Research Statement

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1. Introduction

Quantum topology/algebra has developed over the last few decades since the discovery of Jones polynomial [J]. Later in 1988 E. Witten invented the notion of a topological quantum field theory(TQFT) and related the Jones polynomial to a 2-dimensional modular functor arising in conformal field theory [Wi1, Wi2]. Quantum topology/algebra is closely related to representation theory of quantum groups [K], theory of von Neumann algebra, condensed matter physics, and topological quantum computation [Wa].

The fundamental notions are modular tensor category(MTC) and TQFT. MTCs provide the algebraic data to build up TQFTs from which one obtains invariants of 3-manifolds, namely quantum invariants [Tu1]. In this sense, the study of MTC is more or less equivalent to the study of TQFT. So the classification of MTCs is an interesting and important subject in both of algebraic and topological point of view. MTCs are also used to describe anyonic properties of certain quantum systems and topological quantum computing is encoded by braiding non-abelian anyons, in which sense MTCs also form the algebraic base of topological quantum computation [FKLW].

I am working on a classification of MTCs and more generally fusion categories. I am also studying a generalized version of Yang-Baxter operator in the connection of this operator to braid group representations, link invariants, and the structure of ribbon categories.

2. Classification of fusion categories

A **Fusion category** is a tensor category with certain conditions [BK, ML], among them pentagon axioms(a.k.a. Elliott-Biedenharn identity) $(\mathrm{id}_x \otimes \alpha_{y,z,w}) \circ \alpha_{x,yz,w} \circ (\alpha_{x,y,z} \otimes \mathrm{id}_w) = \alpha_{x,y,zw} \circ \alpha_{xy,z,w}$ allows associativity on tensor products and reduces the number of fusion categories to exist on any given set of fusion rules. **MTC** is a fusion category with additional structures such as braiding, twist, and non-degeneracy of S-matrix.

2.1. **Classification by solving pentagon equations.** In most cases, it often takes a long time to solve pentagon equations. However once one has all pentagon solutions in hands, many of the other structures can be obtained relatively quickly. My work [HH] with T. Hagge was done by doing so. Based on the pentagon solutions we completely classified fusion categories on a specific set of fusion rules, and which would complete the classification of rank 3 fusion categories if a conjecture of V. Ostrik is true [O2]. Furthermore, this work was the first classification on the fusion rules with multiplicity > 1.

The pentagon solutions which appeared in [HH] were used again in the construction of a MTC by the so called quantum double construction [M] (a.k.a. Drinfeld center) in [HRW].

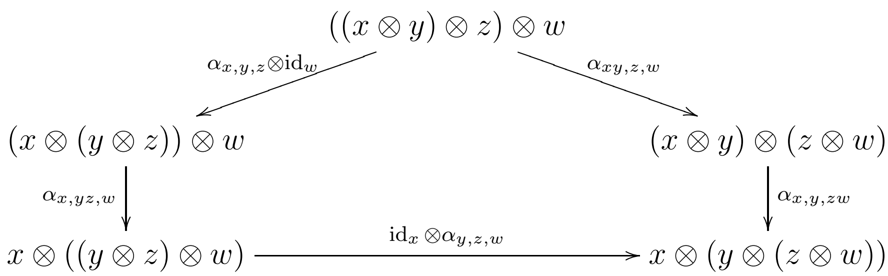


FIGURE 1. Pentagon Axiom

In the paper we studied two MTCs which are believed to be exotic in a certain sense. I have solved pentagon equations for a few other categories as well. Pentagon solutions for the category $SO(N)_2$ were used to construct a family of generalized Yang-Baxter operators in [H2]. A category of rank 10 was considered in [BGHKNNPR] and my pentagon solutions were used to show the existence and some structures of the category. Another pentagon solution appears in [CHW], where we consider a universal quantum computation and the main example is the quantum double of S_3.

2.2. **Classification of MTCs of low rank.** Despite a great progress, there is no complete classification of fusion categories yet. One feasible approach is doing it upon **rank** which is the number of simple objects. Until 2009, classification of MTCs had been done for rank 2, 3, and 4 [CP, O1, O2, RSW] (Recently a progress has been made in this direction [BNRW]). As the rank grows up, classification becomes much more challenging in the full generality. It was why E. Rowell and I imposed another constraint, non-self-dual condition, for the next case, rank 5. Each simple object has its dual object which is automatically simple again. We considered (pseudo-unitary) MTCs of rank 5 for which some object is not isomorphic to its dual. As a result there would be at least two distinct simple objects that are dual to each other. This condition induces more symmetries in S-matrix, called the Galois symmetry [CG, RSW], and thus reduces the number of possibilities to be considered. We applied the Galois symmetry and developed a symbolic computational approach to obtain a complete classification of such MTCs of rank ≤ 5 [HR].

2.3. **Classification of (weakly) integral MTCs.** Frobenius-Perron dimension is defined for each object and for each category. A fusion category is said to be **integral** if the dimension is integer valued for all simple objects. If the dimension of category is an integer, we call the category **weakly integral**. Group-theoretical categories are obtained from finite groups and well-understood. Many integral MTCs are known to be group-theoretical. Indeed, any integral MTC of dimension p^n, pq, pqr, pq^2 or pq^3 is group-theoretical [EGO, DGNO, NR, GNN]. On the other hand, examples of non-group-theoretical integral MTCs of dimension $4q^2$ were considered [GNN, NR].

In [BGHKNNPR], we classified integral MTCs of dimension pq^4 and p^2q^2. We applied some general results concerning the structure of integral MTCs such as universal grading, pointed subcategory, and centralizer of subcategory. We showed that integral MTCs of dimension pq^4 or odd p^2q^2 are all group-theoretical. We also classified non-group-theoretical ones of dimension $4q^2$. Among those categories, some categories of dimension 36 were new.

2.4. **Project on classification of fusion categories.** As stated above (pseudo-unitary) MTCs of rank at most 5 have been classified and integral MTCs have been done upon several Frobenius-Perron dimensions. A rank-wise approach to classifying (weakly) integral MTCs is also interesting. Currently I am working on this project for categories of low rank. In particular we are looking at *strictly* weakly integral categories which are weakly integral but not integral. Thus the Frobenius-Perron dimension of at least one object is equal to square root of an integer for some square-free integer. In such a case one may apply another structural tool, so called the Gelaki-Nikshych grading. This project is also related to a conjecture concerning the image of braid group representation (see section 3.1 below).

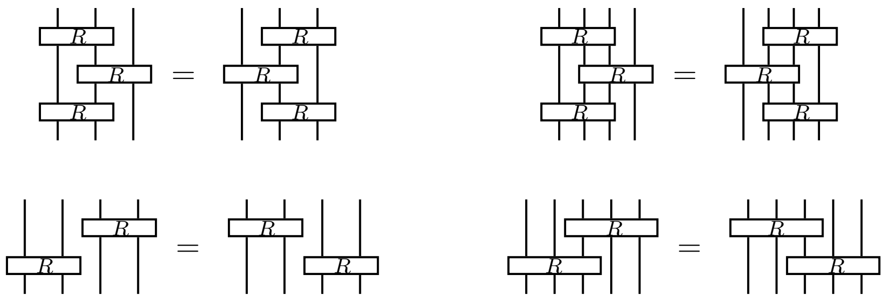
3. Representation of braid groups and invariants of links

Topological quantum computation is realized based on topological phases of matter. Information is stored in anyonic systems and processed by braiding non-abelian anyons. It is believed that this model is inherently fault tolerant [FKLW, Wa]. Inspired by topological quantum computation, I studied representation of braid groups and related topics such as Yang-Baxter operators(YB-operators) and invariants of links.

A YB-operator $R : V^{\otimes 2} \to V^{\otimes 2}$ is a solution to the (quantum) Yang-Baxter equation $(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)$. It is well known that each YB-operator gives rise to a representation of braid groups in a natural way.

3.1. **Localization of representation of braid groups.** The localization problem lies on a connection between two different sources of braid group representations: one is associated to a YB-operator R via $\sigma_i \mapsto \mathrm{id}_V^{\otimes i-1} \otimes R \otimes \mathrm{id}_V^{\otimes n-i-1}$ and the other is to a braided category using the braiding structure. The former representation is explicitly *local.* E. Rowell and Z. Wang studied the connection between such two methods and examined whether or not it is possible to have an equivalence between two representations. They called it a *localization* and showed that certain unitary Jones representation is localizable by a unitary 9×9 R-matrix [RW].

In [GHR] we extended the above result in two directions: one is generalized–localization and the other quasi–localization. My contribution to this work is limited to the generalized–localization for which we considered *generalized* YB-operators $R : V^{\otimes k} \to V^{\otimes k}, k \geq 2$ (gYB-operators) which satisfy generalized Yang-Baxter equation and additionally a far-commutativity condition (see Figure 2). One of the motivations of this consideration is that certain representation of braid groups is *not* localizable by *ordinary* YB-operators while it is localizable by a gYB-operator. Such an example is explicitly discussed in the paper. The main result is that if a representation obtained from an object of a braided category is localizable by a



(a) YB-operator $R : V^{\otimes 2} \to V^{\otimes 2}$ (b) gYB-operator $R : V^{\otimes 3} \to V^{\otimes 3}$

FIGURE 2. Each strand stands for a vector space V. Upper diagrams are (a) Yang-Baxter equation and (b) its generalized version for $k = 3$. Lower diagrams are far-commutativity. (a) Ordinary YB-operator trivially satisfies the far-commutativity, while (b) gYB-operator does not so automatically since two operators possibly act on the same tensor factor.

gYB-operator then the Frobenius-Perron dimension of the object is a square root of integer. This is a partial proof of a conjecture that the conditions are indeed equivalent.

3.2. **gYB-operators and invariants of links.** In 1988 V. Turaev introduced a notion of *enhanced* YB-operator and from which he constructed invariants of links [Tu2]. This construction is based on the Alexander's theorem: Every link can be obtained by closing some braids. The invariant is obtained essentially from the trace of braid image under the representation associated to a YB-operator. The trace itself however does not directly give rise to an invariant of links because there are infinitely many braids whose closures are all isotopic to the given link. Such braids are related by Markov moves $\beta \mapsto \beta\sigma_n^{\pm 1}, \beta \mapsto \eta^{-1}\beta\eta$ where $\beta, \eta \in B_n$ and thus the trace needs to be normalized in a certain way to be preserved under Markov moves (see Figure 3). Enhancement comes up in this context.

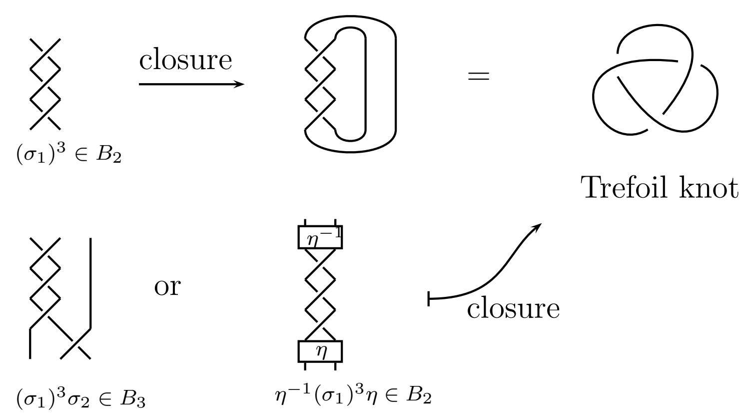


FIGURE 3. Upper diagram is an example of Alexander's theorem (braid closure). Lower diagram illustrates Markov moves for the same example. The two braids are obtained from $(\sigma_1)^3$ by applying each of Markov moves and result in the same link which is Trefoil knot in this example.

In [H1] I extended the enhancing method to gYB-operators. However, Turaev's definition does not generalize immediately: there is a somewhat subtle *orthogonality* condition that must be satisfied in the definition of enhanced gYB-operators. This modification is indeed unavoidable because none of the known examples of gYB-operators can be enhanced otherwise. The essential part lies in the Markov moves again and I was able to enhance all known examples upon the definition. Resulting link invariants were discussed and identified as specializations of either HOMFLY-PT or Kouffman polynomials.

Far-commutativity of gYB-operators is highly nontrivial and there had been only few examples known until 2012 [RZWG, GHR, Ch]. Following the idea in [KW] I constructed a *new* family of gYB-operators from ribbon categories $SO(N)_2$ [H2]. These operators act on $V^{\otimes 3}$ for a 2-dimensional vector space V, and thus are given by 8×8 matrices with the parameter N in them. Furthermore I addressed the following question: One may consider two different approaches to link invariant, one is directly from a ribbon category and the other from a gYB-operator. If a gYB-operator is obtained from a ribbon category, then would the two resulting link invariants be necessarily equivalent? The answer is YES. The essential part is that the enhancement comes *canonically* from the twist structure in the ribbon category.

3.3. **Project on representation of the loop braid groups.** The loop braid group LB_n is the motion group of n loops within a 3-dimension and directly related to 4d-BF theory, describing exotic statistics of string-like defects [BWC]. There are two types of motions: one is an exchange of two adjacent loops, and the other is passing one loop through another. We note that the motions of the first type satisfy the braid relation and form the braid group $B_n \subset LB_n$. Therefore one may obtain representation of loop braid groups by lifting that of braid groups. Based on the result by I. Tuba and H. Wenzl [TW], we are considering a classification of representations of LB_3 for dimension ≤ 5. Also we are looking at the images of those representations and trying to identify those which have finite images.

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