the range of the function f is the set $\{a, b\}$.

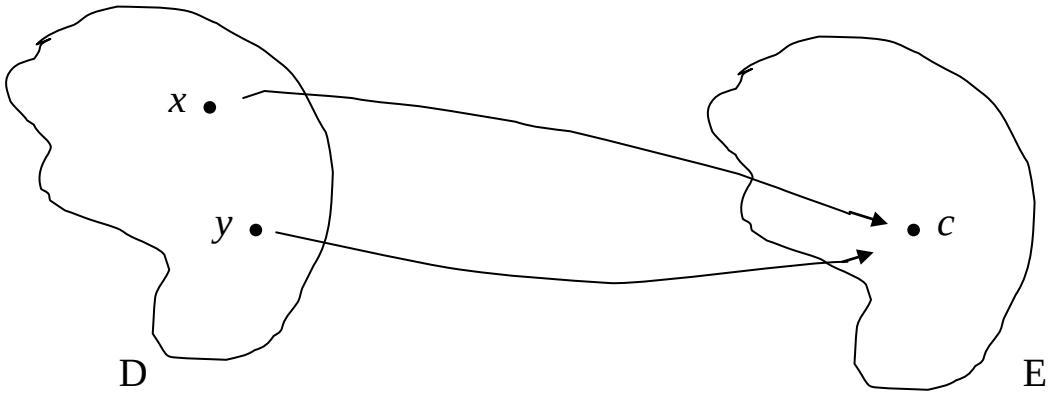
The function f is not an onto function. In order for a function to be an onto function, every element in the set E must be used. For the given function f, the element c in the set E was not used. **In order for a function to have an inverse, it must be an onto function.** If a function is not an onto function, then the lack of this needed condition is easy to fix. To fix the lack of the onto condition, replace the set E by the range of the function.



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Now, this function has an inverse function. Notice the following relationship above: The domain of the inverse function f^{-1} is equal to the range of the function f.

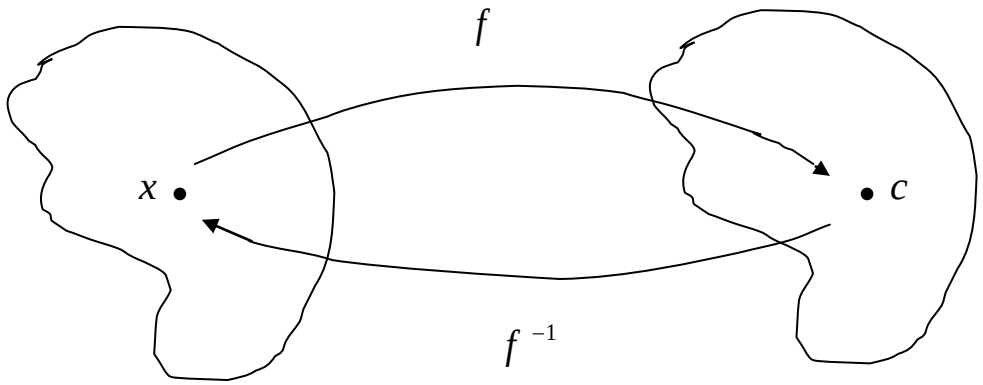
Example Consider the following function f. Does f have an inverse?



Note that the domain of the function f is the set D, which is the set $\{x, y\}$ and the range of the function f is the set $\{c\}$. Also, note that the function f is an onto function. The set E is the range of the function f. The function f maps *x* in the set D to *c* in the set E. The function f also maps *y* in the set D to *c* in the set E. So, the inverse function would map *c* back to either *x* or *y*. Which one do you use? The problem here is that the function f is not a one-to-one function. In order for a function to be a one-to-one function, you may only use each element in the set E once. **In order for a function to have an inverse, it must be a one-to-one function.** If a function is not a one-to-one function, then the lack of this needed condition is not as easy to fix as the lack of the onto condition. In order to fix the lack of the one-to-one condition, you must put a restriction on the domain of the function. In other words, you must eliminate elements from the set D. What elements in the set D are you going to choose to eliminate? This is the reason that fixing the lack of the one-to-one condition is harder. For the function f, the domain is the set D = $\{x, y\}$. Thus, we will either eliminate *x* or *y*. Each restricted domain will produce an inverse function. Thus, these two choices for the restricted domain will produce two inverse functions.

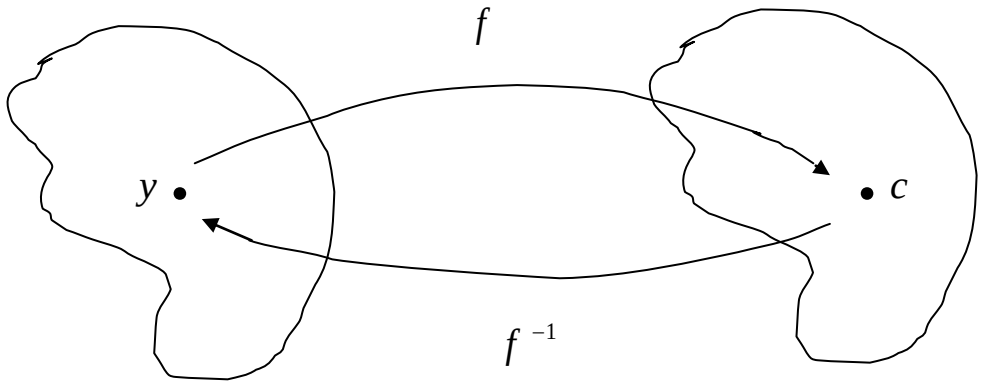
If we eliminate *y*, then we get the following inverse function:

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Restricted domain of $f = \{x\}$ Range of $f = \{c\}$

If we eliminate *x*, then we get the following inverse function:



Restricted domain of $f = \{y\}$ Range of $f = \{c\}$

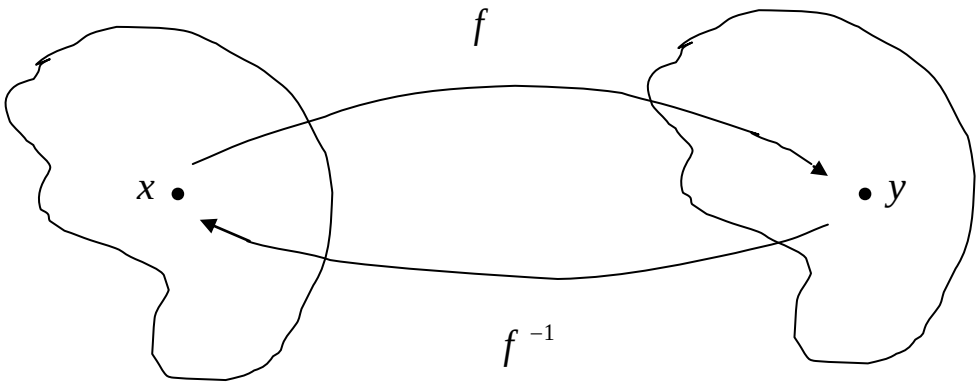
The function f has two possible inverse functions depending on the restricted domain that is chosen.

Notice the following relationship for both inverse functions above: The restricted domain of the function f is equal to the range of the function f^{-1}.

Theorem A function f has an inverse function, denoted by f^{-1} if and only if the function f is one-to-one and onto.

We have the following relationships:

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Restricted domain of f = Range of f^{-1}

Range of f = Domain of f^{-1}

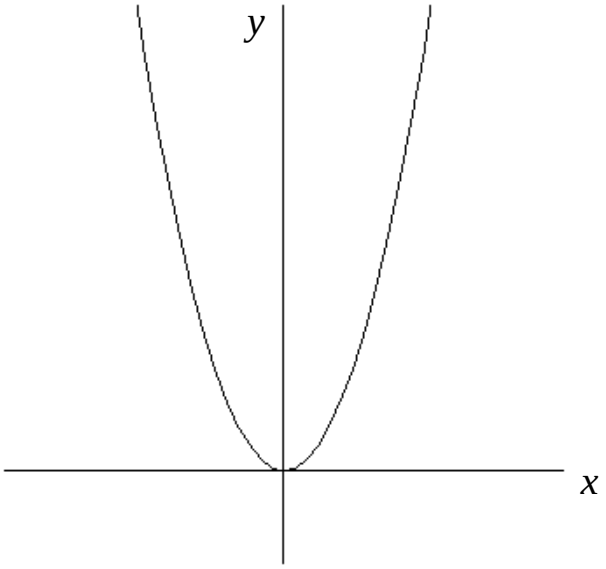
We also have the following two relationships between a function and its inverse function:

1. $f^{-1}(f(x)) = x$ for all x in the **restricted domain of** f

2. $f(f^{-1}(y)) = y$ for all y in the **domain of** f^{-1}

Example Find the inverse function of the function $f(x) = x^2$.

First, consider the graph of $f(x) = x^2$.



Information about the domain of the function f can be determined by the *x*-coordinate of the points on the graph. Since the value of the *x*-coordinates range in value from negative infinity to positive infinity, then the domain of f is all real numbers. Information about the range of the function f can be determined by the *y*-coordinate of the points on the graph. Since the value of the *y*-coordinates range in value from zero to positive infinity, then the range of f is all real numbers greater than or equal to zero. In interval notation, we have the following:

$$\text{Domain of } f = (-\infty, \infty)$$

$$\text{Range of } f = [0, \infty)$$

Since the range of f is $[0, \infty)$, then by the discussion above, the domain of f^{-1} is $[0, \infty)$.

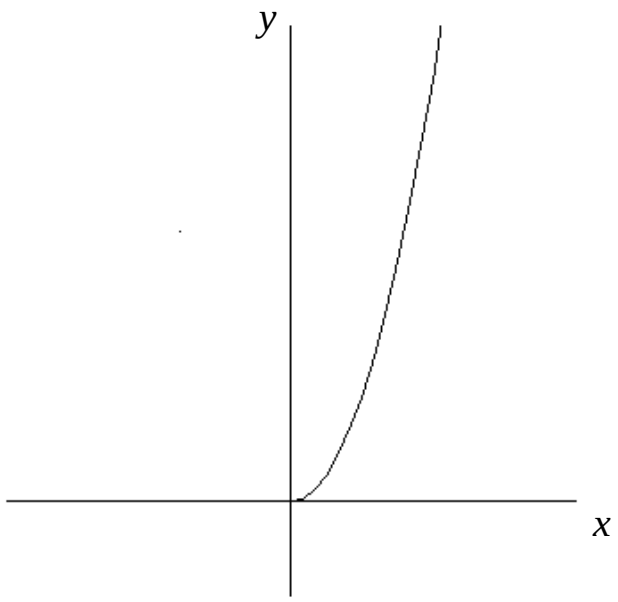
Recall the following test for checking the graph of a function for being one-to-one:

The Horizontal Line Test: If a horizontal line intersects the graph of a function in more than one place, then the function is not one-to-one.

By the horizontal line test, the function f is not one-to-one. We will have to put a restriction on the domain of the function in order to fix this. We have the following two choices for the restricted domain: 1) the interval of numbers $[0, \infty)$; that is, the set of numbers greater than or equal to zero **or** 2) the interval of numbers $(-\infty, 0]$; that is, the set of numbers less than or equal to zero. Each restricted domain will produce an inverse function. Thus, these two choices for the restricted domain will produce two inverse functions.

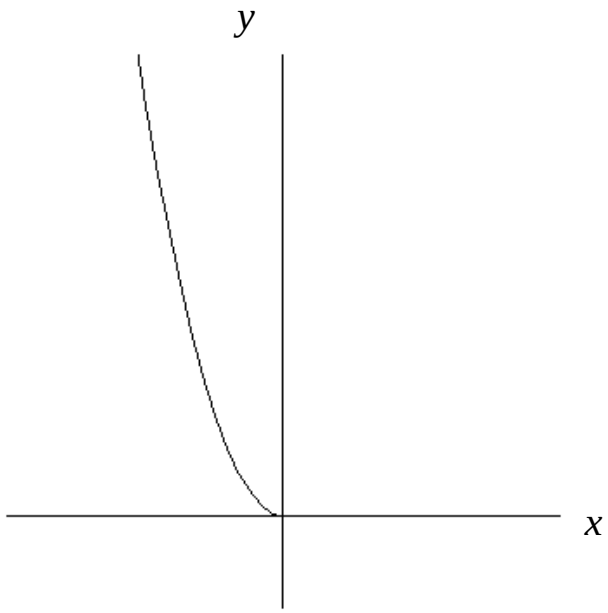
For the first choice of $[0, \infty)$, our restricted domain is $[0, \infty)$ and the graph of the function f on the restricted domain looks like the following:

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Since the restricted domain of f is $[0, \infty)$, then by the discussion above, the range of f^{-1} is $[0, \infty)$.

For the second choice of $(-\infty, 0]$, our restricted domain is $(-\infty, 0]$ and the graph of the function f on the restricted domain looks like the following:



Since the restricted domain of f is $(-\infty, 0]$, then by the discussion above, the range of f^{-1} is $(-\infty, 0]$.

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Now, let's find the inverse function(s) algebraically.

Set $f(x) = y$: $\qquad y = x^2$

Solve for x in terms of y:

$$y = x^2 \Rightarrow x^2 = y \Rightarrow \sqrt{x^2} = \sqrt{y} \Rightarrow |x| = \sqrt{y} \Rightarrow x = \pm\sqrt{y}$$

Thus, either $f^{-1}(y) = \sqrt{y}$ if the restricted domain of f is $[0, \infty)$

or $f^{-1}(y) = -\sqrt{y}$ if the restricted domain of f is $(-\infty, 0]$

Notice, as predicted above the domain of f^{-1} is $[0, \infty)$ and the range of f^{-1} is $[0, \infty)$ for the first inverse function and the range of f^{-1} is $(-\infty, 0]$ for the second inverse function.

Now, let's verify the two relationships for this function and its two inverse functions are true.

1. For the first restricted of domain of $[0, \infty)$ for the function $f(x) = x^2$, we have that $f^{-1}(x) = \sqrt{x}$:

   a. $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x| = x$. Note that since x belongs to the restricted domain of $[0, \infty)$, then x is a positive number. The absolute value of a positive number is itself.

   b. $f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$. Note that since x is in the domain of f^{-1} and the domain of f^{-1} is $[0, \infty)$, then x is greater than or equal to zero. Thus, the square root of x is defined.

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2. For the second restricted of domain of $(-\infty, 0]$ for the function $f(x) = x^2$, we have that $f^{-1}(x) = -\sqrt{x}$:

   a. $f^{-1}(f(x)) = f^{-1}(x^2) = -\sqrt{x^2} = -|x| = -(-x) = x$. Note that since x belongs to the restricted domain of $(-\infty, 0]$, then x is a negative number. The absolute value of a negative number is the negative of itself.

   b. $f(f^{-1}(x)) = f(-\sqrt{x}) = (-\sqrt{x})^2 = x$. Note that since x is in the domain of f^{-1} and the domain of f^{-1} is $[0, \infty)$, then x is greater than or equal to zero. Thus, the square root of x is defined.

Thus, the two relationships for this function and its two inverse functions hold.

Definition A function f is said to be monotonic on the open interval (a, b) if f is either increasing on (a, b) or decreasing on (a, b).

Theorem If a function f is defined and continuous on an open interval (a, b), then f has an inverse on the interval if and only if f is monotonic on the interval.

Proof The prove will be provided later.

Theorem If a function f is differentiable and monotonic on an open interval (a, b), then f is differentiable at $y = f(x)$ and $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))} = \dfrac{1}{f'(x)}$ for all x in (a, b).

Proof Since f is differentiable on the interval (a, b), then by a theorem proved in Pre-Class Problems 8, f is continuous on the interval (a, b). Since f is monotonic on the interval (a, b), then by the theorem stated above, the function f

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has an inverse function defined on the interval (a, b). Let x be in the interval (a, b). Since f has an inverse on the interval (a, b), then it is one-to-one and onto on this interval. Thus, there is a unique y in the range of f such that $x = f^{-1}(y)$. Thus, $y = f(x)$. Using implicit differentiation to differentiate both sides of this equation with respect to y, we obtain that $1 = f'(x)\frac{dx}{dy}$. Solving for $\frac{dx}{dy}$, we have that $\frac{dx}{dy} = \frac{1}{f'(x)}$. Since $x = f^{-1}(y)$, then $\frac{dx}{dy} = (f^{-1})'(y)$. Thus, $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(x)}$ since $x = f^{-1}(y)$. Since x was chosen arbitrarily in the open interval (a, b), then we have that $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(x)}$ for all x in (a, b).

Back to the problems

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