1. Identify and sketch the graph of the surface \(4x^2 + y^2 - z^2 = 1\).

This is an elliptic hyperboloid of one sheet. The planes perpendicular to the \(z\)-axis (parallel to the \(xy\)-plane) intersect the surface in ellipses. Perpendicular to the \(x\) and \(y\)-axis the cross sections are hyperbolas.

2. The position of a particle at time \(t\) is \(\vec{r}(t) = t \ln(t + 1) \hat{i} + (t + 1)^{1/2} \hat{j} + (t + 1)^{3/2} \hat{k}\)

(a) Find the velocity and acceleration vectors.

The velocity is

\[ \vec{v}(t) = \vec{r}'(t) = (\ln(t + 1) + \frac{t}{t + 1}) \hat{i} + \frac{1}{2}(t + 1)^{-1/2} \hat{j} + \frac{3}{2}(t + 1)^{1/2} \hat{k} \]

and the acceleration is

\[ \vec{a}(t) = \vec{v}'(t) = \left( \frac{1}{t + 1} + \frac{(t + 1) - t}{(t + 1)^2} \right) \hat{i} - \frac{1}{4}(t + 1)^{-3/2} \hat{j} + \frac{3}{4}(t + 1)^{-1/2} \hat{k} \]

(b) Find an equation for the tangent line to the path (in part (a)) of the particle when \(t = 0\).

A tangent vector is \(\vec{v}(0) = \frac{1}{2} \hat{j} + \frac{3}{2} \hat{k}\). We want the tangent line at \(\vec{r}(0) = \hat{j} + \hat{k}\) and that line is

\[ \vec{r}(t) = \hat{j} + \hat{k} + t\left( \frac{1}{2} \hat{j} + \frac{3}{2} \hat{k} \right) \]

3. Solve the initial problem for \(\vec{r}\) as a vector function of \(t\) if

Differential equation: \(\frac{d\vec{r}}{dt} = te^{-t} \hat{i} + e^{-2t} \hat{j} - e^{-2t} \hat{k}\)

Initial Condition: \(\vec{r}(0) = -\hat{i} + \hat{j} + 2\hat{k}\)
We integrate:
\[ \vec{r}(t) = \int te^{-t} \, dt \hat{i} + \int e^{-t} \, dt \hat{j} - \int e^{-2t} \, dt \hat{k} = \]
The first integral can be done by parts (\( \int u \, dv = uv - \int v \, du \) with \( u = t, \, dv = e^{-t} \, dt \))
\[ \vec{r}(t) = (-te^{-t} - e^{-t}) \hat{i} - e^{-t} \hat{j} + \frac{1}{2} e^{-2t} \, dt \hat{k} + \vec{C} \]
It remains to find \( \vec{C} \). By the initial condition
\[ -\hat{i} + \hat{j} + 2\hat{k} = \vec{r}(0) = -\hat{i} - \hat{j} + \frac{1}{2} \hat{k} + \vec{C} \]
so that \( \vec{C} = 2\hat{j} + (3/2)\hat{k} \) and therefore
\[ \vec{r}(t) = (-te^{-t} - e^{-t}) \hat{i} + (2 - e^{-t}) \hat{j} + \frac{1}{2} (e^{-2t} + 3) \, dt \hat{k} \]

4. Find the length of the curve \( \vec{r} = \frac{1}{3} t^{3/2} \hat{i} + 2t \hat{j} + \frac{1}{\sqrt{3}} t^{3/2} \hat{k}, \, 0 \leq t \leq 5 \).

We need the speed. Differentiate
\[ \vec{v}(t) = \vec{r}'(t) = \frac{1}{3} \frac{3}{2} t^{1/2} \hat{i} + 2 \hat{j} + \frac{1}{\sqrt{3}} \frac{3}{2} t^{1/2} \hat{k} = \frac{1}{2} t^{1/2} \hat{i} + 2 \hat{j} + \frac{\sqrt{3}}{2} t^{1/2} \hat{k} \]
so that \( |\vec{r}'(t)| = \sqrt{t/4 + 4 + 3t/4} = (4 + t)^{1/2} \). The length is therefore
\[ \int_{0}^{5} (4 + t)^{1/2} \, dt = \frac{2}{3} (4 + t)^{3/2} \bigg|_{0}^{5} = \frac{2}{3} \left[ (9)^{3/2} - (4)^{3/2} \right] = \frac{38}{3} \]

5. (a) Find and sketch the domain of the function \( f(x, y) = \sqrt{x^2 + 4y^2} - 9 \).

The domain is the set \( \{(x, y) : x^2 + 4y^2 \geq 9\} \) which is the exterior of an ellipse.

(b) Find an equation for and sketch the graph of the level curve of the function \( f(x, y) \) of part (a) that passes through the point (3,-1).
Since $f(3, -1) = 2$ we are looking for the level curve $f(x, y) = 2$ which is $x^2 + 4y^2 = 13$ which is an ellipse.

6. Find the limit. \[ \lim_{(x,y) \to (2,-2)} \frac{x + y}{x^2 - y^2} \]

This limit is of the form 0/0 and so we look for a cancellation in the fraction

\[ \lim_{(x,y) \to (2,-2)} \frac{x + y}{x^2 - y^2} = \lim_{(x,y) \to (2,-2)} \frac{x + y}{(x + y)(x - y)} = \lim_{(x,y) \to (2,-2)} \frac{1}{x - y} = \frac{1}{4} \]

7. By considering different paths of approach, show that \( f(x, y) \) has no limit as \( (x, y) \to (0,0) \).

\[ f(x, y) = \frac{y^2 + xy}{x^2 + y^2} \]

Consider

\[ f(r \cos \theta, r \sin \theta) = \frac{(r \sin \theta)^2 + r \cos \theta r \sin \theta}{r^2} = (\sin \theta)^2 + \cos \theta r \sin \theta \]

and so

\[ \lim_{r \to 0} f(r \cos \theta, r \sin \theta) = (\sin \theta)^2 + \cos \theta r \sin \theta \]

and this limit depends on \( \theta \) which means it depends on the path of approach to \( (0,0) \) and so \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist.

8. Find all the second partial derivatives of \( f(x, y) = \ln(x + y^2) \)

There are three second distinct partial derivatives. We first need the first partials.

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{1}{x + y^2} \\
\frac{\partial f}{\partial y} &= \frac{2y}{x + y^2} \\
\frac{\partial^2 f}{\partial x^2} &= -\frac{1}{(x + y^2)^2} \\
\frac{\partial^2 f}{\partial x \partial y} &= -\frac{2y}{(x + y^2)^2} \\
\frac{\partial^2 f}{\partial y^2} &= -\frac{2(x + y^2) - 4y^2}{(x + y^2)^2}
\end{align*}
\]
and of course the order of differentiation for the second mixed partial derivative is inconsequential.

9. Suppose we substitute \( x = u^2 - v^2 \) and \( y = 2uv \) in a differentiable function \( w = f(x, y) \). Show that

\[
\frac{1}{2} \frac{\partial w}{\partial u} = \frac{\partial f}{\partial x} u + \frac{\partial f}{\partial y} v
\]

and derive a comparable expression for \( \frac{\partial w}{\partial v} \).

By the chain rule in two variables:

\[
\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}
\]

but \( \frac{\partial x}{\partial u} = 2u \) and \( \frac{\partial y}{\partial u} = 2v \) so that

\[
\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x} 2u + \frac{\partial f}{\partial y} 2v
\]

and dividing by 2 gives the desired expression. Similarly we have

\[
\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x} (-2v) + \frac{\partial f}{\partial y} 2u
\]

but \( \frac{\partial x}{\partial v} = -2v \) and \( \frac{\partial y}{\partial v} = 2u \) so that

\[
\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x} (-2v) + \frac{\partial f}{\partial y} 2u
\]

and so we have

\[
\frac{1}{2} \frac{\partial w}{\partial v} = -\frac{\partial f}{\partial x} v + \frac{\partial f}{\partial y} u
\]

10. (a) Find the gradient of the function \( f(x, y, z) = xy + 2yz + 3xz \).

The gradient is \( \nabla f = (y + 3z)\hat{i} + (x + 2z)\hat{j} + (2y + 3x)\hat{k} \)

(b) Find the (directional) derivative of \( f \) at \( P_0(1, 3, -1) \) in the direction of \( \vec{u} = \hat{i} - \hat{j} + 2\hat{k} \).

At \( P_0(1, 3, -1) \), we have \( \nabla f(1, 3, -1) = -\hat{j} + 3\hat{k} \) so that the directional derivative in the direction of \( f \) in the direction \( \vec{u} \) is

\[
D_{\vec{u}} f(1, 3, -1) = \frac{\nabla f(1, 3, -1) \cdot \vec{u}}{|\vec{u}|} = \frac{7}{\sqrt{6}}
\]