1 1.5 Solution Sets of Linear System

Definition 1 A system of linear equations is called homogeneous if it’s of the form $Ax = 0$ where $A$ is a $m \times n$ matrix and $x$ is a $n \times 1$ vector ad 0 is the zero vector in $\mathbb{R}^n$.

Remark: $Ax = 0$ always has at least one trivial solution, i.e. $x = 0$ (the zero vector in $\mathbb{R}^n$). For a given homogeneous equation, we are more interested in the nontrivial solution. The following Theorem is obvious.

Theorem 1 The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable.

Let’s look at the following examples.

Example 1 Determine whether the following equation has a nontrivial solution.

(A) \[
\begin{align*}
    x_1 + 2x_2 + 2x_3 &= 0 \\
    x_1 + 3x_2 + 3x_3 &= 0 \\
    x_1 + 2x_2 + 3x_3 &= 0.
\end{align*}
\]

Solution: The augmented matrix for (A) is \[
\begin{bmatrix}
    1 & 2 & 2 & 0 \\
    1 & 3 & 3 & 0 \\
    1 & 2 & 3 & 0
\end{bmatrix}
\]
It is row equivalent to \[
\begin{bmatrix}
    1 & 2 & 2 & 0 \\
    0 & 1 & 1 & 0 \\
    0 & 0 & 1 & 0
\end{bmatrix}
\]
so it has only the trivial solution $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$. Note that there is
no free variable in this case.

2° The augmented matrix is 
\[
\begin{bmatrix}
1 & 1 & -1 & 0 \\
2 & 1 & -3 & 0 \\
4 & 3 & -5 & 0 \\
\end{bmatrix}
\sim (r_2 := r_2 - 2r_1, r_3 := r_3 - 4r_1)
\]

\[
\begin{bmatrix}
1 & 1 & -1 & 0 \\
0 & -1 & -1 & 0 \\
0 & -1 & -1 & 0 \\
\end{bmatrix}
\sim (r_2 := -r_2, r_3 := r_3 - r_2)
\begin{bmatrix}
1 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\sim (r_1 := r_1 - r_2)
\]

\[
\begin{bmatrix}
1 & 0 & -2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

So \( x_3 \) is the free variable and \( x_1 - 2x_3 = 0, x_2 + x_3 = 0 \).

This gives \( x_1 = 2x_3, \, x_2 = -x_3 \) and 
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \] (this is called the parametric vector form).

Next we will see the relation between the solution of homogeneous equation \( Ax = 0 \) and inhomogeneous equation \( Ax = b \) where \( b \neq 0 \).

**Example 2** Find the solution of the following inhomogeneous equation.

\[ Ax = b \text{ where } A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 4 & 3 & -5 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}. \]

Solution: 2° The augmented matrix is 
\[
\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix}
1 & 1 & -1 & 0 \\
2 & 1 & -3 & -1 \\
4 & 3 & -5 & -1 \\
\end{bmatrix}
\sim (r_2 := r_2 - 2r_1, \, r_3 := r_3 - 4r_1)
\begin{bmatrix}
1 & 1 & -1 & 0 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1 \\
\end{bmatrix}
\sim (r_2 := -r_2, \, r_3 := r_3 - r_2)
\begin{bmatrix}
1 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\sim (r_1 := r_1 - r_2)
\begin{bmatrix}
1 & 0 & -2 & -1 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

So \( x_3 \) is the free variable and \( x_1 - 2x_3 = -1, \, x_2 + x_3 = -1 \). This gives 
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 2x_3 \\ -1 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \] (this is called the parametric vector form).

**Remark:** From this example and previous example, we can see that the row reduction process to solve \( Ax = b \) and \( Ax = 0 \) is the same.
We have the following Theorem.

**Theorem 2** Suppose $Ax = b$ is consistent. Then the solution set of $Ax = b$ is of the form $x = p + x_h$ where $Ap = b$ and $x_h$ is any solution of the homogeneous equation $Ax = 0$

2 1.7 Linear independence

**Definition 2** A set of vectors $\{v_1, v_2, \cdots, v_p\}$ in $\mathbb{R}^m$ is said to be linearly independent if the vector equation $x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$ (zero vector in $\mathbb{R}^m$) has only trivial solution.

A set of vectors $\{v_1, v_2, \cdots, v_p\}$ in $\mathbb{R}^m$ is said to be linearly dependent if the vector equation $x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$ (zero vector in $\mathbb{R}^m$) has nontrivial solution.

Using the notation in previous equation, we can relate the problem of linear dependence to the solution of homogeneous equation. Recall that the vector equation $x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$ is the same as the matrix equation $Ax = 0$ where $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$

So we have the following result.

**Theorem 3** A set of vectors $\{v_1, v_2, \cdots, v_p\}$ in $\mathbb{R}^m$ is linearly independent iff the matrix equation $Ax = 0$ has only trivial solution where $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$ iff the augmented matrix $[v_1 v_2 \cdots v_p | 0]$ has only trivial solution.

A set of vectors $\{v_1, v_2, \cdots, v_p\}$ in $\mathbb{R}^m$ is linearly dependent iff the matrix equation $Ax = 0$ has nontrivial solution where $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$ iff the augmented matrix $[v_1 v_2 \cdots v_p | 0]$ has nontrivial solution.

**Example 3** Determine whether the following vectors are linearly independent.

\[
\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -5 \end{bmatrix}.
\]
Solution: Let \( v_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \ v_3 = \begin{bmatrix} -1 \\ -3 \\ -5 \end{bmatrix}. \) Consider the augmented matrix \([v_1 \ v_2 \ v_3 \ 0] = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}.\) From Example 1(A) on page 1, we know this system has only trivial solution. So these three vectors are linearly independent.

**Example 4** Determine if the columns of the matrix form a linearly independent set. \( A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 4 & 3 & -5 \end{bmatrix} \)

Solution: Consider the augmented matrix The augmented matrix is \([A \ 0] = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 1 & -3 & 0 \\ 4 & 3 & -5 & 0 \end{bmatrix}.\)  
\~\ (r_2 := r_2 - 2r_1, \ r_3 := r_3 - 4r_1) \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.\)

So \( x_3 \) is the free variable and \( x_1 - 2x_3 = 0, \ x_2 + x_3 = 0. \) This gives \( x_1 = 2x_3, \)
\( x_2 = -x_3 \) and \( x = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}. \) So this system has nontrivial solution.

Thus the columns of \( A \) form a linearly dependent set.

**Example 5** Find the value of \( h \) for which the following vectors are linearly dependent. \( \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \)

Solution: Consider the augmented matrix \begin{bmatrix} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ -3 & 8 & h & 0 \end{bmatrix} \sim (r_2 := r_2 + r_1) \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ -3 & 8 & h & 0 \end{bmatrix} \sim (r_3 := r_3 + 3r_1) \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -7 & h+3 & 0 \end{bmatrix}.\)
\[ r_2 \leftarrow \left( \frac{r_2}{2} \right) \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -7 & h + 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & h + 10 & 0 \end{bmatrix} \]

This system has nontrivial solution if \( h + 10 = 0 \), that is \( h = -10 \). So these three vectors are dependent if \( h = -10 \).