14.7 Extreme Values and Saddle Points: As in Calculus I, we are interested in local maxima and minima. A function $f(x, y)$ has a **local maximum** (resp. **local minimum**) at $(a, b)$ if $f(x, y) \leq f(a, b)$ (resp. $f(x, y) \geq f(a, b)$) for all $(x, y)$ in a disk centered at $(a, b)$.

A disk is the interior of a circle. The disk could be small or large.

**Definition** A function $f(x, y)$ has a critical point at $(a, b)$ if $f$ is not differentiable at $(a, b)$ or $\nabla f(a, b) = \vec{0}$.

**Theorem** If $f$ is defined on some set $D$ and if $f$ has a local max or min at $(a, b)$ in $D$ then either $(a, b)$ is a critical point or $(a, b)$ is a boundary point.

Recall that $(a, b)$ is a **boundary point** of a set $D$ if every open disk centered at $(a, b)$ contains points inside $D$ and outside $D$.

**Example** $f(x, y) = \sqrt{x^2 + y^2}$ where $D$ is a disk of radius $a > 0$ centered at the origin in the $xy$-plane. Then $f$ has exactly one critical point at the $(0,0)$ because the partial derivatives there do not exist. ($f(x,0) = |x|$.) This corresponds to a local minimum. If $D$ contains a boundary point then that is a local maximum. If $D$ has no boundary points $D = \{(x, y) : x^2 + y^2 < a^2\}$ then there are no local max and that is the same if $D$ is the entire $xy$-plane.

**Example** $f(x, y) = y^2 - x^2$ has a critical point at $(0,0)$ but it is neither a local max nor min: it is a **saddle point** because there are points $(x, y)$ arbitrarily near $(0,0)$ so that $f(x, y) > f(0,0)$ and other points so that $f(x, y) < f(0,0)$. In this example $f(0,y) > 0 = f(0,0)$ and $f(x,0) < 0$.

**Closed Bounded Region Method** This is an analogue of the closed interval method of Section 4.1.

Suppose $f(x, y)$ is continuous on a closed and bounded region $R$. (Thus $R$ can be contained in a large enough ball and it contains all its boundary points.) Then $f$ takes on both its absolute maximum value and its absolute minimum value in $R$ and the points are either

1. critical point of $f$ inside $R$ or
2. at local extrema of $f$ on the boundary point $R$

Then compare the values of $f$ at all the points found and discover which is the absolute max and which is the absolute min.

**Example** Consider the function $f(x, y) = 2x^2 + y^2 - 2y$ on the triangle with vertices $(0,0), (2,2)$ and $(-2,2)$ assuming the edges and corners of the triangle are included.

**Solution** Sketch the region. This is a closed bounded region Check for critical points:

$\nabla f(x, y) = 4xi + (2y - 2)j$. The critical points occur where $\nabla f(x, y) = \vec{0}$ or $\nabla f(x, y)$ does
The only critical point is (0,1).

Check next the boundary. The boundary consists of the 3 edges and the 3 corners.

1. Edge from (0,0) to (2,2): Here $y = x$: $f(x, y) = f(x, x) = 2x^2 + x^2 - 2x = 3x^2 - 2x$, $0 \leq x \leq 2$

$$\frac{d}{dx} 3x^2 - 2x = 6x - 2$$

So $(1/3,1/3)$ is a critical point on the edge.

2. Edge from (0,0) to (-2,2): Here $y = -x$: $f(x, y) = f(x, -x) = 2x^2 + x^2 + 2x = 3x^2 + 2x$, $-2 \leq x \leq 0$.

$$\frac{d}{dx} 3x^2 + 2x = 6x + 2$$

So $(-1/3,1/3)$ is a critical point on the edge.

3. Edge from (-2,2) to (2,2). Here $y = 2$: $f(x, 2) = 2x^2$ and

$$\frac{d}{dx} 2x^2 = 4x$$

so that there is a critical point at (0,2).

4. The vertices (0,0), (2,2) and (-2,2).

Finally we have isolated all possible points where $f$ could take on its absolute max and min values and we need only evaluate.

<table>
<thead>
<tr>
<th>Point $P$</th>
<th>$f(P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 1)</td>
<td>$f(0,1) = -1$</td>
</tr>
<tr>
<td>(1/3, 1/3)</td>
<td>$f(1/3,1/3) = -1/3$</td>
</tr>
<tr>
<td>(-1/3, 1/3)</td>
<td>$f(-1/3,1/3) = -1/3$</td>
</tr>
<tr>
<td>(0, 2)</td>
<td>$f(0,2) = 0$</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>$f(0,0) = 0$</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>$f(2,2) = 8$</td>
</tr>
<tr>
<td>(-2, 2)</td>
<td>$f(-2,2) = 8$</td>
</tr>
</tbody>
</table>

The absolute maximum value 8 occurs at (2,2) and (-2,2) and the absolute minimum value $-1$ occurs at (0,1).

If the region is not closed and bounded then there is no foolproof method for finding absolute extrema but we do have a criterion for the local extrema.

Recall the:

**The Second Derivative Test for functions $f(x)$ of a single variable.** If $f'(a) = 0$ then $f''(a) > 0$ implies $f$ has a local minimum at $a$ and $f''(a) < 0$ implies $f$ has a local max
at $a$ and $f''(a) = 0$ or $f''(a)$ does not exist then this test is indeterminant. (We don’t have a clue.)

**The Second Derivative Test for Functions** $f(x, y)$ If $\nabla f(a, b) = \vec{0}$ and if the second partials exist near $(a, b)$ and if $D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$ (the **Discriminant**) then

1. if $D > 0$ and $f_{xx}(a, b) < 0$ then $f$ has a local maximum at $(a, b)$.
2. if $D > 0$ and $f_{xx}(a, b) > 0$ then $f$ has a local minimum at $(a, b)$.
3. if $D < 0$ then $f$ has a saddle at $(a, b)$.
4. if $D = 0$ or if the second partials do not exist near $(a, b)$ then the test is “indeterminant” (does not work).

**Example**

1. $f(x, y) = x^2 + y^2$ has a local minimum at $(0,0)$
2. $f(x, y) = x^3 + y^3$ has a local saddle at $(0,0)$ but the test is indeterminant ($D = 0$).
3. $f(x, y) = x^4 + y^4$ has a local minimum at $(0,0)$ but the test is indeterminant.

**Example:** Find all the local maxima, minima and saddle points of $f(x, y) = x^3 + 12xy + 8y^3$

**Solution:** Find all critical points $\nabla f = (3x^2 + 12y)i + (12x + 24y^2)j$. $F$ is differentiable everywhere and so the only critical points are where $\nabla f = \vec{0}$ We solve the (non linear!) system

$$3x^2 + 12y = 0, 12x + 24y^2 = 0$$

from which we see $x^2 = -4y$ and $x = -2y^2$. Square both sides of the first equation $x^2 = 4y^4$ and combined with the first equation implies $4y^4 = -4y$ or $y(y^3 + 1) = 0$. Therefore $y = 0$ or $y = -1$. Substituting to find $x$ we have $(0,0)$ and $(-2,-1)$ are the two critical points. (We check that they satisfy the equation $\nabla f = \vec{0}$.) Next we classify the critical points using the second derivative test. Compute the second partial derivatives: $f_{xx} = 6x$, $f_{yy} = 48y$ and $f_{xy} = f_{yx} = 12$ and so $D = 288xy - 144$. We only need the value of $D$ at the critical points.

1. $(0,0)$: Here $D = -144 < 0$ and so $(0,0)$ is a saddle point.
2. $(-2,1)$: Here $D > 0$ and $f_{xx}(-2, -1) = -12 < 0$ and so $(-2,1)$ is a local maximum.

There is a similar test for functions $f(x, y, z)$ of three variables.