My research interests are several complex variables and operator theory on domains in $\mathbb{C}^n$. In particular, I use the methods of several complex variables to study problems in operator theory, with my main focus being Hankel operators and Toeplitz operators on the Bergman spaces of bounded domains in $\mathbb{C}^n$.

Operator theory is the study of linear operators between vector spaces. These vector spaces may be finite or infinite dimensional. For example, an $n \times n$ matrix defines a linear operator from the finite dimensional spaces $\mathbb{R}^n$ to $\mathbb{R}^n$ with respect to a certain basis. Notions such as compactness and continuity of such operators are not very interesting if the domain is finite dimensional, so most of the modern operator theory concerns the study of linear transformations on infinite dimensional spaces. An example of such infinite dimensional spaces is the Bergman space. Let $\Omega \subset \mathbb{C}^n$ be an open non-empty connected set (a domain). Then, the Bergman space $A^2(\Omega)$ is the space of square integrable holomorphic functions on $\Omega$. The Bergman space is an example of a very special normed vector space called a Hilbert space (where we have notions of orthogonality of vectors).

Hankel operators on Bergman spaces are an important class of operators in operator theory. The Hankel operator $H_\phi$ with symbol $\phi \in L^\infty(\Omega)$ is defined as

$$H_\phi(g) = (I - P)(\phi g)$$

for all $g \in A^2(\Omega)$. Here, $P$ is the Bergman projection and $I$ is the identity operator. In particular, my interest lies in the compactness of such operators in relation to the behaviour of the symbol $\phi$ on the boundary of the domain $\Omega$. Let $X$ and $Y$ be Banach spaces. A linear operator $T : X \to Y$ is said to be compact if the image $T(\{x \in X : \|x\| < 1\})$ is relatively compact in the norm topology on $Y$.

Hankel operators have applications in the study of matrices. A Hankel matrix is a square matrix with constant left to right skew diagonals. A Hankel operator on a Hilbert space is an operator whose matrix representation with respect to an orthonormal basis (possibly countably infinite) is a Hankel matrix. Some applications of Hankel matrices include Markov chains and signal processing. See [Par88].

The domains one studies in $\mathbb{C}^n$ usually have certain properties. For example, a domain $\Omega \subset \mathbb{C}^n$ with the property that if $(z_1, z_2, ..., z_n) \in \Omega$ then

$$(r_1 e^{i\theta_1} z_1, r_2 e^{i\theta_2} z_2, ..., r_n e^{i\theta_n} z_n) \in \Omega$$

for all $r_j \in [0,1]$ and for all $\theta_j \in \mathbb{R}$ is called a complete Reinhardt domain. For example, the polydisk in $\mathbb{C}^n$ and the ball in $\mathbb{C}^n$ are examples of a complete Reinhardt domains, however, there are others. These domains have certain boundary geometries that are important for our result.

# Main Results

In one complex dimension, Axler [Axl86] showed that for $f \in L^\infty(\mathbb{D})$ and holomorphic, $H_{\overline{f}}$ is compact if and only if $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. The space of holomorphic functions on $\mathbb{D}$ with the aforementioned growth condition on the derivative is called the little Bloch space. In higher dimensions, Peloso [Pel94] has a similar result on the weighted Bergman spaces of strongly pseudoconvex domains.

Recently Čučković and Şahutoğlu in [ČŞ14], and [ČŞ09] gave a different characterization for symbols that are smooth up to to boundary. Their characterization is given in terms of the holomorphicity of the symbol along the analytic disks in the boundary of the domain. Le [Le10] has a result for symbols only continuous up to the boundary of the domain, but the class of domains is restricted to the product of disks. The characterization given in [Le10] depends upon decomposing the symbol as a sum of a holomorphic function and another function that vanishes on the boundary. As a definition, $\Delta \subset b\Omega$ is said to be an analytic disk if there exists a function $f : \mathbb{D} \to b\Omega$ such that each component of $f$ is holomorphic and $f(\mathbb{D}) = \Delta$. The function $f$ is identified with its image $f(\mathbb{D})$.

Our motivation comes from trying to understand if regularity of the symbol played a role. That is, how we could generalize the following theorem by weakening the regularity assumption on the symbol $\phi$ at the cost of perhaps a more restricted class of domains.

We were motivated by the following theorem.

**Theorem** ([ČŞ09]). *Let $\Omega \subset \mathbb{C}^2$ be a smooth bounded convex domain and $\phi \in C^\infty(\overline{\Omega})$. Then $H_\phi$ is compact on the Bergman space $A^2(\Omega)$ if and only if $\phi \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$.*

We noticed we could relax the smoothness of the symbol and the smoothness of the boundary if the domain $\Omega \subset \mathbb{C}^2$ is bounded, convex, and Reinhardt. Our main result was the following.

**Theorem** ([CŞ17]). *Let $\Omega \subset \mathbb{C}^2$ be a bounded convex Reinhardt domain. Suppose $\phi \in C(\overline{\Omega})$. Then, the Hankel operator $H_\phi$ is compact on the Bergman space $A^2(\Omega)$ if and only if $\phi \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$.*

In [Le10], Le characterizes compactness of Hankel operators with conjugate holomorphic symbols on the Bergman space of the polydisk in $\mathbb{C}^n$. We wish to generalize this result to other domains with similar rotational symmetries. The following questions were investigated and partially answered in [Clo17].

Questions: Let $\Omega \subset \mathbb{C}^2$ be a bounded convex Reinhardt domain. Suppose $\phi \in A^2(\Omega)$ so that $H_{\overline{\phi}}$ is compact on $A^2(\Omega)$. Can one use the rotational symmetry of the domain to infer properties of $\phi$? Is $\phi$ necessarily constant? Do the analytic disks in the boundary of the domain play a role?

In [CŞ17] we studied the geometry of non-constant analytic disks in the boundary of bounded convex Reinhardt domains in $\mathbb{C}^2$. We showed if there is a non-constant analytic disk in the boundary of a bounded convex Reinhardt domain in $\mathbb{C}^2$, it must be affine and horizontal or vertical. This led to the following notion.

We say a bounded domain $\Omega \subset \mathbb{C}^2$ satisfies the disk condition if there exists non-constant holomorphic functions $f : \mathbb{D} \to \mathbb{C}$, $g : \mathbb{D} \to \mathbb{C}$ so that $F(\mathbb{D}) \subset b\Omega$ and $G(\mathbb{D}) \subset b\Omega$, where $F(z) = (f(z), c_1)$ and $G(z) = (c_2, g(z))$ for some constants $c_1, c_2 \in \mathbb{C}$.

**Theorem** ([Clo17]). *Let $\Omega \subset \mathbb{C}^2$ be a bounded convex Reinhardt domain. Let $\phi \in A^2(\Omega)$ so that $H_{\overline{\phi}}$ is compact on $A^2(\Omega)$. If the disk condition is satisfied, then $\phi$ is identically constant.*

The proof uses the rotational symmetry of the domain in a significant way.

If we consider only bounded pseudoconvex complete Reinhardt domains in $\mathbb{C}^2$, we need smoothness of the boundary. However, non-constant disks in the boundary of such domains are not necessarily affine and horizontal or vertical. If such non-affine disk occurs in the boundary of such domain, we have the following theorem.

**Theorem** ([Clo17])**.** *Let $\Omega \subset \mathbb{C}^2$ be a smooth bounded pseudoconvex complete Reinhardt domain. Suppose $\phi \in A^2(\Omega)$ so that the Hankel operator $H_{\overline{\phi}}$ is compact on $A^2(\Omega)$. If either of the following two conditions hold, then $\phi$ is identically constant.*

1. *The disk condition holds.*
2. *There exists non-constant holomorphic functions $f_1, f_2 : \mathbb{D} \to \mathbb{C}$ so that $f = (f_1, f_2)$ and $f(\mathbb{D}) \subset b\Omega$.*

# Future Work

Future work may include relaxing the assumption on domain to include convex domains that are not necessarily Reinhardt. Another possible direction is to investigate the compactness of Hankel operators with continuous symbols on the Bergman spaces of bounded convex Reinhardt domains in $\mathbb{C}^n$ for $n \geq 3$.

It is well known that the product of Hankel operators can be written as the semi-commutant of Toeplitz operators. See for example [ČŞ14]. Perhaps one can study the compactness of such Toeplitz operators if the product of Hankel operators are known to be compact. There is the famous Axler-Zheng theorem connecting the compactness of sums of Toeplitz operators with the Berezin transform. See [AZ98]. It would be interesting to connect it to compactness of Hankel operators.

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