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TRUNCATION AND THE INDUCTION THEOREM

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Abstract. A key result in a 2004 paper by S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg compares the bounded derived category $D^b(block(\mathbb{U}))$ of finite dimensional modules for the principal block of a Lusztig quantum algebra $\mathbb{U}$ at an ℓ^{th} root of unity with a special full subcategory $D_{triv}(\mathbb{B})$ of the bounded derived category of integrable type 1 modules for a Borel part $\mathbb{B} \subset \mathbb{U}$. Specifically, according to this "Induction Theorem" [ABG04, Theorem 3.5.5] the right derived functor of induction $\mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}$ yields an equivalence of categories $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} : D_{triv}(\mathbb{B}) \xrightarrow{\sim} D^b(block(\mathbb{U}))$. Some restrictions on ℓ are required–e.g., $\ell > h$, the Coxeter number. It is suggested briefly [ABG04, Remark 3.5.6] that an analog of this equivalence carries over to characteristic $p > 0$ representations of algebraic groups. Indeed, the authors of the present paper have verified, in a separate preprint [HKS13], that there is such an equivalence $\mathrm{RInd}_B^G : D_{triv}(B) \xrightarrow{\sim} D^b(block(G))$ relating an analog of $D_{triv}(\mathbb{B})$, defined using a Borel subgroup B of a simply connected semisimple algebraic group G, to the bounded derived category of the principal block of finite dimensional rational G-modules. The proof is not without difficulty and supplies new, previously missing details even in the quantum case. The present paper continues the study of the modular case, taking the derived category equivalence as a starting point. The main result here is that, assuming $p > 2h - 2$, the equivalence behaves well with respect to certain weight poset "truncations," making use of a variation by Woodcock [W97] on van der Kallen's "excellent order" [vdK89]. This means, in particular, that the equivalence can be reformulated in terms of derived categories of finite dimensional quasi-hereditary algebras. We expect that a similar result holds in the quantum case.

1. Introduction

Suppose G is a semisimple, simply connected algebraic group defined and split over $\mathbb{F}_p$, with Coxeter number h and Borel subgroup B. Take $\mathcal{C}_G^f$ to be the finite dimensional rational modules for G, and within it, $block(G)$, to be the principal block of G.

It had been known for some time that $block(G) \subseteq \mathcal{C}_G^f$ fully embeds via module restriction into the category $\mathcal{C}_B^f$ of finite dimensional rational B−modules. However, while there are some abstract characterizations in [PSW00], there is no known explicit description of the image of $block(G)$ in terms of B-modules. The full embedding even yields a full embedding of $D^b(block(G))$ in $D^b(\mathcal{C}_B^f)$, for the bounded derived categories of

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block(G) and $\mathcal{C}_B^f$ ([CPSvdK77]); there again, the question of how to explicitly describe the image under restriction of $D^b(block(G))$ in $D^b(\mathcal{C}_B^f)$ remains open. Nevertheless, in contrast, there is now an explicit description of $D^b(block(G))$ inside $D^b(\mathcal{C}_B^f)$, given by the triangulated category $D_{triv}(B)$, as suggested by [ABG04] and described in more detail below. A thematic feature of the approach taken in [ABG04], and adapted to the characteristic p algebraic groups case in [HKS13], is to focus on the right derived functor RInd of induction, that is, the right adjoint of the restriction functor, rather than restriction itself.

Specifically, within $D^b(\mathcal{C}_B^f)$, take $D_{triv}(B)$ to be the full triangulated subcategory consisting precisely of all objects having cohomology both of finite dimension and also with weights all expressible in the form $p\lambda$ for some weight λ in the root lattice.

Theorem 1. *Assume $p > h$. Then the functor* $\mathrm{RInd}_{\mathrm{B}}^{\mathrm{G}}$ *induces an equivalence of triangulated categories*

$$D_{triv}(B) \to D^b(block(G)).$$

Arkhipov, Bezrukavnikov, and Ginzburg state this remarkable property in a quantum version of Theorem 1 as [ABG04, Thm 3.5.5] for quantum groups at an ℓ^{th} root of unity. Called there the "induction theorem", it has essentially the same statement as Theorem 1. The groups B and G are replaced by their quantum enveloping algebra, and p is replaced by ℓ. The positive integer ℓ need not be prime, but, in addition to the requirement $\ell > h$, is required to be odd and not divisible by three when the underlying root system has a compnoent of type G_2 [ABG04, (2.3)]. It seems the authors of [ABG04] believed Theorem 1 to be true, and that some of the ingredients of the proof of its quantum analog [ABG04, Thm 3.5.5] could be applied to its proof. This much, at least, is confirmed by the proof of Theorem 1 in [HKS13], though new ingredients beyond the proof of [ABG04, Thm 3.5.5] are also required.[1]

The functor giving the equivalence in Theorem 1 is obtained from the induction functor $\mathrm{RInd}_B^G : D^b(\mathcal{C}_B) \to D^b(\mathcal{C}_G)$ by restricting its domain and range. (In particular, $D^b(block(G))$ is the strict image of $D_{triv}(B)$ under the induction functor Rind_B^G.) Here we have removed the superscript f from $\mathcal{C}_B^f$ and $\mathcal{C}_G^f$ to indicate categories of all (possibly infinite dimensional) rational $G-$modules, to allow the standard construction of RInd_B^G using complexes of injective modules.

In the process of providing a detailed proof of Theorem 1 [HKS13], we have observed the following finiteness property: when also $p \geq 2h - 2$, the equivalence in Theorem 1 can be written as a union of equivalences of triangulated categories associated to highest weight categories in the sense of [CPS88], each having a finite weight poset.

[1]It should also be recorded that the proof in [ABG04] for the quantum case [ABG04, Thm 3.5.5] appears to be incomplete, in the sense that the proof of one key lemma, [ABG04, Lem. 4.1.1(ii)], is inadequate. Fortunately, the lengthy argument for a corresponding result given in [HKS13] applies in both the modular and quantum cases.

To give some detail, for each positive integer m, we will set Λ_m to be a particular subset in the root lattice $\mathbb{Y}$ of G; then let $\Gamma_m = \Lambda_m \cap (W_p \cdot 0)^+$ be those elements of Λ_m that are among the dominant weights $(W_p \cdot 0)^+$ indexing the irreducible modules in *block*(G). The set Γ_m turns out to be a poset ideal in $(W_p.0)^+$ with respect to the dominance order, and we let $block(G)_{\Gamma_m}$ be the image of *block*(G) under the associated standard dominant weight poset truncation by Γ_m. The subset Λ_m turns out also to be a weight poset, but with respect to an order associated to both the "excellent order" and "antipodal excellent order" highest weight categories of B-modules [W97], terminology introduced (with different weight posets) by van der Kallen [vdK89], [vdK93]. There is a corresponding associated notion of poset truncation for the distribution algebra $Dist(B)$ of B, relative to Λ_m. Using these notions, we can state the main result of this paper:

Theorem 2. *Assume $p > 2h-2$. Then the equivalence* $\mathrm{RInd}_{\mathrm{B}}^{\mathrm{G}} : D_{triv}(B) \to D^b(block(G))$ *of Theorem 1 induces, for each integer $m > 0$, equivalences of full triangulated subcategories*

$$D_{triv}(Dist(B)_{\Lambda_m}) \to D^b(block(G)_{\Gamma_m}).$$

Also, $D_{triv}(B)$ is naturally equivalent to the direct union of its full triangulated subcategories $D_{triv}(Dist(B)_{\Lambda_m})$, $m > 0$ an integer, and $D^b(block(G))$ is similarly naturally equivalent to the direct union of the various $D^b(block(G)_{\Gamma_m})$.

We remark that for each positive integer m, the categories $Dist(B)_{\Lambda_m}$ and $block(G)_{\Gamma_m}$ are equivalent to categories of finite dimensional modules for finite dimensional quasi-heriditary algebras. We believe this "finiteness property" of the equivalence in Theorem 1, as given by Theorem 2, is of sufficient significance to be worth recording on its own.

2. Notation, Conventions, and Other Preliminaries

2.1. **Basics.** We provide below the list of notation for this paper. Most of the less standard terminology is repeated in the main text.

- p a prime
- G a semisimple, simply connected algebraic group defined and split over $\mathbb{F}_p$;
- R fixed root system for G; with $R = R^- \cup R^+$ for a fixed set of negative (resp., positive) roots R^- (resp., R^+)
- W Weyl group of G w.r.t. R
- W_p (also denoted W_{aff}) the affine Weyl group $W_p \cong p\mathbb{Z}R \rtimes W$ of G with respect to R
- h Coxeter number for G
- $B = B^- \subset G$ Borel subgroup associated to R^- (resp., B^+ positive Borel associated to R^+)
- $\mathbb{X}$ weight lattice for G
- $\mathbb{X}^+$ dominant weights, with usual dominance order $\leq$

- $\mathbb{Y}$ root lattice for G
- $\rho = \frac{1}{2}\Sigma_{\alpha\in R^+}\alpha$
- $w\cdot\lambda := w(\lambda+\rho)-\rho$, defines the "dot action" of W on λ for $\lambda\in\mathbb{X}$
- $w_o \in W$, the longest word in W
- x^+ (resp., x^-) the unique dominant (resp., antidominant) weight in the W-orbit Wx of $x\in\mathbb{X}$
- $\preceq$ a partial order defined on $\mathbb{X}$ by $x\preceq x'$ if either the condition $x^+ < x'^+$ holds (in the usual dominance order), or else $w\leq w'$ (in the Bruhat-Chevalley order), where $w, w'\in W$ are the unique elements of minimum length with $x = wx^-$ and $x' = w'x'^-$
- $x\preceq^\circ x'$ a partial order defined on $\mathbb{X}$ by $x\preceq^\circ x'$ iff $x'\preceq x$ (iff $w_0x\preceq w_0x'$)
- $\mathcal{C}_G$ (resp., $\mathcal{C}_B$) category of rational G-modules (resp., rational B-modules)
- $\mathcal{C}_G^0\subseteq\mathcal{C}_G$ full subcategory of rational G-modules with high weights in $(W_p\cdot 0)^+$
- $\mathcal{C}_B^0\subseteq\mathcal{C}_B$ full subcategory of rational B-modules with weights in $\mathbb{Y}$
- $\mathcal{C}_G^f$ (reps., $\mathcal{C}_B^f$) category of finite dimensional rational G-modules (resp., finite dimensional rational B-modules)
- $\mathcal{C}^{0,f}(G)\subseteq\mathcal{C}_G^f$ the full subcategory consisting of all finite dimensional rational $G-$modules for which the weights are in $(W_p\cdot 0)^+\cap\mathbb{Y}$; by definition, this is the principal block of G, also denoted $block(G)$, and $block(G) = \mathcal{C}^0(G) = \mathcal{C}^{0,f}(G)$
- $\mathcal{C}^{0,f}(B)\subseteq\mathcal{C}_B^f$ the full subcategory consisting of all finite dimensional rational $B-$modules for which the weights are in $\mathbb{Y}$; thus $\mathcal{C}^{0,f}(B) = block(B)$, the principal block of B (although we shall not explicitly use this latter fact)
- $D^b(\mathcal{A})$ the bounded derived category of an appropriate category $\mathcal{A}$
- $D_{triv}(B)\subset D^b(\mathcal{C}_B^f)$ the full triangulated subcategory consisting precisely of all objects having cohomology both of finite dimension and also with weights all expressible in the form $p\lambda$ for some weight λ in $\mathbb{Y}$
- $\mathcal{C}_B^{0,f}[\Lambda]\subseteq\ \mathcal{C}_B^0$, the full subcategory of all finite-dimensional objects whose weights all belong to Λ, for any $\Lambda\subseteq\mathbb{Y}$ a finite poset ideal (with respect to either order $\preceq$ or $\preceq^\circ$)
- $\mathcal{C}_G^{0,f}[\Gamma]$ the full subcategory of $\mathcal{C}_G^{0,f}$ consisting of those objects whose composition factors all have highest weights in Γ, for $\Gamma\subseteq(W_{aff}\cdot 0)^+ = (W_p\cdot 0)^+$ a finite poset ideal under the dominance order.

2.2. **Highest Weight Categories, Poset Orders, and Truncation.** The category $\mathcal{C}_B$ of all rational $B-$modules is a highest weight category with respect to either the "excellent" or "antipodal excellent" partial orders on weights in [vdK89], Following [W97], we will use variations, respectively denoted here by $\preceq$ and $\preceq^\circ$, of these orders, which give essentially the same [2] respective highest weight category structures (give the same costandard and standard modules). More precisely, for $x \in \mathbb{X}$, let x^+ denote the unique dominant member of its (undotted) Weyl group orbit, and let x^- denote the unique antidominant member of this orbit. Then $x \preceq x'$ is defined, for $x, x' \in \mathbb{X}$, to mean that either the condition $x^+ < x'^+$ holds (in the usual dominance order), or else $w \leq w'$ (in the Bruhat-Chevalley order), where $w, w' \in W$ are the unique elements of minimum length with $x = wx^-$ and $x' = w'x^-$. (Notice that $x^+ = x'^+$ implies $x^- = x'^-$.) Define also $x \preceq^\circ x'$ iff $x' \preceq x$ (iff $w_0x \preceq w_0x'$). In particular, the action of w_0 interchanges $\preceq$ and $\preceq^\circ$. Both orders can be used for either $\mathcal{C}_B$ or $\mathcal{C}_{B^+}$ The latter category was used in [vdK89], but [W97] uses the former, as we do here.[3] In fact, we use $\mathcal{C}_B^0$, which inherits a highest weight category structure from $\mathcal{C}_B$. Similarly, if $\Lambda \subseteq \mathbb{Y}$ is a finite poset ideal (with respect to either order), then the full subcategory $\mathcal{C}_B^{0,f}[\Lambda] \subseteq \mathcal{C}_B^0$, of all finite-dimensional objects whose weights all belong to Λ, inherits a highest weight category structure [CPS88].

Proposition 3. *Let m be any positive real number (we will just use the integer case), and put $\Lambda_m = \{y \in \mathbb{Y} | \; |(y, \alpha^\vee)| \leq mp$ for all $\alpha \in R^+\}$. Then Λ_m is a poset ideal in $\mathbb{Y}$ with respect to either of the orders $\preceq$ or $\preceq^\circ$*

Proof. First, note that R^+ can be replaced by $R = R^+ \cup -R^+$ in the definition of Λ_m; consequently, the latter set is stable under the action of W. In particular, it is stable under w_0, so it suffices to treat the order $\preceq$. Also, the stability implies, for $y \in \mathbb{Y}$, that $y \in \Lambda_m$ iff $y^+ \in \Lambda_m$. The latter holds iff $(y^+, \alpha^\vee) \leq mp$ for all $\alpha \in R^+$, which holds iff $(y^+, \alpha_{0,}^\vee) \leq mp$, where $\alpha_{0,}$ denotes the maximal short root. Let $y \preceq y'$ with $y, y' \in \mathbb{Y}$ and $y' \in \Lambda$. Then $y^+ \leq y'^+$ in the dominance order, which implies $(y^+, \alpha_{0,}^\vee) \leq (y'^+, \alpha_{0,}^\vee) \leq mp$. Hence, $y \in \Lambda_m$, and the proposition is proved. □

There is an easy analog of the proposition for dominant weights. We will just use those in the weight poset $(W_{aff} \cdot 0)^+$ of dominant weights in $W_{aff} \cdot 0$. These are the dominant weights which occur as highest weights for irreducible modules in *block*(G).We alert the reader that we will later write $(W_{aff} \cdot 0)^+ = (W_p \cdot 0)^+$. We record the following result, whose proof is immediate.

Proposition 4. *Let m be a positive real number (as in the previous proposition), and put $\Gamma_m = \{y \in (W_{aff} \cdot 0)^+ | \; (y, \alpha^\vee) \leq mp$ for all $\alpha \in R^+\}$. Then Γ_m is a poset ideal in $(W_{aff} \cdot 0)^+$ with respect to the dominance order, and $\Gamma_m = \Lambda_m \cap (W_{aff} \cdot 0)^+$.*

[2]This fact, implicitly suggested in [W97] and used in [PSW00], does not seem to be explicitly proved in the literature. We will provide a proof in (an appendix to) a paper in preparation (with Brian Parshall) on quantum analogs of the results in this paper. In the mean time, the reader can simply rely on [W97] for the fact that these orders each define highest weight categories.

[3]It is instructive to note that [W97] refers to $\preceq$ as the "antipodal excellent order" and to $\preceq^\circ$ as the "excellent" order, terminology choices which readers familiar with [vdK93] might expect to have been reversed. The apparent explanation is that [W97] prefers B to B^+. Conjugation by w_0 carries B to B^+ and $\preceq$ to $\preceq^\circ$. So Woodcock does appear to be trying to align [W97] with [vdK93].

There are similar (easier) truncations for $\mathcal{C}_G^{0,f} = block(G)$. The category $\mathcal{C}_G^0$ is a highest weight category with respect to the poset $(W_{aff} \cdot 0)^+$ of dominant weights in $W_{aff} \cdot 0$ using several orders, all equivalent in the sense of giving the same costandard and standard modules.. See [Ja03, 1.5,1.8].We will just use the dominance order. We take this opportunity to note that W_{aff}, in its affine action on X, is denoted W_p in [Ja03, 1.5,1.8], with the p reminding us that W_{aff} acts on X as the semidirect product of W, acting linearly (before the "dot" is introduced), with translations by elements of $p\mathbb{Y}$. If $\Gamma \subseteq (W_{aff} \cdot 0)^+ = (W_p \cdot 0)^+$ is a finite poset ideal, let $\mathcal{C}_G^{0,f}[\Gamma]$ denote the full subcategory of $\mathcal{C}_G^{0,f}$ consisting of those objects whose composition factors all have highest weights in Γ. Then $\mathcal{C}_G^{0,f}[\Gamma]$ inherits a highest weight category structure from $\mathcal{C}_G^0$.

3. Main Results: Bounded Derived Categories, Truncation, and the Induction Theorem

Because $\mathcal{C}_G^{0,f}[\Gamma]$ inherits a highest weight category structure from $\mathcal{C}_G^0$, there is a natural full embedding of triangulated categories $D^b(\mathcal{C}_G^{0,f}[\Gamma]) \subseteq D^b(\mathcal{C}_G^{0,f}) = D^b(block(G))$. To simplify notation, we write $D^b(\mathcal{C}_G^{0,f}[\Gamma]) = D^{b,f}(Dist(G)_\Gamma)$.[4] Thus, the previous strict full embedding is now written $D^{b,f}(Dist(G)_\Gamma) \subseteq D^b(block(G))$. With abuse of notation, we will sometimes identify $D^{b,f}(Dist(G)_\Gamma)$ with its strict image in $D^b(block(G))$. Similarly, if $\Lambda \subseteq \mathbb{Y}$ is a finite poset ideal with respect to either $\preceq$ or $\preceq^\circ$, there is a natural full embedding $D^b(\mathcal{C}_B^{0,f}[\Lambda]) \subseteq D^b(\mathcal{C}_B^{0,f})$. We write $D^b(\mathcal{C}_B^{0,f}[\Lambda]) = D^{b,f}(Dist(B)_\Lambda)$ and also let $D_{triv}(Dist(B)_\Lambda)$ denote the full subcategory of $D^{b,f}(Dist(B)_\Lambda)$ whose cohomology has only weights py with $y \in Y$ and $py \in \Lambda$. Then the full embedding $D^b(\mathcal{C}_B^{0,f}[\Lambda]) \subseteq D^b(\mathcal{C}_B^{0,f})$ gives a full embedding $D_{triv}(Dist(B)_\Lambda) \subseteq D_{triv}(B)$. The strict (or "essential") image[5] of $D_{triv}(Dist(B)_\Lambda)$ is the (full) subcategory of objects in $D_{triv}(B)$ represented by complexes which have cohomology with all high weights in Λ. This subcategory of $D_{triv}(B)$ is certainly interesting in its own right, and its interpretation here as a strict image gives a (non-obvious) way of viewing it inside the more "finite" $D^{b,f}(Dist(B)_\Lambda)$.

Next, it makes sense to ask when the induction equivalence $\mathrm{RInd}_\mathrm{B}^\mathrm{G} : \mathrm{D_{triv}(B)} \to \mathrm{D^b(block(G))}$ takes $D_{triv}(Dist(B)_\Lambda)$ into $D^{b,f}(Dist(G)_\Gamma)$, and when the strict image

[4]This notation, in addition, suggests the (correct) fact that $\mathcal{C}_G^{0,f}[\Gamma]$ is naturally equivalent to the category of finite dimensional modules for $Dist(G)_\Gamma$. The latter is a (finite dimensional) quasi-hereditary algebra, defined as the quotient of $Dist(G)$ by the ideal which is the annihilator of all rational $G-$modules whose composition factors have high weights only in Γ.

Similar remarks may be made regarding the notation $D^{b,f}(Dist(B)_\Lambda)$ in the next paragraph. We leave the fairly routine proofs in both cases to the interested reader. These identifications, though informative, are not required for our main results.

[5]There seems to be no standard terminology here. If $F{:}\mathcal{A} \to \mathcal{B}$ is a (triangulated) functor between (triangulated) categories, we define the strict image of F, or of $\mathcal{A}$ under F, to be the smallest full (triangulated) subcategory of $\mathcal{B}$ contaning, for each object X in A, each object of $\mathcal{B}$ isomorphic to $F(X)$.

of $D_{triv}(Dist(B)_\Lambda)$ contains $D^{b,f}(Dist(G)_\Gamma)$. There are easy combinatorial sufficient conditions in each case. Let $\mathbb{X}^+ \subseteq \mathbb{X}$ denote the set of dominant integral weights.

Proposition 5. *Let Λ, Γ be as above.*

1) If $(W\cdot(\Lambda\cap p\mathbb{Y}))\cap\mathbb{X}^+ \subseteq \Gamma$, then $\mathrm{RInd}_{\mathrm{B}}^{\mathrm{G}}$ takes $D_{triv}(Dist(B)_\Lambda)$ into $D^{b,f}(Dist(G)_\Gamma) \subseteq D^b(block(G))$.

2) If $W\cdot\Gamma\cap p\mathbb{Y} \subseteq \Lambda$, then the strict image of $\mathrm{RInd}_{\mathrm{B}}^{\mathrm{G}}\mathrm{D}_{\mathrm{triv}}(\mathrm{Dist}(\mathrm{B})_\Lambda) \subseteq D^b(block(G))$ contains $D^{b,f}(Dist(G)_\Gamma)$.

Proof. Consider a complex M representing an object $[M]$ in $D_{triv}(Dist(B)_\Lambda) \subseteq D_{triv}(B)$. To prove $\mathrm{RInd}_{\mathrm{B}}^{\mathrm{G}}\ [M]$ belongs to $D^{b,f}(Dist(G)_\Gamma)$, it suffices, by standard truncation methods [BBD82]), to take M concentrated in a single cohomological degree, which may taken to be 0, and isomorphic to its cohomology. Thus, M has a finite filtration with sections one dimensional $B-$modules, each identified with a weight $py \in \Lambda\cap p\mathbb{Y}$. Without loss, M is itself one dimensional, identifying with such a py. As is well known, there is a unique dominant weight γ in $W\cdot py$ (a verification is included in the proof of the lemma below), which must belong to Γ when the hypothesis of 1) holds. By Andersen's strong linkage theory (available in [Ja03]), each composition factor of any cohomology group of $\mathrm{RInd}_{\mathrm{B}}^{\mathrm{G}}\ [M]$ must then have highest weight $\lambda \leq \gamma$ (even in the strong linkage order). Thus, λ belongs to the poset ideal Γ. This proves assertion 1).

Andersen's theory also guarantees, under the hypothesis that γ dominant and $\gamma = w\cdot py$ with $py \in \Lambda\cap p\mathbb{Y}$, that the irreducible module $L(\gamma)$ appears with multiplicity 1 in the cohomology of $\mathrm{RInd}_B^G\ [py]$, again identifying py with the associated one dimensional B-module. This gives an easy proof of assertion 2) by induction: Suppose, the hypothesis of 2) holds, and all irreducible modules $L(\gamma')$ belong to the strict image of $\mathrm{RInd}_B^G D_{triv}(Dist(B)_\Lambda)$ whenever $\gamma' < \gamma$. Then $\mathrm{RInd}_B^G\ [py]$ belongs to $\mathrm{RInd}_B^G D_{triv}(Dist(B)_\Lambda)$ and, as remarked, has $L(\gamma)$ as a composition factor of its cohomology with multiplicity 1. All other composition factors $L(\gamma')$ satisfy $\gamma' < \gamma$ and so belong to the strict image of $\mathrm{RInd}_B^G D_{triv}(Dist(B)_\Lambda)$. It then follows that $L(\gamma)$ belongs to this strict image (see the remark below), and 2) follows by induction. This proves the proposition. □

Lemma 6. *Let m be a positive real number, and put $\Lambda = \Lambda_m$, $\Gamma = \Gamma_m$. Recall our standing hypothesis $p > h$. The both of the following hold.*

1) The sets Λ, Γ satisfy the hypothesis (and conclusion) of part 1 of the previous proposition.

2) If, in addition, m is an integer and $p \geq 2h-2$, the sets Λ, Γ satisfy the hypothesis (and conclusion) of part 2 of the previous proposition.

Proof. First, we need a claim (which does not involve m and assumes only $p \geq h$). Let $y \in \mathbb{Y}$, and put $\nu = py$. Let $w \in W$ with $w\cdot\nu + \rho$ dominant. (At least one such w always exists; and the dominant weight $w\cdot\nu + \rho = w(\nu+\rho)$ is the unique dominant weight in $W(\nu+\rho.)$ We claim w is unique, and the weights $w\nu$ and $w\cdot\nu$ are also dominant. In addition w is also the unique element in W with $w.\nu$ dominant.

To prove the claim, note first that $0<|(\rho,\alpha^\vee)|\leq h-1<p$ for every root a. Since $(w\rho,\alpha^\vee)=(\rho,(w^{-1}\alpha)^\vee)$, it follows that the dominant weight $w(\nu+\rho)=p(wy)+w\rho$ is regular: That is, $(p(wy)+w\rho,\,\alpha^\vee)\neq 0$ for all roots α. So $w(\nu+\rho$ has trivial stabilizer in W. Thus, w is unique. Next, for any simple root α, apply $(-,\alpha^\vee)$ to both sides of the equation

$$w\cdot\nu+\rho=p(wy)+w\rho$$

If we ever had $(wy,\alpha^\vee)<0$, then, using the bound $|(w\rho,\alpha^\vee)|<p$, the above equation would give $(w\cdot\nu+\rho,\alpha^\vee)<0$, contradicting the dominance of $w\cdot\nu+\rho$. Consequently, $(wy,\alpha^\vee)\geq 0$ for all simple roots α, and so wy must be dominant. Also, $w\nu=pwy$ must be dominant. Next, since $w\cdot\nu{+}\rho={}w(\nu+\rho)$ is both dominant and regular (in the sense above), the weight $w\cdot\nu$ must be dominant. (In particular this gives the verification promised in the proof of the previous proposition.) Finally, if $w'\in W$ is such that $w'\cdot\nu$ is dominant, then $w'(\nu+\rho)=w'\cdot\nu+\rho$ is also dominant. Thus, $w'(\nu+\rho)=w(\nu+\rho)$,which gives $w=w'$ by regularity of $w(\nu+\rho)$, as previously noted. This completes the proof of the claim.

Note that, as a consequence, one can deduce from dominance of $w.\nu.$ when $\nu\in p\,\mathbb{Y}$ and $w\in W$, that $w\nu$ is dominant. We will often use below the claim in this way.

Next, suppose ν belongs to Λ, as well as to $p\mathbb{Y}$, and suppose $w\in W$ is such that $w\cdot\nu$ is dominant. Thus, $w\nu$ is also dominant, by the claim. From the inequality $w\cdot\nu=w\nu+w\rho-\rho\leq w\nu$, we obtain, for each positive root α, the inequality $(w\cdot\nu,\alpha^\vee)\leq(w\cdot\nu,\alpha_0^\vee)\leq(w\nu,\alpha_0^\vee)=|(\nu,w^{-1}\alpha_0^\vee)|\leq mp$. (Recall the definition of $\Lambda=\Lambda_m$.) Thus $w\cdot\nu\in\Lambda_m\cap(W_p\cdot 0)^+=\Gamma_m=\Gamma$. We have shown $W\cdot(\Lambda\cap p\mathbb{Y})\cap\mathbb{X}^+\subseteq\Gamma$, the hypothesis of part 1 of the previous proposition. Part 1 of the lemma follows.

Finally, suppose ν belongs to $W\cdot\Gamma$, as well as to $p\mathbb{Y}$, and let $w\in W$ be such that $w\cdot\nu\in\Gamma$. We want to show $\nu\in\Lambda$, as required in part 2 of the lemma. Assume m is a (positive) integer and that $p\geq 2h-2$, as given in the hypothesis of part 2. Note that, since also $p>h\geq 2$, we have the strict inequality $p>2h-2$. Write $\nu=py$ with $y\in\mathbb{Y}$. By the claim, the weight $w\nu$ is dominant. To prove $\nu\in\Lambda{=}\Lambda_m$ we just need to show $|(\nu,\alpha^\vee)|\leq mp$ for all roots α, But $|(\nu,\alpha^\vee)|=|(w\nu,w\alpha^\vee)|=|(w\nu,\pm w\alpha^\vee)|\leq(w\nu,\alpha_0^\vee)$, the last inequality following from the dominance of $w\nu$. Also, we have $(w\nu,\alpha_0^\vee)=(wpy,\alpha_0^\vee)=p(wy,\alpha_0^\vee)$ and

$$p(wy,\alpha_0^\vee)=(w\nu,\alpha_0^\vee)=(w\cdot\nu,\alpha_0^\vee)+(\rho-w\rho,\alpha_0^\vee)\leq mp+2h-2<(m+1)p.$$

Since m and $(wy,\alpha_0^\vee)$ are integers, we have $(wy,\alpha_0^\vee)\leq m$, and so $(w\nu,\alpha_0^\vee)=p(wy,\alpha_0^\vee)\leq pm$. Thus $\nu\in\Lambda_m=\Lambda$, as required. The hypothesis $W\cdot\Gamma\cap p\mathbb{Y}\subseteq\Lambda$ of part 2 of the preceding proposition is now verified, and so both parts of the lemma have been proved. $\square$

The hypothesis on p in the purely combinatorial result below is essentially the same as $p \geq 2h-2$, given our standing hypothesis $p > h$. There are no other hypotheses, except for the stated one on m.

Corollary 7. *Let m be a positive integer and assume $p > 2h-2$. Then there is a 1-1 correspondence between $\Lambda_m \cap pY$ and Γ_m , given by sending an element $\nu \in \Lambda_m \cap pY$ to the unique dominant weight γ in $W \cdot \nu$. This weight γ is in Γ. In the inverse direction, a weight $\gamma \in \Gamma$ is sent to the unique weight ν in $W \cdot \gamma$ of the form py with $y \in \mathbb{Y}$. This weight ν is in $\Lambda_m \cap p\mathbb{Y}$.*

Proof. We will use the previous lemma and also quote the claim in its proof.. If $\nu \in \Lambda_m \cap p\mathbb{Y} \subseteq$, then part 1 of the lemma above implies $W \cdot \nu \cap X^+ \subseteq \Gamma_m$. In addition, the claim in the proof shows there is only one $w \in W$ with $w \cdot \nu \in X^+$. Thus $\nu \mapsto w.\nu$ gives a well-defined map $\Lambda_m \cap pY \to \Gamma_m$. Next, if we start with a $\gamma \in \Gamma_m$, then part 2 implies $W \cdot \gamma \cap p\mathbb{Y} \subseteq \Lambda_m \cap p\mathbb{Y}$. If we have $w \cdot \gamma = py$ and $w' \cdot \gamma = py'$ for some $w, w' \in W$ and $y, y' \in Y$, then regularity of the action of W_p on $W_p \cdot 0$ forces $w = w'$ and $y = y'$ In particular, the assignment $\gamma \mapsto \nu = w \cdot \gamma = py$ gives a well-defined map $\Gamma_m \to \Lambda_m \cap p\mathbb{Y}$. By construction of the latter map, we have $\gamma = w^{-1} \cdot \nu$, and so the composite $\Gamma_m \to \Lambda_m \cap p\mathbb{Y} \to \Gamma_m$ is the identity. Also, if we start with $\nu = py \in \Lambda_m \cap p\mathbb{Y}$ and send ν to $w \cdot \nu \in X^+$, as in the definition of $\Lambda_m \cap p\mathbb{Y} \to \Gamma_m$ above, we have $w \cdot v \in \Gamma$, and $w^{-1} \cdot (w \cdot v) = \nu = w^{-1} \cdot (w \cdot v) = py$. Thus, the composite $\Lambda_m \cap p\mathbb{Y} \to \Gamma_m \to \Lambda_m \cap p\mathbb{Y}$ is also the identity, and the proof is complete. $\square$

Remark 8. *It is easy to deduce from the above three results that the functor $\mathrm{RInd}_{\mathrm{B}}^{\mathrm{G}}$ induces an isomorphism of Grothendieck groups*

$$K_0(D_{triv}(B)) \to K_0(D^b(block(G))).$$

This isomorphism may be regarded as a "shadow" of Theorem 1, though with a more restrictive bound $p > 2h-2$ on p

We do not know a proof of that theorem based on this isomorphism, even with the stricter bound, though it remains natural to look for such an argument.

But here we quote Theoren 1, together with the lemma above, **to prove our main result, Theorem 2.**

Proof. By part 1 of the previous proposition and part 1 of the previous lemma, if $N \in D_{triv}(B)$ is isomorphic to an object in $D_{triv}(Dist(B)_{\Lambda_m}) \subseteq D_{triv}(B)$, then $\mathrm{RInd}_B^G N$ is isomorphic to an object in $D^{b,f}(Dist(G)_{\Gamma_m}) \subseteq D^b(block(G))$. Also, by part 2, the full triangulated subcategory generated by all objects $\mathrm{RInd}_B^G N$ contains an isomorphic copy of each object in $D^{b,f}(Dist(G)_{\Gamma_m})$. However, since $\mathrm{RInd}_B^G : D_{triv}(B) \to D^b(block(G))$ is an equivalence, the collection $\mathcal{E}_m$ of all these objects $\mathrm{RInd}_B^G N$ is already a full triangulated subcategory of $D^b(block(G))$, equivalent to $D_{triv}(Dist(B)_{\Lambda_m})$. Since $\mathcal{E}_m \subseteq D^{b,f}(Dist(G)_{\Gamma_m})$, up to isomorphism of objects, and every object of $D^{b,f}(Dist(G)_{\Gamma_m})$ is isomorphic to an object in $\mathcal{E}_m$, the inclusion of $\mathcal{E}_m$ in the full subcategory $\mathcal{F}_m$ of objects in $D^b(block(G))$ isomorphic to an object in (the image in $D^b(block(G))$) of $D^{b,f}(Dist(G)_{\Gamma_m})$ is an equivalence of triangulated categories. So is the natural functor $D^{b,f}(Dist(G)_{\Gamma_m}) \to \mathcal{F}_m$. To summarize, the functor RInd_B^G directly induces an equivalence $D_{triv}(Dist(B)_{\Lambda_m}).\to \mathcal{E}_m$, while the latter triangulated category is equivalent to $D^{b,f}(Dist(G)_{\Gamma_m})$ through the composite of the equivalence $\mathcal{E}_m \subseteq \mathcal{F}_m$ and an inverse for the equivalence $D^{b,f}(Dist(G)_{\Gamma_m}) \to \mathcal{F}_m$. This proves the first assertion of the theorem.

The second assertion, regarding direct unions, follows from general derived category "recollement" considerations in highest weight category theory [CPS88], together with the obvious facts that

$$\mathbb{Y} = \bigcup_{m>0} \Lambda_m \text{ and } W_p \cdot 0 = \bigcup_{m>0} \Gamma_m$$

with the subscripts m always taken to be positive integers. This completes the proof of the theorem. □

References

[ABG04] S. Arkhipov, R. Bezrukavnikov, V. Ginzburg, *Quantum groups, the loop Grassmannian, and the Springer resolution,* J. Amer. Math. Soc. **17** (2004), 595–678 (electronic version April 13, 2004).

[BBD82] A. Beilinson, J. Bernstein, P. Deligne, *Analyse et topologie sur les espaces singuliers (1)*, Asterisque 100 (1982), 172pp.

[CPS88] E. Cline, B. Parshall, and L. Scott, *Finite dimensional algebras and highest weight categories*, J. reine u. angew. Math. 391 (1988), 85-99..

[CPSvdK77] E. Cline, B. Parshall, L. Scott, and W. van der Kallen, *Rational and generic cohomology*, Invent. math. 39 (1977), 143-169.

[HKS13] T. Hodge, P. Karuppuchamy , L. Scott, *Remarks on the ABG induction theorem*, preprint arXiv 1603.05699, 46pp.

[Ja03] J. C. Jantzen, *Representations of algebraic groups, 2nd ed.*, American Mathematical Society (2003).

[PSW00] B. Parshall, L. Scott, J.-p. Wang, *Borel subalgebras redux with examples from algebraic and quantum groups*, Algebras and Rep. Theory **3** (2000), 213–257

[vdK89] W. van der Kallen, *Longest weight vectors and excellent filtrations*, Math. Zeit. **201** (1989), 19–31.

[vdK93] W. van der Kallen, *Lectures on Frobenius Splittings and B-modules (Notes by S. P. Inamdar)*, Tata Institute Notes/Springer.

[W97] D. Woodcock, Schur Algebras and Global Bases: New Proofs of Old Vanishing Theorems, *Journal of Algebra* **191** (1997) 331-370.

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