1. Find the directions in which the function \( f(x, y, z) = \frac{x}{y} - yz \) increase and decrease the most rapidly at \( P_0(4, 1, 1) \). Then find the derivatives of the function in these directions. (5)

We need \( \nabla f = \left( \frac{1}{y} \right) \mathbf{i} - x \mathbf{j} - y \mathbf{k} \) and \( \nabla f(4, 1, 1) = 5 \mathbf{i} - 5 \mathbf{j} - \mathbf{k} \). The direction of most rapid increase at \( P_0 \) is

\[
\frac{\nabla f}{|\nabla f|} = \frac{5\mathbf{i} - 5\mathbf{j} - \mathbf{k}}{\sqrt{27}}
\]

and the direction of most rapid decrease is the opposite direction

\[
-\frac{\nabla f}{|\nabla f|} = -\frac{5\mathbf{i} - 5\mathbf{j} - \mathbf{k}}{\sqrt{27}}
\]

and the most rapid rate of increase at \( P_0 \) is \( |\nabla f(4, 1, 1)| = \sqrt{27} = 3\sqrt{3} \) and the most rapid rate of decrease is \(-|\nabla f(4, 1, 1)| = -3\sqrt{3} \).

2. Find the linearization \( L(x, y, z) \) of \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \) at \((1,2,2)\). (4)

We compute \( \nabla f = (x^2 + y^2 + z^2)^{-1/2} (2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}) \) and so \( \nabla f(1,2,2) = \frac{1}{3} (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \). We also need \( f(1,2,2) = 3 \). Therefore

\[
L(x, y, z) = 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) + \frac{2}{3}(z-2)
\]

3. Find equations for the

(a) tangent plane and

(b) normal line at the point \( P_0(0,1,2) \) on the given surface.

\[
\cos \pi x - x^2 y + e^{xy} + yz = 4
\]

We need \( \nabla f \) for \( f(x, y, z) = \cos \pi x - x^2 y + e^{xy} + yz \) because the given surface is a level surface of \( f \).

\[
\nabla f = (-\pi \sin \pi x - 2xy + ye^{xy}) \mathbf{i} + (-x^2 + xe^y) \mathbf{j} + y \mathbf{k}
\]

At \( P_0 \) we have \( \nabla f(0,1,2) = \mathbf{i} + \mathbf{k} \). The tangent plane is \( x + (z-2) = 0 \) because \( \mathbf{i} + \mathbf{k} \) is normal to the plane. The normal line is \( x = t, y = 1 \) and \( z = 2 + t \).

4. Find an equation for the plane that is tangent to the surface \( z = \ln(x^2 + y^2) \) at \((1,0,0)\). (4)

We need \( \nabla z = (x^2 + y^2)^{-1}(2x \mathbf{i} + 2y \mathbf{j}) \) and at \((1,0,0)\) we have \( \nabla z = 2\mathbf{i} \) and so the tangent plane is \( z = z_0 + f_x(1,0)(x-1) + f_y(1,0)(y-0) = 0 + 2(x-1) + 0(y-0) \) or \( z = 2x - 2 \)