14.2 Limits and Continuity:

We now extend the concepts of limits and continuity to the case that \( f \) is a real valued function defined on a set \( D \) in \( \mathbb{R}^2 \) (or \( \mathbb{R}^3 \)).

**Definition:** We say that the limit of \( f(x, y) \) as \( (x, y) \) approaches \( (x_0, y_0) \) is \( L \) if \( |f(x, y) - L| \) can be made arbitrarily small by choosing \( 0 < |(x, y) - (x_0, y_0)| \) small (but not 0). We write
\[
\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L.
\]

The subtlety here, that did not arise in functions of a single variable, is that \( f(x, y) \) must get close to \( L \) no matter how \( (x, y) \) goes to \( (x_0, y_0) \). For example it is quite possible that \( \lim_{x \to x_0} f(x, y_0) \) exists but \( \lim_{(x,y) \to (x_0,y_0)} f(x, y) \) does not.

**Examples:**
\[
\begin{align*}
\lim_{(x,y) \to (1,3)} \sin x + \frac{y}{x - 2} &= \sin 1 - 3 \\
\lim_{(x,y) \to (2,2)} \frac{x^2 - y^2}{y - x} &= \lim_{(x,y) \to (2,2)} -(x + y) = -4 \\
\lim_{(x,y) \to (2,0)} \frac{x \sin y}{y} &= -4 \\
\end{align*}
\]

**Counterexample:**
\[
\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}
\]
does not exist

The reason is that if we approach the origin along the straight line \( y = mx \) (where \( m \) is the slope) then \( xy/(x^2 + y^2) = mx^2/(x^2 + m^2 x^2) = m/(1 + m^2) \) and so the limiting value depends on the path of approach and that means there is no limit as \( (x, y) \to (0,0) \).

**Curvature:** The curvature is a measure of how fast a curve turns. It is the reciprocal of the radius of the “osculating” circle. The curvature of the curve \( \vec{r} \) at \( \vec{r}(t) \) is
\[
\kappa(t) = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|^3}
\]
Example: Consider $\vec{r} = \cos 2\pi t \vec{i} + \sin 2\pi t \vec{j} + t \vec{k}$. Then

$$\vec{v}(t) = -2\pi \sin 2\pi t \vec{i} + 2\pi \cos 2\pi t \vec{j} + \vec{j}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{\sqrt{4\pi^2 + 1}}$$

$$\vec{a}(t) = -4\pi^2 \cos 2\pi t \vec{i} - 4\pi^2 \sin 2\pi t$$

$$\vec{N}(t) = -\cos 2\pi t \vec{i} - \sin 2\pi t$$

$$\vec{B}(t) = \frac{1}{\sqrt{4\pi^2 + 1}} \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\pi \sin 2\pi t & 2\pi \cos 2\pi t & 1 \\ -\cos 2\pi t & -\sin 2\pi t & 0 \end{bmatrix}$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}(t)|^3} = \frac{4\pi^2 \sqrt{4\pi^2 + 1}}{(\sqrt{4\pi^2 + 1})^3} = \frac{4\pi^2}{4\pi^2 + 1}$$