1. Consider the surface \( x^4 + 3xz + z^2 + \cos(\pi xy) = -2 \) and the point \( P_0(-1, 1, 2) \) on that surface. Find an equation of

(a) the tangent plane at \( P_0 \)

For \( F(x, y, z) = x^4 + 3xz + z^2 + \cos(\pi xy) \), we know that \( \nabla F \) is perpendicular to the level surface.

\[
\nabla F(x, y, z) = (4x^3 + 3z - \sin(\pi xy)\pi y)\vec{i} - \sin(\pi xy)\pi x\vec{j} + (3x + 2z)\vec{k}
\]

\( \nabla F(-1, 1, 2) = 2\vec{i} + \vec{k} \)

so that the tangent plane \( 2(x - (-1)) + (z - 2) = 0 \) or \( 2x + z = 0 \)

(b) the normal line to the surface at \( P_0 \).

The normal line is \( \vec{r}(t) = -\vec{i} + \vec{j} + 2\vec{k} + t(2\vec{i} + \vec{k}) \)

2. Sketch the region of integration and write an equivalent double integral with the the order of integration reversed. Do NOT evaluate.

\[
\int_e^1 \int_0^{\ln x} x^2y \, dy \, dx
\]

The region is \( 0 \leq y \leq \ln x, 1 \leq x \leq e \). Sketch. The region is also \( 1 \leq x \leq e^y \), \( 0 \leq y \leq 1 \):

\[
\int_1^e \int_0^{\ln x} x^2y \, dy \, dx = \int_0^1 \int_{e^y}^e x^2y \, dx \, dy
\]

3. Test the function \( f(x, y) = 3xy - x^3 - y^3 \) for local maxima, minima and saddle points.

Check for critical points. These occur when \( \nabla f = \vec{0} \) or \( f \) is not differentiable but \( f \) is differentiable everywhere (because the partial derivatives exist and are continuous). Compute \( \nabla f = (3y - 3x^2)\vec{i} + (3x - 3y^2)\vec{j} \). Set \( \nabla f = \vec{0} \) gives \( 3y - 3x^2 = 0 \) and \( 3x - 3y^2 = 0 \). Therefore \( y = x^2 \) and \( x = y^2 \) so that \( x = (x^2)^2 = \)
Therefore $x(1 - x^3) = 0$ which says $x = 0$ (in which case $y = 0$) or $x = 1$ (in which case $y = 1$). The critical points are $(0,0)$ and $(1,1)$.

Consider first $(0,0)$. We have $f_{xx} = -6x$, $f_{xy} = 3$, $f_{yy} = -6y$ At $(0,0)$, $f_{xx} = 0 = f_{yy}$ so that $f_{xx}f_{yy} - f_{xy}^2 = -9$ so that $(0,0)$ is a saddle point by the second derivative test.

Consider next $(1,1)$. At $(1,1)$, $f_{xx} = -6$, $f_{yy} = -6$, so that $f_{xx}f_{yy} - f_{xy}^2 = 27$. Since $f_{xx} < 0$ this says $(1,1)$ is a local maximum.

4. Find the absolute maximum and minimum values of $f(x, y) = 2x - 2xy + y + 2$ on the closed triangular region $R$ in the first quadrant, bounded by the lines $y = 0$ $x = 0$ and $x + y = 3$. (Show all your work.)

Sketch the triangular region. First we check the interior for critical points. We see

$$\nabla f = (2 - 2y)i + (1 - 2x)j$$

Set $\nabla f = 0$ and we see $x = 1/2$ and $y = 1$ so that $(1/2, 1)$ is a critical point. Since $f$ is differentiable everywhere this is the only critical point.

Next we check the edges. Consider first the line $y = 0$ ($0 \leq x \leq 4$) $f(x, 0) = 2x + 2$ and this function is increasing in $x$ (its derivative is 2) and it has no critical points.

The second edge is $x = 0$ $0 \leq y \leq 3$ where $f(0, y) = y + 2$ and this is an increasing function of $y$ (its derivative is 1) and so it has no critical points.

The third and final edge is $x + y = 3$, $0 \leq x \leq 3$ where $f(x, 3 - x) = 2x - 2x(3 - x) + 3 - x + 2 = 2x^2 - 5x + 5$ (or in terms of $y$ $f(3 - y, y) = 6 - 2y - (3 - y)y + y + 2 = 2y^2 - 7y + 8$). This function has a critical point where its derivative $4x - 5$ is zero and that is $x = 5/4$. We must therefore include $(5/4, 7/4)$ in our analysis.

The corners are $(0, 3), (3, 0)$ and $(0, 0)$.

Evaluate $f$ at all 5 points .

<table>
<thead>
<tr>
<th>Point $P$</th>
<th>$f(P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1/2, 1)$</td>
<td>$f(1/2, 1) = 3$</td>
</tr>
<tr>
<td>$(5/4, 7/4)$</td>
<td>$f(5/4, 7/4) = 15/8$</td>
</tr>
<tr>
<td>$(0, 3)$</td>
<td>$f(0, 3) = 5$</td>
</tr>
<tr>
<td>$(3, 0)$</td>
<td>$f(3, 0) = 8$</td>
</tr>
<tr>
<td>$(0, 0)$</td>
<td>$f(0, 0) = 2$</td>
</tr>
</tbody>
</table>
We see that \( f \) takes its absolute maximum value of 8 at \((3,0)\) and its absolute minimum value of \(15/8\) at \((5/4,7/4)\).

5. The temperature at a point \((x, y)\) on a metal plate is \( T(x, y) = 10xy \) (in degrees Celsius). Use Lagrange multipliers (or otherwise) to find the points on the curve \(4x^2 + 9y^2 = 72\) that are the hottest and coldest.

Here we look for points where the level curves of \( T \) meet the constraint curve \( g(x, y) = 72 \) where \( g(x, y) = 4x^2 + 9y^2 \) tangentially, that is \( \nabla T = \lambda \nabla g \). Componentwise we have

\[
\begin{align*}
10y &= 8\lambda x \\
10x &= 18\lambda y \\
4x^2 + 9y^2 &= 72
\end{align*}
\]

Multiply the first equation by \( x \) and the second by \( y \) and subtract: \( 8\lambda x^2 - 18\lambda y^2 = 0 \). Further we know \( \lambda \neq 0 \) because otherwise the first two equations would say \( x = 0 \) and \( y = 0 \) and \((0,0)\) is not on our constraint curve. We can cancel the \( \lambda \) above to get \( 8x^2 - 18y^2 = 0 \). Consequently \( 9y^2 = 4x^2 \) and therefore the equation for the constraint curve becomes \( 4x^2 + 4x^2 = 72 \) so that \( x = \pm 3 \) and so \( y = \pm 2 \) and we get 4 points: \((3,2),\ (3,-2),\ (-3,2),\ (-3,-2)\). Evaluate

\[
\begin{align*}
T(3, 2) &= 60 \\
T(3, -2) &= -60 \\
T(-3, 2) &= -60 \\
T(-3, -2) &= 60
\end{align*}
\]

So the hottest points are \((3,2),\) and \((-3,-2)\) where the temperature is 60 degrees and the coldest points are \((-3,2)\) and \((3,-2)\) where the temperature is -60.

6. Find the volume of the wedge in the first octant bounded by the parabolic cylinder \( z = 8 - 2x^2 \) and the plane \( x + y = 2 \) and the coordinate planes.

The wedge is the region below \( z = 8 - 2x^2 \) but above the triangle \( R \) in \( xy\)-plane bounded by \( x + y = 2 \) and the coordinate axes. We sketch \( R \). The volume is
\[ \int \int_R 8 - 2x^2 \, dA = \int_0^2 \int_0^{2-x} 8 - 2x^2 \, dy \, dx \]
\[ = \int_0^2 (8 - 2x^2)(2 - x) \, dx \]
\[ = \int_0^2 16 - 8x - 4x^2 + 2x^3 \, dx \]
\[ = 16x - \frac{4}{3}x^3 + \frac{1}{2}x^4 \bigg|_0^2 \]
\[ = 32 - 16 - \frac{32}{3} + 8 = \frac{40}{3} \]

7. Change the Cartesian integral to an equivalent polar integral. Then evaluate the polar integral

\[ \int_0^2 \int_0^{\sqrt{4-x^2}} \frac{xe^{x^2+y^2}}{\sqrt{x^2+y^2}} \, dy \, dx \]

(14)

The region of integration is bounded by \(0 \leq y \leq \sqrt{4 - x^2}\) and \(0 \leq x \leq 2\) which is a quarter of a circle \((y = \sqrt{4 - x^2})\) of radius 2 centered at (0,0) and lying in the first quadrant. To convert we describe this region as \(0 \leq r \leq 2\) and \(0 \leq \theta \leq \pi/2\).

We also have \(x^2 + y^2 = r^2\) and \(x = r \cos \theta\) and \(dA = r \, dr \, d\theta\) so that

\[ \int_0^2 \int_0^{\sqrt{4-x^2}} \frac{xe^{x^2+y^2}}{\sqrt{x^2+y^2}} \, dy \, dx = \int_0^{\pi/2} \int_0^2 \frac{r \cos \theta e^{r^2}}{r} \, r \, dr \, d\theta \]
\[ = \int_0^{\pi/2} \cos \theta \int_0^2 e^{r^2} \, r \, dr \, d\theta \]

Substitute \(u = r^2\) so that \(du = 2r \, dr\) and we have the above equals

\[ \int_0^{\pi/2} \cos \theta \int_0^2 e^{r^2} \, r \, dr \, d\theta = \int_0^{\pi/2} \cos \theta \int_0^4 e^u \, (1/2) \, du \, d\theta \]
\[ = \int_0^{\pi/2} \cos \theta (1/2) e^{u} \bigg|_0^4 \, d\theta \]
\[ = \frac{e^4 - 1}{2} \int_0^{\pi/2} \cos \theta d\theta = \frac{e^4 - 1}{2} \sin \theta \bigg|_0^{\pi/2} = \frac{e^4 - 1}{2} \]