Toric degeneration of Bott-Samelson-Demazure-Hansen varieties

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Abstract

In this paper we construct a degeneration of Bott-Samelson-Demazure-Hansen varieties to toric varieties in an algebraic family and study the geometry of the resulting toric varieties. We give a natural set of torus invariant curves that generate the Chow group of 1-cycles of the limiting toric variety and express the ample cone of this toric variety as a sub-cone of the ample cone of the corresponding Bott-Samelson-Demazure-Hansen variety. We also give a description of Extremal and Mori rays and determine when this toric variety is Fano.

1 Introduction

Let G be an almost simple, simply connected, affine algebraic group defined over an algebraically closed field k of arbitrary characteristic. Let B be a Borel subgroup of G. Then G acts from the left on the flag variety G/B. The

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B-invariant closed subvarieties of G/B are called *Schubert varieties*. Every Schubert variety is uniquely represented by an element of the Weyl group. After choosing a reduced expression for Weyl group elements as product of simple reflections, one constructs certain smooth birational modifications of the corresponding Schubert varieties. These desingularizations used to be called as Bott-Samelson varieties and the constructions were first described by Demezure and indepently by Hansen ([J], Chapetr 13, page 353). These constructions (cf. [D] or [J]) extend naturally to any given sequence of simple reflections. The resulting varieties are referred as Bott-Samelson-Demazure-Hansen varieties. We often abbreviate this long name and call these as BSDH varieties.

For $k = \mathbb{C}$, the field of complex numbers, the authors Grossberg and Karshon (cf. [GK]) obtained a family of complex structures on BSDH variety as differentiable manifold which degenerate to a different complex structure, with resulting manifold having a toric variety structure. We give an algebraic degeneration by constructing a smooth family of varieties parametrised by the affine line with general fibre isomorphic to the BSDH variety and the special fibre isomorphic to a smooth toric variety (cf. Section 3).

In another direction it may be desirable to obtain degenerations of Schubert varieties using the degenerations of their BSDH resolutions. Using Standard Monomial basis Gonciulea and Lakshmibai [GL] degenerated Flag varieties and some Schubert varieties to toric varieties. Later Caldero [C] had constructed degeneration of Schubert varieties into toric varieties using Lusztig's canonical basis.

Lauritzen and Thomsen have given an ampleness criterion for line bundles on these BSDH varieties (cf. [LT], Theorem 3.1). In fact they gave a set of line bundles and showed that the ample cone is the strict positive cone generated by these line bundles (cf. section 7). Here we describe the ample cone of the toric variety. We also give a necessary and sufficient condition for the anti canonical bundle to be ample, i.e. whether this toric variety is a Fano variety.

The paper is arranged as follows. In section 2 we fix some notations and recall some basic results. The main observation here is the relation between the self intersection number of a BSDH surface constructed using two simple reflections and the pairing between the corresponding roots (Lemma 3). Section 3 is devoted to obtain the degeneration of BSDH variety to a toric variety in an algebraic family (Theorem 9). In section 4 we study the Chow group of cycles of dimension 1 (curves) of the limiting toric variety. We label certain torus invariant curves and obtain some basic relation between them (Proposition 17, Remark 21). In section 5, we obtain a basis for the Chow group of one cycles of the limiting toric variety such that all torus invarian curves are non negative linear combination of elelments of this basis (Theorem 22). We also give two algorithms to find these torus invariant curves (Lemmas 23 and 24). In section 6 we study the Extremal rays and Mori rays and describe them completely in the limiting toric variety (Theorem 30, Theorem 35). The last section deals with description of the ample cone of these toric varieties as a subcone of the ample cone of the corresponding BSDH variety (Theorem 39).

2 Preliminaries

Let G be an almost simple, simply connected, affine algebraic group defined over an algebraically closed field k of arbitrary characteristic. Fix a maximal torus $T \subset G$. Then the Weyl group is defined as $N(T)/T$, where $N(T)$ is the normaliser of T in G. If we denote the character group of T by $X(T)$, then the Weyl group W has a faithful representation on the real vector space $X(T) \otimes \mathbb{R}$. Let $(\ , \ )$ be a non-degenerate W invariant pairing on $X(T) \otimes \mathbb{R}$.

Let S, Φ^+, $\Phi^- \subset X(T)$ be simple roots, positive roots and negative roots respectively. Let U_α be the root subgroup corresponding to the root α and B be the Borel subgroup of G generated by the root subgroups corresponding to the negative roots and maximal torus T. Let $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$ be the co-root of α. For a given simple root $\alpha \in S$, let s_α denote the simple reflection on $X(T) \otimes \mathbb{R}$ defined by $s_\alpha(x) = x - (x, \alpha^\vee)\alpha$. Then W is generated by $\{s_\alpha \mid \alpha \in S\}$. For a simple root α, the minimal parabolic subgroup P_α is defined to be the subgroup generated by B and the root subgroup U_α. The fundamental weight corresponding to the simple root α is denoted by ω_α.

Recall the following well-known description of parabolic subgroups of G. Let $\xi : \mathbb{G}_m \longrightarrow G$ be a one parameter subgroup of G. Define a $\mathbb{G}_m$-action on G by $x * g = \xi(x)g\xi(x)^{-1}$. Then we have the following (cf. [S] p. 148):

Lemma 1.. *The set* $P(\xi) := \{g \in G \ \mid \lim_{x\to 0} \ x * g \text{ exists}\}$ *is a parabolic subgroup and the unipotent radical* $R_u(P(\xi))$ *of* $P(\xi)$ *is given by* $\{g \in G \ \mid \lim_{x\to 0} \ x * g = \text{identity}\}$.

Moreover any parabolic subgroup of G *is of the form* $P(\xi)$ *for some* $\xi : \mathbb{G}_m \longrightarrow G$. $\diamondsuit$

We fix a $\xi : \mathbb{G}_m \longrightarrow T$ once and for all such that $B = P(\xi)$ and the unipotent radical B_U of B is $\{g \in G \mid \lim_{x\to 0} x * g = \text{identity }\}$. Then we have the following:

Lemma 2.. *Let* $\mathbb{A} :=$ *spec* $k[t]$ *denote the affine line over* k. *Define a family of homomorphisms* $\phi_x : B \to B$ *parametrised by* $x \in \mathbb{A}$ *as follows:*

$$\phi_x(b) = \xi(x)b\xi(x)^{-1} \text{ if } x \neq 0 \qquad \text{and} \qquad \phi_0(b) = \lim_{x\to 0} \ \xi(x)b\xi(x)^{-1}$$

Then ϕ_x *is an automorphsm for* $x \neq 0$ *and* ϕ_0 *is the natural Levi projection* $B \to T$ *with unipotent radical* B_U *of* B *as the kernel.* $\diamondsuit$

Let $\mathbf{w} = (s_1, s_2, \cdots, s_m)$ be a sequence of simple reflections, where m is any positive integer. Each simple reflection s_j is defined by a simple root, say α_j. We call m the length of the sequence and we denote it by $l(\mathbf{w})$. Length of an empty sequence is defined to be 0. Note that this lenth is just the number of terms in the sequence. We also have a lenth function on the Weyl group W. Every element w of W can be written as a product of simple reflections, $w = s_{j_1}s_{j_2}\cdots s_{j_r}$. If w can not be written as of less than r number of simple reflections then this expression is called a *reduced expression* and r is called the the length of w. We have a natural map from (finite) sequences of simple relfections to the the Weyl group W. This maps a sequence $\mathbf{w} = (s_1, s_2, \cdots, s_m)$ to the element $w = s_1 s_2 \cdots s_m$. But this representation may not be reduced and hence the length of the sequnce $\mathbf{w}$ may not be equal to the length of the corresponding Weyl group element w.

A sequence of integers $I = (i_1, \cdots, i_r)$ is called *m-admissible* if $1 \leq i_1 < i_2 < \cdots < i_r \leq m$. The entries i_1 and i_r of $I = (i_1, \cdots, i_r)$ are called *initial* and *final* entries respectively. Define the subsequence $\mathbf{w}_I$ of $\mathbf{w} = (s_1, s_2, \cdots, s_m)$ for every m-admissible sequence $I = (i_1, \cdots, i_r)$ by $\mathbf{w}_I := (s_{i_1}, \cdots, s_{i_r})$. For $0 \leq r \leq m$ we define the truncated m-admissible sequences $I[r] := (1,\ 2,\ \cdots, r)$ and $[r]I := (r+1, r+2, \cdots, m)$, we denote the corresponding subsequence of simple reflections $\mathbf{w}_{I[r]}$ and $\mathbf{w}_{[r]I}$ by $\mathbf{w}[r]$ and $[r]\mathbf{w}$ respectively. Note that $I[0] = [m]I$ is the empty sequnce of integers and $\mathbf{w}[m] = \mathbf{w} = [0]\mathbf{w}$ and $[m]\mathbf{w} = \mathbf{w}[0]$ is the empty sequence of simple reflections.

Let $P^{\mathbf{w}}$ and $B^{\mathbf{w}}$ denote the products $P_1 \times \cdots \times P_m$ and $B \times \cdots \times B$ (m copies) respectively, where P_j denotes the minimal parabolic subgroup P_{α_j}. The Bott-Samelson-Demazure-Hansen (BSDH) variety, $Z_{\mathbf{w}}$, is defined (see Definition 5 and Definition 7 below for a twisted version) as the quotient

$$Z_{\mathbf{w}} := \frac{P_1 \times P_2 \times \cdots \times P_m}{B \times B \times \cdot \times B} = \frac{P^{\mathbf{w}}}{B^{\mathbf{w}}}$$

where the $B^{\mathbf{w}}$ acts on $P^{\mathbf{w}}$ from the right as follows:

$$(p_1, p_2, p_3 \cdots p_m)(b_1, b_2, b_3 \cdots b_m) = (p_1 b_1, b_1^{-1} p_2 b_2, b_2^{-1} p_3 b_3 \cdots b_{m-1}^{-1} p_m b_m)$$

The BSDH variety also has the following inductive geometric construction. The induction is on the length of the sequence $\mathbf{w} = (s_1, s_2, \cdots, s_m)$. We construct the BSDH variety $Z_{\mathbf{w}[r]}$ and a map $f_r : Z_{\mathbf{w}[r]} \longrightarrow G/B$ inductively for all $0 \leq r \leq m$.

When $l(\mathbf{w}) = 0$ i.e. $\mathbf{w} = \mathbf{w}[0]$, we define the corresponding BSDH variety to be the unique B fixed point, the identity coset eB, of G/B. The map $f_0 : Z_{\mathbf{w}[0]} \longrightarrow G/B$ is the inclusion.

When the $l(\mathbf{w}) = 1$ i.e. $\mathbf{w} = \mathbf{w}[1] = s_1$, we define the corresponding BSDH variety $Z_{\mathbf{w}[1]} \cong P_1/B$ as the fiber product $Z_{\mathbf{w}[0]} \times_{B/P_1} G/B$ Observe that our construction gives the BSDH variety $Z_{\mathbf{w}[1]}$ with the map $f_1 : Z_{\mathbf{w}[1]} \longrightarrow G/B$, the projection $\psi_1 : Z_{\mathbf{w}[1]} \longrightarrow Z_{\mathbf{w}[0]}$, and the section $\sigma_0 : Z_{\mathbf{w}[0]} \longrightarrow Z_{\mathbf{w}[1]}$.

Suppose the BSDH variety $Z_{\mathbf{w}[r]}$ together with a map $f_r : Z_{\mathbf{w}[r]} \longrightarrow G/B$ is already constructed. Now the fiber product $Z_{\mathbf{w}[r]} \times_{B/P_{r+1}} G/B$ defines the BSDH variety $Z_{\mathbf{w}[r+1]}$ and the map f_{r+1} and ψ_{r+1} are canonical projections.

A section σ_r to this projection ψ_{r+1} is equivalent to giving a lift of the map $Z_{\mathbf{w}[r]} \longrightarrow G/P_{r+1}$ to G/B. Our inductive procedure already provided us such a map, f_r (see diagram below).

$$\begin{tikzcd}
Z_{\mathbf{w}[r+1]} \arrow[r, "f_{r+1}"] \arrow[d, "\psi_{r+1}"] \arrow[dr, phantom, "\square"] & G/B \arrow[d, "\pi_{r+1}"] \\
Z_{\mathbf{w}[r]} \arrow[r, "\pi_{r+1} \circ f_r"] \arrow[u, bend left, "\sigma_r"] & G/P_{r+1}
\end{tikzcd}$$

In summary, we have indcutively constructed the following:

i) The BSDH variety $Z_{\mathbf{w}[r]}$ with the map $f_r : Z_{\mathbf{w}[r]} \longrightarrow G/B$, for all r.

ii) The projection $\psi_r : Z_{\mathbf{w}[r]} \longrightarrow Z_{\mathbf{w}[r-1]}$, for $1 \leq r \leq m$.

iii) The section $\sigma_r : Z_{\mathbf{w}[r]} \longrightarrow Z_{\mathbf{w}[r+1]}$, for $0 \leq r \leq m-1$.

The next two Lemmas appear in [[K], Lemma 3(a), Lemma 2(3)] in slightly different notations. Since these Lemmas are crucial for our computations we give a proof of one of these lemmas for the convenience of the reader.

Lemma 3.. *Let $\mathbf{u} = (s_1, s_2)$ be a sequence of simple reflections. Then the self intersection number of the section $\sigma_1(Z_{s_1})$ in the surface $Z_{\mathbf{u}}$ is $(\alpha_2,\ \alpha_1^\vee)$. By abuse of notation we sometimes denote this number $(\alpha_2,\ \alpha_1^\vee)$ by $(2, 1)$.*

Proof: The self intersection number of the section $\sigma_1(Z_{s_1})$ in the surface $Z_{\mathbf{u}}$ is by definition the degree of the normal bundle $N_{\sigma_1(Z_{s_1})/Z_{\mathbf{u}}}$. But the normal bundle of a section in a fibration can be identified with the restriction of the relative tangent bundle. Since this fibration is a fibre product of $Z_{s_1} \to G/P_2$ with the natural fibration $\pi_2 : G/B \to G/P_2$, the relative tangent bundle is the pull back of the relative tangent bundle of π_2

$$\begin{tikzcd}
Z_{s_2} \arrow[r, equal] \arrow[d] & P_2/B \arrow[d] \\
Z_{\mathbf{u}} \arrow[r, "f"] \arrow[d, "\psi_2"] & G/B \arrow[d, "\pi_2"] \\
Z_{s_1} \arrow[r] \arrow[u, bend left, "\sigma_1"] & G/P_2
\end{tikzcd}$$

Now the relative tangent bundle of π_2 is canonically identified with L_{α_2}, the line bundle on G/B associated to the character α_2. Hence it suffices to prove that the degree of L_{α_2} restricted to $f(\sigma_1(Z_{s_1}))$ is $(\alpha_2, \alpha_1^\vee)$, as f defines an embedding. In fact $f(\sigma_1(Z_{s_1})) = P_1/B \subset G/B$.

Using Part II, Proposition 5.2 of [J], we obtain the individual cohomology groups of the restriction of L_λ for any character λ on P_1/B, which in turn determine the line bundle. The Lemma will then follow by substituting the character α_2 for λ.

Proposition 5.2 (b) implies when $(\lambda, \alpha_1^\vee) = -1$ then $H^i(P_1/B, L_\lambda) = 0$ for all $i \geq 0$, hence $L_\lambda \mid_{P_1/B}$ is isomorphic to $\mathcal{O}_{P_1/B}(-1)$, whose degree is $-1 = (\lambda, \alpha_1^\vee)$,

Proposition 5.2 (c) implies when $(\lambda, \alpha_1^\vee) \geq 0$, then $H^1(P_1/B, L_\lambda) = 0$ and $H^0(P_1/B, L_\lambda)$ has dimension $(\lambda, \alpha_1^\vee)+1$. Hence the line bundle $L_\lambda \mid_{P_1/B}$ is $\mathcal{O}_{P_1/B}((\lambda, \alpha_1^\vee))$.

Proposition 5.2 (d) implies when $(\lambda, \alpha_1^\vee) \leq -2$, then $H^0(P_1/B, L_\lambda) = 0$ and $H^1(P_1/B, L_\lambda)$ has dimension $-(\lambda, \alpha_1^\vee)-1$. Hence the line bundle $L_\lambda \mid_{P_1/B}$ is $\mathcal{O}_{P_1/B}((\lambda, \alpha_1^\vee))$. $\diamondsuit$

In fact the proof provides a more general result. Consider the following diagramme with natural maps

$$\begin{tikzcd}
Z_{s_{r+1}} \arrow[d] & P_{r+1}/B \arrow[d] \\
Z_{\mathbf{w}[r+1]} \arrow[r, "f_{r+1}"] \arrow[d, "\psi_{r+1}"] & G/B \arrow[d, "\pi_{r+1}"] \\
Z_{\mathbf{w}[r]} \arrow[r] \arrow[ur, "f_r"] \arrow[u, bend left, "\sigma_r"] & G/P_{r+1}
\end{tikzcd}$$

then we have:

Lemma 4.. *The relative tangent bundle of ψ_{r+1} is $f_{r+1}^*(L_\alpha)$, where L_α is the line bundle on G/B associated to the character α, and α is the simple root corresponding to s_{r+1}.* $\diamondsuit$

3 Construction of the Degeneration

Consider a sequence $\mathbf{w} = (s_1, s_2, \cdots, s_m)$ of simple reflections. Let $\mathcal{B}$ denote $B \times \mathbb{A}$ and $\mathcal{P}_j$ denote $P_j \times \mathbb{A}$ for $1 \leq j \leq m$. Then $\mathcal{B}^{\mathbf{w}} = \mathcal{B} \times_{\mathbb{A}} \mathcal{B} \cdots, \times_{\mathbb{A}} \mathcal{B} = B^{\mathbf{w}} \times \mathbb{A}$ and $\mathcal{P}^{\mathbf{w}} = \mathcal{P}_1 \times_{\mathcal{B}} \mathcal{P}_2 \cdots \times_{\mathcal{B}} \mathcal{P}_m = P^{\mathbf{w}} \times \mathbb{A}$. Both $\mathcal{B}^{\mathbf{w}}$ and $\mathcal{P}^{\mathbf{w}}$ are group schemes over $\mathbb{A}$.

Definition 5.. *Define the following twisted action of $\mathcal{B}^{\mathbf{w}}$ on $\mathcal{P}^{\mathbf{w}}$ over $\mathbb{A}$ as follows:*

$$\begin{aligned}&[(p_1,\ p_2,\ \cdots,\ p_m),\ x] \cdot [(b_1,\ b_2,\ \cdots,\ b_m),\ x] \\ &\quad = [(p_1\ b_1,\ \phi_x(b_1)^{-1} p_2 b_2,\ \cdots,\ \phi_x(b_{m-1})^{-1} p_m b_m),\ x]\end{aligned}$$

where $\phi_x : B \to B$ is the family of homomorphisms defined in Lemma 2.

Lemma 6.. *The action of $\mathcal{B}^{\mathbf{w}}$ on $\mathcal{P}^{\mathbf{w}}$ over $\mathbb{A}$ is* free.

Prooof: Recall that an action $\sigma : G \times_S X \longrightarrow X$ of a group scheme G over a scheme S on a Scheme X over S is said to be free if the map $(\sigma, \pi_2) : G \times_S X \longrightarrow X \times_S X$ is a closed immersion. [[MFK], Page 10].

We show the map $\mathcal{P}^{\mathbf{w}} \times_{\mathbb{A}} \mathcal{B}^{\mathbf{w}} \longrightarrow \mathcal{P}^{\mathbf{w}} \times_{\mathbb{A}} \mathcal{P}^{\mathbf{w}}$ is injective.
Suppose

$$((p_1, p_2, \cdots, p_3) \cdot (b_1, b_2, \cdots, b_m), (p_1, p_2, \cdots, p_m))$$

$$= ((p_1', p_2', \cdots, p_m') \cdot (b_1', b_2', \cdots, b_m'), (p_1', p_2', \cdots, p_m'))$$

over a point x. Then $p_1 = p_1',\ p_2 = p_2',\ \cdots\ p_m = p_m'$, and (refer Definition 5) $p_1 b_1 = p_1 b_1',\ \phi_x(b_1)^{-1} p_2 b_2 = \phi_x(b_1')^{-1} p_2 b_2',\ \ \phi_x(b_2)^{-1} p_2 b_3 = \phi_x(b_2')^{-1} p_2 b_3'$, $\cdots,\ \ \phi_x(b_{m-1})^{-1} p_m b_m = \phi_x(b_{m-1}')^{-1} p_m b_m'$. Which successively implies $b_1 = b_1',\ b_2 = b_2',\ \cdots\ b_m = b_m'$. Using the fact that the B orbits in P_i are closed embedding, one can show that the map $\mathcal{P}^{\mathbf{w}} \times_{\mathbb{A}} \mathcal{B}^{\mathbf{w}} \longrightarrow \mathcal{P}^{\mathbf{w}} \times_{\mathbb{A}} \mathcal{P}^{\mathbf{w}}$ is a closed immersion. $\diamondsuit$

Definition 7.. *Since the action of $\mathcal{B}^{\mathbf{w}}$ on $\mathcal{P}^{\mathbf{w}}$ over $\mathbb{A}$ is* free, *the quotien $\mathcal{Z}_{\mathbf{w}} = \mathcal{P}^{\mathbf{w}}/\mathcal{B}^{\mathbf{w}}$ exists as an algenraic space over $\mathbb{A}$ [[KM], Theorem 1.1]. Let $\pi : \mathcal{Z}_{\mathbf{w}} \longrightarrow \mathbb{A}$ denote the defining morphism.*

Remark 8.. Note that the projection $\mathcal{P}^{\mathbf{w}[r]} \to \mathcal{P}^{w[r-1]}$ is equivariant for the actions of $\mathcal{B}^{\mathbf{w}[r]}$ and $\mathcal{B}^{w[r-1]}$ on $\mathcal{P}^{w[r]}$ and $\mathcal{P}^{\mathbf{w}[r-1]}$ respectively. Hence this projection descend to give a morphism of the quotient spaces $\psi_{r,\mathbb{A}} : \mathcal{Z}_{\mathbf{w}[r]} \longrightarrow \mathcal{Z}_{\mathbf{w}[r-1]},$. It is a P_r/B fibration. There is also a section $\sigma_{r,\mathbb{A}} : \mathcal{Z}_{\mathbf{w}[r]} \hookrightarrow \mathcal{Z}_{\mathbf{w}[r+1]}$ to the projection $\psi_{r+1,\mathbb{A}}$ induced by the inclusion

$$\mathcal{P}^{\mathbf{w}[r]} \cong P_1 \times P_2 \times \ldots \times P_r \times \{1\} \times \mathbb{A} \hookrightarrow P_1 \times P_2 \times \ldots \times P_{r+1} \times \mathbb{A} = \mathcal{P}^{\mathbf{w}[r+1]}$$

Now we describe a scheme structure on the algebraic space $\mathcal{Z}_{\mathbf{w}}$.

Theorem 9.. *The morphism $\pi : \mathcal{Z}_{\mathbf{w}} \longrightarrow \mathbb{A}$ has the following properties:*

1. *π is a smooth projective morphism*
2. *The fiber over* 1, *$\mathcal{Z}_{\mathbf{w}}^1 := \pi^{-1}(1)$ is the BSDH variety $Z_{\mathbf{w}}$ and $\mathcal{Z}_{\mathbf{w}}^x := \pi^{-1}(x)$ is isomorphic to $Z_{\mathbf{w}}$ for $x \neq 0$*
3. *$\mathcal{Z}_{\mathbf{w}}^0 := \pi^{-1}(0)$ is a smooth toric variety.*

Proof of (1) We construct $\mathbb{A}$ schemes $\mathcal{Y}_i$ and smooth morphisms $\psi_{i,\mathbb{A}} : \mathcal{Y}_i/B \longrightarrow \mathcal{Y}_{i-1}/\mathcal{B}$, and sections $\sigma_{i-1,\mathbb{A}} : \mathcal{Y}_{i-1}/B \longrightarrow \mathcal{Y}_i/\mathcal{B}$ for $i = 1, 2, \cdots, m$ inductively. Set $\mathcal{Y}_0 = \mathcal{B}$, $\mathcal{Y}_1 = \mathcal{P}_1$ and $\psi_{1,\mathbb{A}}$ is the defining morphism from $\mathcal{Z}_{\mathbf{w}[1]} = \mathcal{P}_1/\mathcal{B} \longrightarrow \mathbb{A}$. For $i \geq 2$, define

$$\mathcal{Y}_i := \mathcal{Y}_{i-1} \times^{\mathcal{B}} \mathcal{P}_i$$

where the $\mathcal{B}$ action is defined by $(y, p) \cdot b = (yb, \phi_x(b)^{-1}p)$ for $y \in \mathcal{Y}_{i-1}$, $p \in \mathcal{P}_i$, $b \in \mathcal{B}$, and $x \in \mathbb{A}$. There is still an action of $\mathcal{B}$ (in fact $\mathcal{P}_i$) on $\mathcal{Y}_i$ coming from right multiplication on $\mathcal{P}_i$ and $\mathcal{Y}_i/B \cong \mathcal{Z}_{\mathbf{w}[i]}$. Note that the $\mathcal{P}_i$ action is free and $\mathcal{Y}_i/\mathcal{P}_i \cong \mathcal{Z}_{\mathbf{w}[i-1]}$. In fact, the scheme $\mathcal{Y}_i$ is a principal $\mathcal{P}_i$ bundle over $\mathcal{Z}_{\mathbf{w}[i-1]}$.

We define $\psi_{i,\mathbb{A}}$ to be the composite of the following maps

$$\psi_{i,\mathbb{A}} : \mathcal{Z}_{\mathbf{w}[i]} = \mathcal{Y}_i/\mathcal{B} \longrightarrow \mathcal{Y}_i/\mathcal{P}_i \cong \mathcal{Y}_{i-1}/\mathcal{B} = \mathcal{Z}_{\mathbf{w}[i-1]}$$

The map sending y to the class $[y, 1]$ from $\mathcal{Y}_{i-1}$ to $\mathcal{Y}_i$ descend to give

$$\sigma_{i-1,\mathbb{A}} : \mathcal{Y}_{i-1}/\mathcal{B} \longrightarrow \mathcal{Y}_i/\mathcal{B}$$

Since G is simply connected group, there exists an irreducible rank 2 representation V_i of P_i which trivially extends over the base $\mathbb{A}$ to $\mathcal{P}_i$. This gives rise to a rank 2 vector bundle $\mathcal{V}_i$ on $\mathcal{Z}_{\mathbf{w}[i-1]}$ such that the $\mathcal{B}$ quotient $\mathcal{Y}_i/\mathcal{B} \cong \mathcal{Z}_{\mathbf{w}[i]}$ is cannonically isomorphic to $\mathbf{P}(\mathcal{V}_i)$. We get the following diagramme:

$$\begin{tikzcd} \mathcal{Z}_{\mathbf{w}[i]} \arrow[r, "\cong"] \arrow[d, "\psi_{i,\mathbb{A}}"'] & \mathbf{P}(\mathcal{V}_i) \arrow[dl] \\ \mathcal{Z}_{\mathbf{w}[i-1]} & \end{tikzcd}$$

Since each $\psi_{i,\mathbb{A}}$ is smooth proper morphism as it is isomorphic to the projective bundle of a vector bundle, the composition $\pi = \psi_{m,\mathbb{A}} \circ \psi_{m-1,\mathbb{A}} \circ \cdots \circ \psi_{1,\mathbb{A}}$ is a smooth projective morphism.

Proof of (2): For $x \neq 0$, consider the map $f_x : P_1 \times \ldots \times P_m \longrightarrow P_1 \times \ldots \times P_m$ given by $f_x((p_1, \ldots, p_m)) = (p_1, \xi(x) \cdot p_2, \ldots, \xi(x) \cdot p_m)$, where $\xi(x) \cdot p_i$ is the multiplication in P_i. We show that this is a B^w equivarient map:
$f_x((p_1, p_2 \ldots p_m) \cdot (b_1, b_2 \ldots b_m)) = f_x((p_1 b_1, \phi_x(b_1)^{-1} p_2 b_2 \ldots \phi_x(b_{m-1})^{-1} p_m b_m))$
$= (p_1 b_1, \xi(x)\phi_x(b_1)^{-1} p_2 b_2 \ldots \xi(x)\phi_x(b_{m-1})^{-1} p_m b_m)$ (Refer Lemma 2 for ϕ_x)
$= (p_1 b_1, b_1^{-1}\xi(x) p_2 b_2 \ldots, b_{m-1}^{-1}\xi(x) p_m b_m)) = f_x((p_1, p_2 \ldots p_m)) \cdot (b_1, b_2 \ldots b_m)$
This $B^{\mathbf{w}}$ equivarient isomorphism descends to give a well defined isomorphism $\overline{f_x} : Z_{\mathbf{w}} \longrightarrow \mathcal{Z}^x_{\mathbf{w}}$. Note that the map f_1 is the identity map. Which proves the claim $\mathcal{Z}^1_{\mathbf{w}} = Z_{\mathbf{w}}$.

Proof of (3):

Note that $\mathcal{Z}^0_{\mathbf{w}[i]}$ can also be viewed as

$$\mathcal{Z}^0_{\mathbf{w}[i]} = \mathcal{Y}^0_{i-1} \times^B (P_i/B)$$

where B acts on P_i/B via its projection to the maximal torus T. We observe that the action of the maximal torus T on P_i/B factors through the action of the multiplicative group on the projective line P_i/B via the character α_i. We denote this quotient of T by $\mathcal{T}_i$. We define $\mathcal{T}_{\mathbf{w}[r]} := \mathcal{T}_1 \times \cdots \times \mathcal{T}_r$, for all $1 \leq r \leq m$. Then one see that the action of $T \times \cdots \times T$ on $\mathcal{Z}^0_{\mathbf{w}[i]}$ factors through $\mathcal{T}_{\mathbf{w}[i]}$.

For $i \geq 2$, consider the principal B-fibration $\mathcal{Y}^0_{i-1} \longrightarrow \mathcal{Y}^0_{i-1}/B = \mathcal{Z}^0_{\mathbf{w}[i-1]}$. Let $E_i := \mathcal{Y}^0_{i-1} \times^B \mathcal{T}_i) \longrightarrow \mathcal{Z}^0_{\mathbf{w}[i-1]}$ be the principal $\mathcal{T}_i$ bundle obtained using the associated construction with the quotient homomorphism $B \to \mathcal{T}_i$. Then E_i is a $\mathcal{T}_{\mathbf{w}[i]}$ variety with $E_i \longrightarrow \mathcal{Z}^0_{\mathbf{w}[i-1]}$ is a $\mathcal{T}_{\mathbf{w}[i-1]}$ equivariant map. Then we have

$$\mathcal{Z}^0_{\mathbf{w}[i]} = \mathcal{Y}^0_{i-1} \times^B P_i/B = (\mathcal{Y}^0_{i-1} \times^B \mathcal{T}_i) \times^{\mathcal{T}_i} P_i/B = E_i \times^{\mathcal{T}_i} P_i/B$$

Now the Theorem follows as $\mathcal{Z}^0_{\mathbf{w}[i]}$ has a dense open orbit for the action of $\mathcal{T}_{\mathbf{w}[i-1]} \times \mathcal{T}_i \cong \mathcal{T}_{\mathbf{w}[i]}$. Hence $\mathcal{Z}^0_{\mathbf{w}[i]}$ is a smooth toric variety. $\diamondsuit$

Notice that P_{r+1}/B has two $\mathcal{T}_{r+1}$-fixed points, one is the B-fixed point called the *Schubert point* and the other called *non-Schubert point*. These give rise to two sections

$$\sigma^0_r \ , \ \sigma^1_r : \ \mathcal{Z}^0_{\mathbf{w}[r]} \to \mathcal{Z}^0_{\mathbf{w}[r+1]}$$

The section σ^0_r corresponding to the B-fixed point is called a *Schubert section* which is the restriction of the section $\sigma_{r,\mathbb{A}}$ to the special fibre. The other $\mathcal{T}_{r+1}$ fixed point of P_{r+1}/B gives the other section σ^1_r disjoint from the Schubert section. This section will be called *non-Schubert section.*

The point $\sigma_{0,\mathbb{A}}(x) \in (\mathcal{P}_1/\mathcal{B})^x \cong \mathcal{Z}^x_{\mathbf{w}[1]}$ is called the Schubert point in $\mathcal{Z}^x_{\mathbf{w}[1]}$ The *Schubert point* of $\mathcal{Z}^x_{\mathbf{w}[r]}$ is defined inductively as the image of the Schubert point under the Schubert section, $\sigma_{r-1,\mathbb{A}} \mid_{\mathcal{Z}^x_{\mathbf{w}[r-1]}}$.

A *Schubert line* in $\mathcal{Z}^0_{\mathbf{w}[r]}$ is defined to be any $\mathcal{T}_{\mathbf{w}[r]}$-invariant curve containing the Schubert point. More generally we may call a face to be a *Schubert face* if it contains the Schubert point.

The following Lemma is standard.

Lemma 10.. *Suppose V be a rank 2 vector bundle over a curve C and $\mathbf{P}(V)$ be the projective bundle. Then the sections $\sigma : C \longrightarrow \mathbf{P}(V)$ are in one to one corresspondence with the line bundle quotients Q of V. Moreover the self intersection number of $\sigma(C)$ in the surface is given by deg $Q -$ deg V, where deg denote the degree of the locally free sheaves.* $\diamondsuit$

We also need the following result which can be extracted from the proof of Theorem 9. We state this as a separate Lemma.

Lemma 11.. *The section $\sigma_{i-1,\mathbb{A}} : \mathcal{Z}_{\mathbf{w}[i-1]} \longrightarrow \mathcal{Z}_{\mathbf{w}[i]}$ provides a line bundle quotient $\mathcal{V}_i \longrightarrow \mathcal{Q}$. Let $\mathcal{S}_i$ be the kernel (a line bundle). For each $x \in \mathbb{A}$ let us denote the restrictions of these bundles on $\mathcal{Z}^x_{\mathbf{w}[i-1]}$ by $\mathcal{V}^x_i$, $\mathcal{S}^x_i$ and $\mathcal{Q}^x_i$ respectively. Then*

$$\mathcal{Z}^x_{\mathbf{w}[i]} \cong \mathbf{P}(\mathcal{V}^x_i)$$

$$\mathcal{Z}^0_{\mathbf{w}[i]} \cong \mathbf{P}(\mathcal{Q}^0_i \oplus \mathcal{S}^0_i)$$

with the projection $\mathcal{V}^x_i \longrightarrow \mathcal{Q}^x_i$ providing the Schubert section $\sigma^0_{i-1}(\mathcal{Z}^x_{\mathbf{w}[i-1]})$ for all $x \in \mathbb{A}$. Moreover the non-Schubert section, $\sigma^1_{i-1}(\mathcal{Z}^0_{\mathbf{w}[i-1]})$, is provided by the quotient $\mathcal{Q}^0_i \oplus \mathcal{S}^0_i \longrightarrow \mathcal{S}^0_i$. $\diamondsuit$

We can now state a generalization of Lemma 3 to the family.

Lemma 12.. *Let $\mathbf{u} = (s_1, s_2)$ be a sequence of simple reflections. Then the self intersection number of the section $\sigma_{1,\mathbb{A}}(\mathcal{Z}^x_{s_1})$ in the surface $\mathcal{Z}^x_{\mathbf{u}}$ is $(\alpha_2,\ \alpha_1^\vee)$, for all $x \in \mathbb{A}$.*

Proof: The above Lemma 10 and Remark 11 shows the the self intersection number of $\sigma_{1,\mathbb{A}}(\mathcal{Z}_{s_1})$ does not change in the family. Over a general fiber this surface is isomorphic to a Schubert surface. For $x \neq 0$ this follows from Lemma 3 and Theorem 9(2). $\diamondsuit$

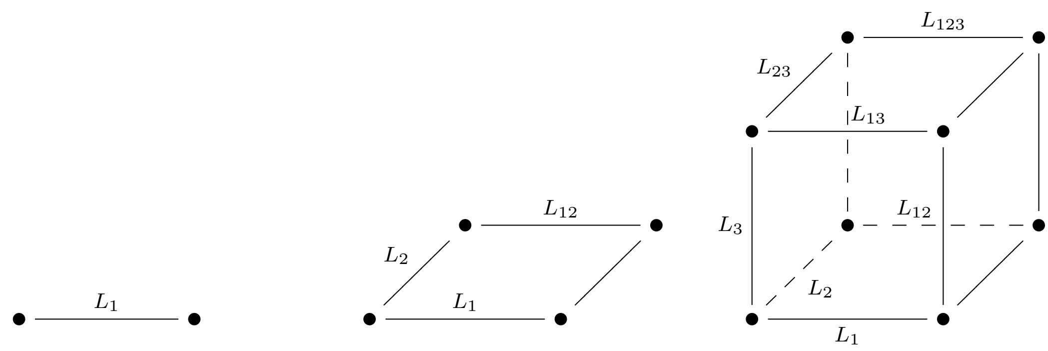
4 Chow group of 1-cycles

Given a variety X defined over k we denote the group of 1-cycles on X modulo numerical equivalence by $\mathbf{N_1}(X)$. Let $\mathbf{A_1}(X)$ denote the real vector space $\mathbf{N_1}(X) \otimes \mathbb{R}$. Similarly let $\mathbf{N^1}(X)$ be the group of line bundles modulo numerical equivalence and $\mathbf{A^1}(X)$ denote the real vector space $\mathbf{N^1}(X) \otimes \mathbb{R}$. Both $\mathbf{A_1}(X)$ and $\mathbf{A^1}(X)$ are finite dimensional by a theorem of Neron-Severi. It is also known that the intersection pairing $\mathbf{A^1}(X) \otimes_{\mathbb{R}} \mathbf{A_1}(X) \to \mathbb{R}$ is perfect.

Let us denote the rational curve $\mathcal{Z}^x_{\mathbf{w}[1]} = \mathcal{Z}^x_{s_1}$ by L_1. The fibre of $\psi_{r,x} = \psi_{r,\mathbb{A}} \mid_{\mathcal{Z}^x_{\mathbf{w}[r]}}$ over the Schubert point is the Schubert line L_r. We index (label) all $\mathcal{T}_{\mathbf{w}}$-invariant curves in $\mathcal{Z}^0_{\mathbf{w}}$, which project to a Schubert point in $\mathcal{Z}^0_{\mathbf{w}[j]}$ for some j, by m-admissible sequences as follows. For $r \geq 2$, let $I = (i_1, \cdots, i_j)$ be an $r-1$ admissible sequence and L_I be the corresponding labelled curve in $\mathcal{Z}^0_{\mathbf{w}[r-1]}$. Then the $\mathcal{T}_{\mathbf{w}[r]}$-invariant curve $\sigma^0_{r-1}(L_I)$ in $\mathcal{Z}^0_{\mathbf{w}[r]}$ is denoted by the same symbol L_I and $\sigma^1_{r-1}(L_I)$ is denoted by L_{Ir}, where Ir denote the r-admissible sequence $(i_1, \cdots, i_j, r)$.

The group $\mathbf{A_1}(\mathcal{Z}^x_{\mathbf{w}})$ is freely generated by the Schubert lines $L_1,\ L_2\ \cdots L_m$ (cf. [Ba], Lemma 1.1). Hence for any m-admissible sequence $I = (i_1, i_2 \cdots, i_r)$ we have $L_I = \sum_{j=1}^m d_j L_j$, for some $d_j \in \mathbb{R}$. In the next proposition we give an explicit formula to write down the coefficients d_j.

Example 1: In the following pictures $\mathcal{T}$-invariant curves are shown for $Z^0_{\mathbf{w}[1]}$, $\mathcal{Z}^0_{\mathbf{w}[2]}$ and $\mathcal{Z}^0_{\mathbf{w}[3]}$ respectively.



Example 2: consider the curve L_{35679} in $\mathcal{Z}^0_{\mathbf{w}[r]}$ for $r \geq 9$. This line project down to L_3 in $\mathcal{Z}^0_{\mathbf{w}[3]}$, i.e $\psi_4\psi_5\psi_6\psi_7\psi_8\psi_9 \cdots \psi_r(L_{35679}) = L_3$ and $\sigma^0_{r-1} \cdots \sigma^0_9\sigma^1_8\sigma^0_7\sigma^1_6\sigma^1_5\sigma^1_4\sigma^0_3(L_3) = L_{35679}$.

The following observations will be useful in later sections:

1. The schubert section $\sigma^0_j(\mathcal{Z}^0_{\mathbf{w}[j]})$ and the non-Schubert section $\sigma^1_j(\mathcal{Z}^0_{\mathbf{w}[j]})$ do not intersect. In the above picture (Example 1), the Schubert surface is placed at the 'bottom' and the non-Schubert surface placed at the 'top'.
2. Let $I = (i_1, i_2, \cdots, , i_r)$ be an m admissible sequence. Then L_I does not exist in $\mathcal{Z}^0_{\mathbf{w}[j]}$ for $j < i_r$, Moreover L_I lies in the non Schubert section of $\mathcal{Z}^0_{\mathbf{w}[j]}$ for $j = i_r$ and in the Schubert section of $\mathcal{Z}^0_{\mathbf{w}[j]}$ for all $j > i_r$

Remark 13. (i) The indexing set consists of 2^m-1 elements as it corresponds to the set of nonempty subsets of the set $\{1, \cdots, m\}$.

(ii) We show that the total number of $\mathcal{T}_{\mathbf{w}}$-invariant curves in $\mathcal{Z}^0_{\mathbf{w}}$ is $m2^{m-1}$.

One can inductively show that the number of $\mathcal{T}_{\mathbf{w[r]}}$ invariant points, pt_r, in $\mathcal{Z}^0_{\mathbf{w}[r]}$ is 2^r. The $\mathcal{T}_{\mathbf{w[r]}}$ invariant curves, l_r, in $\mathcal{Z}^0_{\mathbf{w}[r]}$ can be counted as follows: There are l_{r-1} $\mathcal{T}_{\mathbf{w[r]}}$-invariant curves in the Schubert section $\sigma^0(\mathcal{Z}^0_{\mathbf{w}[r-1]})$ and l_{r-1} $\mathcal{T}_{\mathbf{w[r]}}$-invariant curves in the non-Schubert section $\sigma^1(\mathcal{Z}^0_{\mathbf{w}[r-1]})$. Also there are 2^{r-1} $\mathcal{T}_{\mathbf{w[r]}}$-invariant curves in the fibre which project down to the 2^{r-1}

$\mathcal{T}_{\mathbf{w[r-1]}}$-invariant points of $\mathcal{Z}^0_{\mathbf{w}[r-1]}$. Thus

$$l_r = 2l_{r-1} + pt_{r-1} \quad r \geq 2$$

Now we prove the assertion by induction on m. Clearly the assertion is true when $m = 1$. Assume the result for $m = r$, the number $\mathcal{T}_{\mathbf{w[r]}}$-invariant curves in $\mathcal{Z}^0_{\mathbf{w}[r]}$ is $r2^{r-1}$. By the above observation the number $\mathcal{T}_{\mathbf{w[r+1]}}$-invariant curves in $\mathcal{Z}^0_{\mathbf{w}[r+1]}$ is $2r2^{r-1} + 2^r = 2^r(r+1)$.

We can also count these $\mathcal{T}_{\mathbf{w}}$-invariant curves using our labellings. We have labelled certain curves with non-empty-ordered(increasing order) subsets of the set $\{1, 2, \cdots, m\}$. Clearly there are 2^{m-1} curves with the labelling set starting with 1. Any $\mathcal{T}_{\mathbf{w}}$-invariant curves with the labelling set starting with i, $i \geq 2$ will project down to L_i, The curve L_i will project down to a $\mathcal{T}_{\mathbf{w[i-1]}}$-invariant point (the Schubert point) of $\mathcal{Z}^0_{\mathbf{w}[i-1]}$. The number of labelling set starting with i is 2^{m-i}. But $\mathcal{Z}^0_{\mathbf{w}[i-1]}$ has 2^{i-1} $\mathcal{T}_{\mathbf{w[i-1]}}$-invariant points. Any $\mathcal{T}_{\mathbf{w}}$-invariant curves projecting down to any of these points other than the Schubert points are not labelled. There are 2^{m-i} $\mathcal{T}_{\mathbf{w}}$-invariant curves over each of the $\mathcal{T}_{\mathbf{w[i-1]}}$-invariant points, each of these curves is equivalent to one of those labelled curves. The total number of curves (labelled and un-labelled) which are equivalent to one of the labelled curves with labelling set starting with i is $2^{m-i} \times 2^{i-1} = 2^{m-1}$. This is true for all $1 \leq i \leq m$. Hence the total number $\mathcal{T}_{\mathbf{w}}$-invariant curves is $m2^{m-1}$.

Lemma 14.. *Suppose $Y \subset X$ be smooth projective varieties and C_1, C_2 be two curves in Y. If $a_1C_1 + a_2C_2 = 0$ in $\mathbf{A}_1(Y)$ then $a_1C_1 + a_2C_2 = 0$ in $\mathbf{A}_1(X)$.*

In most of our computations on the limiting toric varieties $\mathcal{Z}^0_{\mathbf{w}[r]}$ we will work on a suitable surface (refer Lemma 12) $\mathcal{Z}^0_u \subseteq \mathcal{Z}^0_{\mathbf{w}[r]}$. The following Lemma is used repeatedly in most of the computation related to curves.

Lemma 15.. *Let $\mathbf{u} = (s_1, s_2)$ be a sequence of simple reflections. Let L_1 denote the section $\sigma^0_1(\mathcal{Z}^0_{s_1})$, L_{12} denote the section $\sigma^1_1(\mathcal{Z}^0_{s_1})$ and L_2 denote the* *fiber $(P_2/B)^0$ over the Schubert point in $\mathcal{Z}^0_{\mathbf{u}}$. Then $L_{12} = L_1 - (2,1)L_2$ in $\mathbf{N}_1(\mathcal{Z}^0_{\mathbf{u}})$. Recall that the notation $(2,1)$ stands for $(\alpha_2, \alpha_1^\vee)$*

Proof: Since the curves L_1, L_2 generate $\mathbf{A}_1(\mathcal{Z}^0_{\mathbf{u}})$ there exist numbers a, b such that $L_{12} = aL_1 + bL_2$. Recall that $L_1 \cdot L_1 = (2,1)$ (by Lemma 12), and $L_2 \cdot L_2 = 0$ (because L_2 is a fiber). Clearly $L_1 \cdot L_{12} = 0$, $L_1 \cdot L_2 = 1 = L_2 \cdot L_{12}$. By intersecting with L_2, $L_{12} \cdot L_2 = aL_1 \cdot L_2 + bL_2 \cdot L_2$, we get $1 = a$. Similarly intersecting with L_1, we get $0 = a(2,1) + b$. Thus $a = 1$ and $b = -(2,1)$

$\diamondsuit$

Recall $s_{\alpha^\vee}(x) := x - (x, (\alpha^\vee)^\vee)\alpha^\vee = x - (x, \alpha)\alpha^\vee$.

Definition 16.. *Set $\alpha^\vee_{j_1 j_2 \cdots j_r} := s_{\alpha^\vee_{j_r}} s_{\alpha^\vee_{j_{r-1}}} \cdots s_{\alpha^\vee_{j_2}}(\alpha^\vee_{j_1})$.*

Proposition 17.. *Let $\mathbf{w} = (s_1, \cdots, s_m)$ be a sequence of simple reflections with $m \geq 2$. For $2 \leq r \leq m$, let $I = (i_1, i_2, \cdots, i_r) := (i_1, i_2, I')$ be an m-admissible sequence and L_I be the corresponding curve. Then in the Chow group $\mathbf{A_1}(\mathcal{Z}^0_{\mathbf{w}})$ of 1-cycles, we have:*

(i) $L_I = L_{i_1 I'} - (i_2, i_1) L_{i_2 I'}$
(ii) $L_I = L_{i_1} + \sum_{k=2}^r \ \sum_{d=2}^k \ \sum_{1=j_1<\cdots<j_d=k} (-1)^{d+1} (i_{j_d}, i_{j_{d-1}}) \cdots (i_{j_2}, i_{j_1}) L_{i_k}$
(iii) Set $d_{i_1} := 1$ and $d_{i_k} = \sum_{j=1}^{k-1} -d_{i_j}(i_k, i_j)$ for $2 \leq k \leq r$. Then $L_I = \sum_{j=1}^r d_{i_j} L_{i_j}$.
(iv) $d_{i_j} = -(\alpha^\vee_{i_1 i_2 \cdots i_{j-1}}, \alpha_{i_j})$ for all $j > 1$

Proof *of (i):* Note that the curves, $L_{i_1 I'}$, $L_{i_2 I'}$ and $L_{i_1 i_2 I'}$ are edges of a surface that is obtained from the Schubert surface $< L_{i_1}, L_{i_2}, L_{i_1 i_2} >$ by successively taking sections and hence are isomorphic. Now (i) follows from Lemma 15.

Proof *of (ii):* By (i) $L_{i_1 i_2 \cdots i_r} = L_{i_1 i_3 \cdots i_r} - (i_2, i_1) L_{i_2 i_3 \cdots i_r}$. By induction we get:

$$L_{i_1 i_3 \cdots i_r} =$$

$$L_{i_1} - \sum_{k=3}^{r}(i_k, i_1)L_{i_k} + \sum_{k=4}^{r}\sum_{d=1}^{k-3}\sum_{2<j_1<\cdots<j_d<k}(-1)^{d+1}(i_k, i_{j_d})(i_{j_d}, i_{j_{d-1}})\cdots(i_{j_1}, i_1)L_{i_k}$$

and $L_{i_2 i_3 \cdots i_r} =$

$$L_{i_2} - \sum_{k=3}^{r}(i_k, i_2)L_{i_k} + \sum_{k=4}^{r}\sum_{d=1}^{k-3}\sum_{2<j_1<\cdots<j_d<k}(-1)^{d+1}(i_k, i_{j_d})(i_{j_d}, i_{j_{d-1}})\cdots(i_{j_1}, i_2)L_{i_k}$$

By multiplying the second equation with $-(i_2, i_1)$ and adding to the first, one can easily check (ii).

Proof *of (iii):* For $k = 1, 2$, the formula follows directly from definition and (ii). For $k \geq 3$, the coefficient of L_{i_k} in the expansion of L_I is given by (ii):

$$\sum_{d=2}^{k}\sum_{1=j_1<\cdots<j_d=k}(-1)^{d+1}(i_{j_d}, i_{j_{d-1}})\cdots(i_{j_2}, i_{j_1}) =$$

$$\sum_{d=2}^{k} -(i_{j_d}, i_{j_{d-1}})\sum_{1=j_1<\cdots<j_{d-1}}(-1)^{d}(i_{j_{d-1}}, i_{j_{d-2}})\cdots(i_{j_2}, i_{j_1})$$

$$= \sum_{d=2}^{k} -(i_{j_d}, i_{j_{d-1}})d_{i_{j_{d-1}}}$$

That proves (iii).

Proof *of (iv):* Proof is by induction on j. Clearly $d_{i_2} = -(i_2, i_1) = -(\alpha_{i_2}, \alpha_{i_1}^{\vee}) = -(\alpha_{i_1}^{\vee}, \alpha_{i_2})$. Assume the result for $d_{i_1}, \cdots, d_{i_{j-1}}$. Now $(\alpha_{i_1 i_2 \cdots i_{j-1}}^{\vee}, \alpha_{i_j}) = (s_{\alpha_{i_{j-1}}^{\vee}}(\alpha_{i_1 i_2 \cdots i_{j-2}}^{\vee}), \alpha_{i_j}) = (\alpha_{i_1 i_2 \cdots i_{j-2}}^{\vee} - (\alpha_{i_1 i_2 \cdots i_{j-2}}^{\vee}, \alpha_{i_{j-1}})\alpha_{i_{j-1}}^{\vee}, \alpha_{i_j}) = (\alpha_{i_1 i_2 \cdots i_{j-2}}^{\vee}, \alpha_{i_j}) - (\alpha_{i_1 i_2 \cdots i_{j-2}}^{\vee}, \alpha_{i_{j-1}})(\alpha_{i_{j-1}}^{\vee}, \alpha_{i_j}) = (\alpha_{i_1 i_2 \cdots i_{j-2}}^{\vee}, \alpha_{i_j}) + d_{i_{j-1}}(\alpha_{i_{j-1}}^{\vee}, \alpha_{i_j})$ (by inducation hypothesis) $= (\alpha_{i_1 i_2 \cdots i_{j-3}}^{\vee}, \alpha_{i_j}) + d_{i_{j-2}}(\alpha_{i_{j-2}}^{\vee}, \alpha_{i_j}) + d_{i_{j-1}}(\alpha_{i_{j-1}}^{\vee}, \alpha_{i_j})$ $= \cdots = (\alpha_{i_1}^{\vee}, \alpha_{i_2}) + d_{i_2}(\alpha_{i_2}^{\vee}, \alpha_{i_j}) + \cdots d_{i_{j-2}}(\alpha_{i_{j-2}}^{\vee}, \alpha_{i_j}) + d_{i_{j-1}}(\alpha_{i_{j-1}}^{\vee}, \alpha_{i_j}) = -d_{i_j}$ by (iii).

$\diamondsuit$

Corollary 18.. *Suppose* $I = (i_1, i_2, \cdots, i_r)$ *with* $\alpha_{i_1} = \alpha_{i_2}$. *Then*

$$L_I = \sum_{j=2}^{r} c_{i_j} L_{i_j}$$

a) $c_{i_2} = -1$ *and for*

$$k > 2,\ c_{i_k} = \sum_{d=2}^{k} \sum_{2=j_1<\cdots<j_d=k} (-1)^d (i_{j_d}, i_{j_{d-1}}) \cdots (i_{j_2}, i_{j_1})$$

b) $c_{i_j} = (\alpha^{\vee}_{i_2 i_3 \cdots i_{j-1}}, \alpha_{i_j})$ *for all* $j > 2$

Proof: Replace i_1 by i_2 and use the fact $(i_2, i_1) = 2$ in Proposition 17 (ii) and (iv).

Remark 19.. The coefficients c_{i_j} in corollary 18 are negative of the coefficients of $L_{I'}$ where $I' = (i_2, i_3, \cdots, i_r)$

Remark 20.. The coefficient of L_k in $L_{i_1 i_2 \cdots i_r}$ vanishes for all $k \notin \{i_1, i_2 \cdots i_r\}$. For any m-admissible sequence $(i_1, \cdots i_r, I)$ the coefficient of L_{i_j} in $L_{i_1 \cdots i_r I}$ and the coefficient of L_{i_j} in $L_{i_1 \cdots i_r}$ are the same for all $j, 1 \leq j \leq r$.

Remark 21.. For an m-admissible sequence $I = (i_1, i_2 \cdots, i_r)$, the coefficient d_{i_j} of L_{i_j} in the expression $L_I = \sum d_{i_j} L_{i_j}$ is the negative of the self intersection of $\sigma^0_{i_j - 1}(L_{i_1 \cdots i_{j-1}})$ in the surface $\psi^{-1}_{i_j}(L_{i_1 \cdots i_{j-1}})$. From the inductive definition of the coefficients of L_I, the coefficients may seem to take arbitrary values but Proposition 17 (iv) shows that these numbers belong to the set $\{(\alpha, \gamma^{\vee}) \mid \alpha \in S$ and $\gamma^{\vee}$ is any dual root. In other words, the coefficients are bounded by *coxeter number* of the dual root system

5 A basis for $\mathbf{A}_1(\mathcal{Z}^0_{\mathbf{w}})$

The main result of this section is the following basis theorem and the two algorithms to find out this basis from the given sequence $\mathbf{w}$.

Theorem 22.. *Let* $\mathbf{w} = (s_1,\ s_2,\ \cdots, s_m)$ *be a sequence of simple reflections. Then there exist a set of* m *linearly independent* $\mathcal{T}_{\mathbf{w}}$ *invariant curves,* $L_j(\mathbf{w})$ $1 \leq j \leq m$*, of* $\mathcal{Z}^0_{\mathbf{w}}$ *which generate* $\mathbf{A}_1(\mathcal{Z}^0_{\mathbf{w}})$ *such that every* $\mathcal{T}_{\mathbf{w}}$ *invariant curve lies in the* $\mathbb{Z}_{\geq 0}$ *span of this set.*

Proof: We choose the generating set inductively. To begin the induction we note that for $\mathcal{Z}^0_{\mathbf{w}[1]} \cong \mathbb{P}^1$, the assertions of the theorem are valid. Suppose we have chosen a generating set $L_j(\mathbf{w}[r])$, $1 \leq j \leq r$ for $\mathcal{Z}^0_{\mathbf{w}[r]}$ such that every $\mathcal{T}_{\mathbf{w}[r]}$ invariant curve in $\mathcal{Z}^0_{\mathbf{w}[r]}$ are non-negatively generated by $L_j(\mathbf{w}[r])$. Then a generating set for $\mathcal{Z}^0_{\mathbf{w}[r+1]}$ can be chosen as

$$L_j(\mathbf{w}[r+1]) := \begin{cases} \sigma^0_r(L_j(\mathbf{w}[r])) & if\ \sigma^0_r(L_j(\mathbf{w}[r]))^2\ \leq 0\ in\ \psi^{-1}_{r+1}(L_j(\mathbf{w}[r])) \\ \sigma^1_r(L_j(\mathbf{w}[r])) & if\ \sigma^0_r(L_j(\mathbf{w}[r]))^2\ > 0\ in\ \psi^{-1}_{r+1}(L_j(\mathbf{w}[r])) \end{cases}$$

for $1 \leq j \leq r$ and $L_{r+1}(\mathbf{w}[r+1]) := L_{r+1}$. Here $\sigma^0_r(L_j(\mathbf{w}[r]))^2$ denote the self intersection number in the surface $\psi^{-1}_{r+1}(L_j(\mathbf{w}[r]))$. By induction we know that any $\mathcal{T}_{\mathbf{w}[r]}$ invariant curve in $\mathcal{Z}^0_{\mathbf{w}[r]}$ is a positive linear combination of $L_j(\mathbf{w}[r])$. First observe from the definition of $L_j(\mathbf{w}[r+1])$, $j = 1, \cdots r$, that the set of curves $\sigma^0_r(L_j(\mathbf{w}[r]))$ and $\sigma^1_r(L_j(\mathbf{w}[r]))$ are non-negative linear combinations of $< L_{r+1}, L_r(\mathbf{w}[r+1]), \cdots L_1(\mathbf{w}[r+1]) >$. Any $\mathcal{T}_{\mathbf{w}[r+1]}$-invariant curve in $\mathcal{Z}^0_{\mathbf{w}[r+1]}$ is either equivalent to the fibre L_{r+1} or lies in either of the sections $\sigma^i_r(\mathcal{Z}^0_{\mathbf{w}[r]})$. Now any $\mathcal{T}_{\mathbf{w}[r+1]}$-invariant curve in either of the sections $\sigma^i_r(\mathcal{Z}^0_{\mathbf{w}[r]})$ are non-negative linear combinations of $\sigma^i_r(L_j(\mathbf{w}[r]))$ by induction hypothesis. Hence they lie in the $\mathbb{Z}_{\geq 0}$ span of $L_j(\mathbf{w}[r+1])$. $\diamondsuit$

Note that the curve $L_j(\mathbf{w})$ is represented by an m-admissible sequence with initial entry j, i.e., $L_j(\mathbf{w}) = L_I$ where $I = (j, \cdots)$.

We have seen that any $\mathcal{T}_{\mathbf{w}}$ invariant curve L_I is a linear combination of Schubert lines L_i with integer coefficients. But the coefficients can be negative. For example if $s_1 = s_2$ then $L_{12} = L_1 - 2L_2$. The idea is to replace L_1 by L_{12} in the generating set whenever $L_{12} = L_1 + d_2 L_2$ and $d_2 < 0$. This observation provides an algorithm to obtain the basis.

Lemma 23.. *Let* $\mathbf{w} = (s_1,\ s_2,\ \cdots, s_m)$ *be a sequence of simple reflections.*

1. *If* $i_2 > 1$ *is the smallest positive integer such that* $s_{i_2} = s_1$ *then* $L_1(\mathbf{w}[j]) = L_1, \forall j, 1 \leq j \leq i_2 - 1$ *and* $L_1(\mathbf{w}[i_2]) = L_{1i_2}$*. If there is no such* i_2 *then* $L_1(\mathbf{w}) = L_1(\mathbf{w}[k]) = L_1$*, for all* $k \geq 1$*.*
2. *Suppose there exist an* i_2 *such that* $s_{i_2} = s_1$*. If* $i_3 > i_2$ *be the smallest positive integer such that* $c_3 := (i_3, i_2) < 0$ *then* $L_1(\mathbf{w}[j]) = L_{1i_2}, \forall j, i_2 \leq j \leq i_3 - 1$ *and* $L_1(\mathbf{w}[i_3]) = L_{1i_2i_3}$*. If there is no such* i_3 *exists then* $L_1(\mathbf{w}) = L_1(\mathbf{w}[k]) = L_{1i_2}$*, for all* $k \geq i_2$*.*
3. *Let* $L_1(\mathbf{w}[i_{r-1}]) = L_{1i_2i_3\cdots i_{r-1}}$ *be chosen inductively for* $r \geq 3$*. If* $i_r > i_{r-1}$ *is the smallest positive integer such that* $c_r = (i_r, i_2) - c_3(i_r, i_3) - c_4(i_r, i_4) - \cdots - c_{r-1}(i_r, i_{r-1}) < 0$ *then* $L_1(\mathbf{w}[j]) = L_1 i_2 \cdots i_{r-1}, \forall j, i_{r-1} \leq j \leq i_r - 1$ *and* $L_1(\mathbf{w}[r]) := L_{1i_2i_3\cdots i_r}$*. If there is no such* i_r *exists then* $L_1(\mathbf{w}) = L_1(\mathbf{w}[k]) = L_{1i_2\cdots i_{r-1}}$*, for all* $k \geq i_{r-1}$*.*
4. *For* $j > 1$*, we repeat these procedures for the truncated word* $[j-1]\mathbf{w}$*. In other words* $L_j(\mathbf{w}) = L_1([j-1]\mathbf{w})$*.*

Proof:

We obtain $L_1(\mathbf{w})$ inductively. Clearly $L_1(\mathbf{w}[1]) = L_1$ gives the required basis for $\mathbf{A_1}(Z^0_{\mathbf{w}[1]})$. By Theorem 22 and Lemma 3, $L_1(\mathbf{w}[2])$ is L_1 or L_{12} (note that $\sigma_1^0(L_1) = L_1$ and $\sigma_1^1(L_1) = L_{12}$) depending on whether $(2, 1)$ is non positive or positive. From the theory of root system we know that $(2, 1)$ is positive only when $s_2 = s_1$ If $s_1 \neq s_j, \forall j > 1$ in the sequence $\mathbf{w} = (s_1, \cdots, s_m)$ then $L_1(\mathbf{w}[k]) = L_1, \forall k \geq 1$

If there exists a $j > 1$ such that $s_j = s_1$ then let $i_2 > 1$ be the smallest positive integer such that $s_1 = s_{i_2}$. As above we have $L_1(\mathbf{w}[k]) = L_1 \forall k < i_2$ and $L_1(\mathbf{w}[i_2]) = L_{1i_2}$. Again by Theorem 22 $L_1(\mathbf{w}[i_2+1])$ is $\sigma^0_{i_2}(L_1(\mathbf{w}[i_2]))$ or $\sigma^1_{i_2}(L_1(\mathbf{w}[i_2]))$ depending on whether $\sigma^0_{i_2}(L_1(w[i_2])^2$ is non positive or positive in the surface $\psi^{-1}_{i_2+1}(L_{1i_2})$. But $\sigma^0_{i_2}(L_1(w[i_2])^2$ is the negative of the

coefficient of L_{i_2+1} in $L_{1i_2(i_2+1)}$ (i.e if $L_{1i_2(i_2+1)} = L_1 + d_{i_2}L_{i_2} + d_{i_2+1}L_{i_2+1}$ then $\sigma^0_{i_2}(L_1(w[i_2])^2$ is $-d_{i_2+1}$).
By Proposition 17(iii) $d_{i_2+1} = -1(1, i_2+1) - d_{i_1}(i_2, i_2+1) = -(i_2, i_2+1) + 2(i_2, i_2+1) = (i_2, i_2+1)$ [as $s_1 = s_{i_2}$ and $d_{i_1} = -2$]
This justifies step 2. A similar argument will work for step 3. Note that c_r is the coefficient of L_{i_r} in $L_{1i_2i_3\cdots i_{r-1}i_r}$, which is negative of the self intersection number, $\sigma^0_{i_r-1}(L_{1i_2i_3\cdots i_{r-1}})^2$, in the surface $\psi^{-1}_{i_r}(L_{1i_2i_3\cdots(i_r-1)})$. Now the proof follows from Theorem 22. $\diamondsuit$

The above algorithm can be written using root data related to the algebraic group G. This will also prove the Remark 20.

Recall also that the height of a positive root $\gamma = \sum_i n_i\gamma_i$, where γ_i's are simple roots, is defined to be the number $\sum_i n_i$ and is denoted by $\mathrm{ht}(\gamma)$ in a root system. We will use this definion for the 'dual roots' $\{\alpha^\vee\}$ in the following lemma. Recall $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ and the simple reflection $s_{\alpha^\vee}(x) := x - (x, (\alpha^\vee)^\vee)\alpha^\vee = x - (x,\alpha)\alpha^\vee$.

Definition 24.. *Set* $\alpha^\vee_{j_1j_2\cdots j_r} := s_{\alpha^\vee_{j_r}} s_{\alpha^\vee_{j_{r-1}}} \cdots s_{\alpha^\vee_{j_2}}(\alpha^\vee_{j_1})$.

Let $\mathbf{w} = (s_1, s_2, \cdots, s_m)$ *be a sequence of simple reflections. We give an algorithm to find the subsequence* $I(\mathbf{w})$

Let $i_2 > 1$ *be the smallest positive integer such that* $s_{i_2} = s_1$. *Let* $i_3 > i_2$ *be the smallest positive integer such that ht(*$\alpha^\vee_{i_2i_3}$*) >ht(*$\alpha^\vee_{i_2}$*).*

Suppose we have chosen i_1, i_{r-1} *inductively with* $r \geq 3$. *Let* $i_r > i_{r-1}$ *be the smallest positive integer such that ht(* $\alpha^\vee_{i_2i_3i_4\cdots i_r}$*) >ht(* $\alpha^\vee_{i_2i_3i_4\cdots i_{r-1}}$*).*

Theorem 25.. *Given a sequence* $\mathbf{w} = (s_1, s_2, \cdots, s_m)$ *the Schuber lines* $L_{I([j-1]\mathbf{w})} = L_j(\mathbf{w})$, $1 \leq j \leq m$.

Proof: By definition $\alpha^\vee_{i_2\cdots i_j} = s_{\alpha^\vee_{i_j}}(\alpha^\vee_{i_2\cdots i_{j-1}}) = \alpha^\vee_{i_2\cdots i_{j-1}} - (\alpha^\vee_{i_2\cdots i_{j-1}},\ \alpha_{i_j})\alpha^\vee_{i_j}$. It is easy to see that $(\alpha^\vee_{i_2\cdots i_{j-1}},\ \alpha_{i_j}) = c_{i_j}$. Note that $\mathrm{ht}(\alpha^\vee_{i_2i_3i_4\cdots i_j})$ >ht($\alpha^\vee_{i_2i_3i_4\cdots i_{j-1}}$) if and only if $c_{i_j} < 0$. Now the Lemma follows from the previous Lemma 23. $\diamondsuit$

Remark 26.. Even though the length of the sequence could be very large, the number of steps in the above algorithm can not be very large. For example if the height of the longest root in the dual Weyl group orbit of $\alpha_1^\vee$ (i.e the unique dominant weight in the orbit containing $\alpha_1^\vee$) is n_1 then the number of steps in the algorithm for $L_1(\mathbf{w})$ will be at most n_1. In other words if $L_1(\mathbf{w}) = L_{1i_1\cdots i_r}$, then $r \leq n_1$.

6 Extremal Rays and Mori Rays

Let X be a normal projective variety defined over k. Then we recall the following definitions (motivated by the definition given for $k = \mathbb{C}$ in Page 254 of [W]).

Definition 27.. *Let $NE(X) \subset A_1(X)$ be the $\mathbb{R}_{\geq 0}$ cone spanned by effective 1-cycles. A ray $R \subset NE(X)$ is an* extremal ray *if given $Z_1, Z_2 \in NE(X)$ such that $Z_1 + Z_2 \in R$, then both $Z_1, Z_2 \in R$.*

Definition 28.. *If an extremal ray R satisfies $R \cdot K_X < 0$, then R is called a* Mori extremal ray *(also referred as a* Mori ray*) where K_X denote the canonical bundle of X.*

Lemma 29.. *Let X be a variety such that $NE(X)$ is nonnegatively generated by a linearly independent set of effective 1-cycles. Then the rays defined by this generating set are precisely the extremal rays.*

Proof: Let $Z_1, \cdots, Z_n$ be the linearly independent set of effective curves generating $A_1(X)$. First we prove that the rays $\mathbb{R}_{\geq 0}Z_i$'s are extremal rays. Suppose $\sum_{a_i \geq 0} a_i Z_i + \sum_{b_i \geq 0} b_i Z_i = cZ_k \in \mathbb{R}_{\geq 0} Z_k$. Then $a_i = b_i = 0$, for $i \neq k$ as Z_i's are linearly independent. Now both $\sum_{a_i \geq 0} a_i Z_i = a_k Z_k$ and $\sum_{b_i \geq 0} b_i Z_i = b_k Z_k$ lie in $\mathbb{R}_{\geq 0} Z_k$.

Consider any extremal ray generated by $R = \sum_{a_i \geq 0} a_i Z_i$. If it is not one of the $\mathbb{R}_{\geq 0} Z_i$'s, then there are at least two nonzero coefficients, say a_k and

a_l. Let $R_1 := \sum_{i \neq l} a_i Z_i$ and $R_2 := a_l Z_l$. Then clearly $R_1 + R_2 = R$ lies in $\mathbb{R}_{\geq 0} R$, but neither R_1 nor R_2 lies in $\mathbb{R}_{\geq 0} R$, a contradiction. Which proves that $\mathbb{R}_{\geq 0} Z_i$'s are the only extremal rays. $\diamondsuit$

Theorem 30.. *The extremal rays of the toric variety $\mathcal{Z}^0_{\mathbf{w}}$ are precisely the curves $L_j(\mathbf{w})$.*

Proof: In view of Lemma 29 it suffices to prove that the effective cone $NE(Z^0_{\mathbf{w}})$ coincides with the positive convex cone generated by the torus invariant curves $L_i(\mathbf{w})$. Since $L_i(\mathbf{w})$ form a basis for $A_1(Z^0_{\mathbf{w}})$, we can write any effective curve C as a linear combination $\sum n_i L_i(\mathbf{w})$. For each $1 \leq l \leq l(\mathbf{w})$, consider the divisor $D(a_l) = \sum_{i \neq l} \tilde{D}_i(\mathbf{w}) + a_l \tilde{D}_l(\mathbf{w})$. Then $D(a_l)$ is ample for all $a_l > 0$ by Theorem 39(a). Hence $D(a_l) \cdot C = \sum_{i \neq l} n_i + a_l n_l > 0$. But $\sum_{i \neq l} n_i + a_l n_l > 0$ for all $a_l > 0$ implies $n_l \geq 0$. Hence C belongs to the cone generated by $L_i(\mathbf{w})$. $\diamondsuit$

Now we give a criterion for an extremal ray to be a Mori ray (cf. Page 254 [W]).

Recall the following standard lemma:

Lemma 31.. *Let Z be a complex manifold and X, Y submanifolds of Z intersecting transversally. Let $\mathcal{N}_{X/Z}$ denote the normal bundle of X in Z. Then $\mathcal{N}_{X/Z} \mid_{X \cap Y} \cong \mathcal{N}_{X \cap Y / Y}$. If X is a divisor in Z then $\mathcal{N}_{X/Z} = \mathcal{O}_Z(X) \mid_X$.*

We recall the boundary of a toric variety as the complement of the dense open orbit or equivalently as the union of all torus invariant divisors. Then the boundary of $Z^0_{\mathbf{w}}$ can be inductively shown to be

$$\partial(Z^0_{\mathbf{w}[r]}) := \psi_r^{-1}(\partial(Z^0_{\mathbf{w}[r-1]})) \cup \sigma^0_{r-1}(Z^0_{\mathbf{w}[r-1]}) \cup \sigma^1_{r-1}(Z^0_{\mathbf{w}[r-1]})$$

Then the canonical bundle $K_{Z^0_{\mathbf{w}[r]}} \cong -\partial(Z^0_{\mathbf{w}[r]})$ (cf. [O]). Given a Schubert line L_r, we compute it's intersection with the boundary components in the following:

Lemma 32.. *For $i \neq r-1$, $(\psi^{i+1})^{-1}\sigma_i^0(\mathcal{Z}^0_{\mathbf{w}[i]}) \cap \langle L_r, L_{i+1}\rangle = L_r$. Moreover*

1. $i < r-1, (\psi^{i+1})^{-1}\sigma_i^0(\mathcal{Z}^0_{\mathbf{w}[i]}) \cdot L_r = 0 = (\psi^{i+1})^{-1}\sigma_i^1(\mathcal{Z}^0_{\mathbf{w}[i]}) \cdot L_r$
2. $i > r-1$, $(\psi^{i+1})^{-1}\sigma_i^0(\mathcal{Z}^0_{\mathbf{w}[i]}) \cdot L_r = (i+1, r)$ and $(\psi^{i+1})^{-1}\sigma_i^1(\mathcal{Z}^0_{\mathbf{w}[i]}) \cdot L_r = 0$
3. $i = r-1, (\psi^{i+1})^{-1}\sigma_i^0(\mathcal{Z}^0_{\mathbf{w}[i]}) \cdot L_r = 1 = (\psi^{i+1})^{-1}\sigma_i^1(\mathcal{Z}^0_{\mathbf{w}[i]}) \cdot L_r$

Proof: Note that $L_{i+1} \not\subseteq (\psi^{i+1})^{-1}\sigma_i^0(\mathcal{Z}^0_{\mathbf{w}[i]})$, in fact, L_{i+1} is normal to $(\psi^{i+1})^{-1}\sigma_i^0(\mathcal{Z}^0_{\mathbf{w}[i]})$ and all other Schubert lines are contained in $(\psi^{i+1})^{-1}\sigma_i^0(\mathcal{Z}^0_{\mathbf{w}[i]})$ When $i \neq r-1$, $L_{i+1} \neq L_r$. Hence L_r is in $(\psi^{i+1})^{-1}\sigma_i^0(\mathcal{Z}^0_{\mathbf{w}[i]})$. One can easily see that $(\psi^{i+1})^{-1}\sigma_i^0(\mathcal{Z}^0_{\mathbf{w}[i]}) \cap \langle L_r, L_{i+1}\rangle = L_r$. Now by Lemma 31 $(\psi^{i+1})^{-1}\sigma_i^0(\mathcal{Z}^0_{\mathbf{w}[i]}) \cdot L_r$ is the self intersection number $L_r \cdot L_r$ in the Schubert surface $\langle L_r, L_{i+1}\rangle$

When $i < r-1$, the Schubert surface $\langle L_r, L_{i+1}\rangle$ maps onto L_{i+1} with firber L_r. Hence $L_r.L_r = 0$ in this surface. Which proves 1.

When $i > r-1$, the Schubert surface $\langle L_r, L_{i+1}\rangle$ maps onto L_r with firber L_{i+1}. Now 2) follows from the Lemma 12.

It is clear that L_r intersect transversally at the Schubert point and at the non Schubert point of L_r transversally. This proves 3.

$\diamondsuit$

Proposition 33.. $K_{Z^0_{\mathbf{w}}} \cdot L_r = -\partial(Z^0_{\mathbf{w}[r]}) \cdot L_r = -2 - \sum_{j=r}^{m-1}(j+1, r)$

Proof: Note that $L_r \subset \sigma_j^0(Z^0_{\mathbf{w}[j]})$, for each $j \geq r$, we see that $\sigma_j^1(Z^0_{\mathbf{w}[j]}) \cdot L_r = 0$ for $j \geq r$. Hence we have

$$K_{Z^0_{\mathbf{w}}} \cdot L_r = -(\partial(Z^0_{\mathbf{w}})) \cdot L_r$$

$$= -(\sigma^0_{r-1}(Z^0_{\mathbf{w}[r-1]}) \ + \sigma^1_{r-1}(Z^0_{\mathbf{w}[r-1]})) \cdot L_r - \sum_{j=r}^{m-1} (\psi^{j+1})^{-1}(\sigma_j^0(Z^0_{\mathbf{w}[j]})) \cdot L_r$$

The restriction of the pull back divisor $(\psi^{j+1})^{-1}(\sigma_j^0(Z^0_{\mathbf{w}[j]}))$ to L_r is isomorphic to the normal bundle of L_r in the surface $< L_r, L_{j+1} >$. Hence the degree,

$(\psi^{j+1})^{-1}(\sigma_j^0(Z^0_{\mathbf{w}[j]})) \cdot L_r = (j,r)$ by Lemma 12. Now

$$K_{Z^0_{\mathbf{w}}} \cdot L_r = -\partial(Z^0_{\mathbf{w}[r]}) \cdot L_r = -2 - \sum_{j=r}^{m-1}(j+1,r)$$

$\diamondsuit$

Lemma 34.. *If an extremal ray in $Z^0_{\mathbf{w}[r]}$ is not a Mori ray then the extremal ray lying over this ray cannot be a Mori ray for any $Z^0_{\mathbf{w}[j]}$, $j > r$.*

Proof: By induction it suffices to prove for $j = r+1$. Let I be an $r+1$-admissible sequence such that L_I is an extremal ray in $Z_{\mathbf{w}[r+1]}$. Assume $\psi_{r+1}(L_I)$ is not a Mori ray. Then we have

$$K_{Z^0_{\mathbf{w}[r+1]}} \cdot L_I = K_{Z^0_{\mathbf{w}[r]}} \cdot \psi_{r+1}(L_I) - d$$

where d is the self intersection of the curve L_I in the surface $\psi_{r+1}^{-1}(\psi_{r+1}(L_I))$. Since $\psi_{r+1}(L_I)$ is not a Mori ray, it follows that $K_{Z^0_{\mathbf{w}[r]}} \cdot \psi_{r+1}(L_I) \geq 0$. By the construction of extremal rays (cf. Theorem 22) it follows that $d \leq 0$. Hence $K^0_{Z_{\mathbf{w}[r+1]}} \cdot L_I \geq 0$ as claimed. $\diamondsuit$

This leads to a criterion for an extremal ray to be a Mori ray.

Theorem 35.. *An extremal ray L_I is a Mori Ray if and only if there exists an $r > 0$ such that L_I is the Schubert line L_r and there is at most one $j > r$ such that $(j,r) < 0$ and it should be -1.*

Proof: A Schubert line L_r is an extremal ray if and only if $(j,r) \leq 0$ for all $j > r$ (refer Lemma 23, Theorem 30). L_r is a Mori ray if and only if it is an extremal ray and $K_{Z^0_{\mathbf{w}}} \cdot L_r = -\sum_{j>r}(j,r) - 2 < 0$ i.e., $\sum_{j>r}(j,r) \geq -1$. So there is at most one $j > r$ such that $(j,r) < 0$ and it must be -1.

Let $I = (i_1, \cdots, i_r)$ such that $L_I = \sum d_{i_j} L_{i_j}$ be a non-Schubert extremal ray and hence $L_I = L_{i_1}(\mathbf{w})$ (by Theorem 30). Notice that $(i_2, i_1) = 2$ and

$(j, i_1) \leq 0$ for all $i_1 < j < i_2$ by Lemma 23. Hence

$$K_{Z^0_{\mathbf{w}[i_2]}} \cdot L_{i_1 i_2} = -\sum_{i_1<j<i_2} (j, i_1) - 2 + (i_2, i_1) = -\sum_{i_1<j<i_2} (j, i_2) \geq 0$$

This shows L_{i_1,i_2} is not a Mori ray in $Z^0_{\mathbf{w}[i_2]}$. Now by Lemma 34 L_I is not a Mori ray in $Z^0_{\mathbf{w}}$. $\diamondsuit$

Recall that a smooth projective variety is called *Fano* if it's anti-canonical bundle is ample.

Corollary 36.. *For a ginve* $\mathbf{w} = (s_1, s_2, \cdots, s_m)$ *The toric variety* $\mathcal{Z}^0_{\mathbf{w}}$ *is Fano if and only if all Schubert lines are Mori.*

Prooof: Suppose all the L_i's are Mori, then by definition of Mori, they are all extremal and hence by Theorem 30 and Theorem 22 they generate all the torus invariant lines positively. Now the ampleness of $-K_{\mathcal{Z}^0_{\mathbf{w}}}$ follows from Toric Nakai Criterion. Conversely if $\mathcal{Z}^0_{\mathbf{w}}$ is Fano then $L_j(\mathbf{w})$ (refer 22) are Mori. By Theorem 35 $L_j(\mathbf{w})$ is the Schubert line L_j. Thus all Schubert lines are Mori. $\diamondsuit$

Example 3: $G = SL_{n+1}$ and the sequence $\mathbf{w}_0 = (1, 2, \cdots, n, 1, 2 \cdots, n-1, \cdots .1, 2, 1)$ is a reduced expression for the longest element w_0 of the Weyl Group. Then there are exactly n Mori rays which are:
$L_n, L_{n+n-1}, \cdots, L_{n+n-1+n-2+\cdots n-r}, \cdots, L_{n(n+1)/2}$.

7 Ample Cone

Let $\mathbf{w} = (s_1, s_2, \cdots, s_m)$ be a sequence of simple reflections and $Z_{\mathbf{w}[i]}$ be the intermediate BSDH variety. Let $f_i : Z_{\mathbf{w}[i]} \to G/B$ be the B-equivariant map. Let $\mathcal{L}(\omega_j)$ denote the line bundle on G/B corresponding to the fundamental weight ω_j.

Let $\psi^i := \psi_{i+1} \circ \cdots \circ \psi_m : Z_{\mathbf{w}} \to Z_{\mathbf{w}[i]}$ be the composite projection. Then define $\mathcal{L}_j := (\psi^j)^*(f_j^*(\mathcal{L}(\omega_j))$. Then Lauritzen and Thomsen [LT] have proved that $\mathcal{L}_j$ for $j = 1, \cdots, m$ form a basis for the Picard group of line bundles, hence a basis for $\mathbf{A}^1(Z_{\mathbf{w}})$. In fact they also proved that the ample cone is the 'strict positive cone' generated by $\mathcal{L}_j$.

Lemma 37..

$$\mathcal{L}_j \cdot L_r = \begin{cases} 0 & \text{if} \quad j < r \quad \text{or} \quad \alpha_j \neq \alpha_r \\ 1 & \text{if} \quad j \geq r \quad \text{and} \quad \alpha_j = \alpha_r \end{cases}$$

Proof:

For $j < r$, choose a section of $\mathcal{L}_j$ that does not contain the Schubert point of $Z_{\mathbf{w}[j]}$. Then the inverse image of this section under ψ^j does not intersect with the Schubert line L_r and hence the $\mathcal{L}_j \cdot L_r = 0$.

For $j \geq r$, f_m restricts to l_r as an embedding wth image is P_r/B. Moreover, $\mathcal{L}_j$ restricted to L_r can be identified with the restriction of line bundle $L(\omega_j)$ to P_r/B. Now the theorem follows from the fact that $L(\omega_j)$ has degree 0 if $\alpha_r \neq \alpha_j$ it is 1 if $\alpha_r = \alpha_j$ $\diamondsuit$

The boundary components $(\psi^j_{\mathbb{A}})^{-1}\sigma_{j-1,\mathbb{A}}(\mathcal{Z}_{\mathbf{w}[j-1]}),\ 1 \leq j \leq m$, are simple normal crossing $\mathbb{A}$ divisors in $\mathcal{Z}_{\mathbf{w}}$. For each $x \in \mathbb{A}$, these divisors form a basis for the Picard group $Pic(\mathcal{Z}^x_{\mathbf{w}})$. In particular $\mathcal{L}_j$ can be expressed uniquely as linear combination of these boundary divisors of $\mathcal{Z}^1_{\mathbf{w}} = Z_{\mathbf{w}}$.

$$\mathcal{L}_j = \sum_{i=1}^{j} a_{ij} (\psi^i)^{-1} \sigma_{i-1}(Z_{\mathbf{w}[i-1]})$$

Now consider the following relative divisor

$$\mathcal{L}_{j,\mathbb{A}} := \sum_{i=1}^{j} a_{ij} (\psi^i_{\mathbb{A}})^{-1} \sigma_{i-1,\mathbb{A}}(\mathcal{Z}_{\mathbf{w}[i-1]})$$

We denote the line bundle $\mathcal{L}^0_j$ given by the restriction of this relative divisor to the special fiber at 0. Now by using Lemma 32, we can show an analogue of Lemma 37 in the toric variety for the line bundle $\mathcal{L}_j^0$.

Lemma 38.. *A relative line bundle $\mathcal{L}$ given by $\sum_{i=1}^m a_i(\psi_{\mathbb{A}}^i)^{-1}\sigma_{i-1,\mathbb{A}}(\mathcal{Z}_{\mathbf{w}[i-1]})$ on $\mathcal{Z}_{\mathbf{w}}$ represents a relatively ample bundle if and only if $\mathcal{L}^0 \cdot L_j(\mathbf{w}) > 0$ for all j, where $\mathcal{L}^0$ is the line bundle given by $\sum_{i=1}^m a_i(\psi^i)^{-1}\sigma_{i-1}(\mathcal{Z}_{\mathbf{w}[i-1]}^0)$*

Proof: Ampleness being an open condition in a flat family to check the relative ampleness it sufices to check ove the special fiber $\mathcal{Z}_{\mathbf{w}}^0$. Note that by our choice of $L_j(\mathbf{w})$ any torus invariant curve can expressed as non negative linear combination of these $L_j(\mathbf{w})$ (refer 22). Now the stated condition is equivalent to the Toric Nakai Criterion for ampleness for a smooth toric variety. $\diamondsuit$

Theorem 39.. *The ample cone of $\mathcal{Z}_{\mathbf{w}}^0$ is a naturally a subcone of the ample cone of the BSDH variety Z_w*

Proof: Using the basis $(\psi_{\mathbb{A}}^i)^{-1}\sigma_{i-1,\mathbb{A}}(\mathcal{Z}_{\mathbf{w}[i-1]})$ for the relative $Pic(\mathcal{Z}_{\mathbf{w}})$ and the openness of the ampleness one can identify the ample cone of $\mathcal{Z}_{\mathbf{w}}^0$ as a subcone of the ample cone of $\mathcal{Z}_{\mathbf{w}}^1$.

We would like to give more computable comparison of this sub cone in terms of the natural generators of the ample cone of the BSDH variety. For this purpose let us denote the coeffecients of $L_j(\mathbf{w})$ by $d_{i_j}([j-1]\mathbf{w})$ for $j = 1, \cdots m$. Then we have

$$L_k(\mathbf{w}) = L_k + \sum_{i_j i_j > k} d_{i_j}([k-1]\mathbf{w}) L_{i_j}$$

Note that $d_{i_j}([k-1]\mathbf{w}) < 0$ for all i_j. Now consider an ample line bundle $\mathcal{L} = \sum a_i \mathcal{L}_i$. Then

$$\mathcal{L} \cdot L_k(\mathbf{w}) = a_k + \sum a_i d_{i_j}([k-1]\mathbf{w})$$

where the sum is taken all indices $i, i_j > k$ and $\alpha_i = \alpha_{i_j}$. We can prove that $a_m > 0$ as $\mathcal{L} \cdot L_m(\mathbf{w}) = a_m$. By (downward) induction we assume that $a_i > 0$ fo all $i > k$. then we observe that $\mathcal{L} \cdot L_k(\mathbf{w}) = a_k + \sum a_i d_{i_j}([k-1]\mathbf{w}) > 0$ implies

$$a_k > -\sum a_i d_{i_j}([k-1]\mathbf{w}) > 0$$

This infact proves that the line bundle $\mathcal{L}$ is ample on the BSDH variety by the Theorem of Lauritzen and Thomsen. This gives an inclusion of the ample cone. $\diamondsuit$

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Appendix

1. $L_{i_1 i_2 i_3 i_4 i_5}$
$= L_{i_1 i_3 i_4 i_5} - (i_2, i_1) L_{i_2 i_3 i_4 i_5}$
$= L_{i_1 i_4 i_5} - (i_3, i_1) L_{i_3 i_4 i_5} - (i_2, i_1) L_{i_2 i_4 i_5} + (i_3, i_2)(i_2, i_1) L_{i_3 i_4 i_5}$
$= L_{i_1 i_4 i_5} - (i_2, i_1) L_{i_2 i_4 i_5} + [-(i_3, i_1) + (i_3, i_2)(i_2, i_1)] L_{i_3 i_4 i_5}$
$= L_{i_1 i_5} - (i_4, i_1) L_{i_4 i_5} - (i_2, i_1)[L_{i_2 i_5} - (i_4, i_2) L_{i_4 i_5}]$
$[-(i_3, i_1) + (i_3, i_2)(i_2, i_1)][L_{i_3 i_5} - (i_4, i_3) L_{i_4 i_5}]$
$= L_{i_1 i_5} - (i_2, i_1) L_{i_2 i_5} + [-(i_3, i_1) + (i_3, i_2)(i_2, i_1)] L_{i_3 i_5} + [-(i_4, i_1) + (i_4, i_2)(i_2, i_1) + (i_4, i_3)(i_3, i_1) - (i_4, i_3)(i_3, i_2)(i_2, i_1)] L_{i_4 i_5}$
$= L_{i_1} - (i_5, i_1) L_{i_5} - (i_2, i_1)[L_{i_2} - (i_5, i_2) L_{i_5}] +$
$[-(i_3, i_1) + (i_3, i_2)(i_2, i_1)][L_{i_3} - (i_5, i_3) L_{i_5}] +$
$[-(i_4, i_1) + (i_4, i_2)(i_2, i_1) + (i_4, i_3)(i_3, i_1) - (i_4, i_3)(i_3, i_2)(i_2, i_1)][L_{i_4} - (i_5, i_4) L_{i_5}]$
$= L_{i_1} - (i_2, i_1) L_{i_2} + [-(i_3, i_1) + (i_3, i_2)(i_2, i_1)] L_{i_3} + [-(i_4, i_1) + (i_4, i_2)(i_2, i_1) + (i_4, i_3)(i_3, i_1) - (i_4, i_3)(i_3, i_2)(i_2, i_1)] L_{i_4} + [-(i_5, i_1) + (i_5, i_2)(i_2, i_1) + (i_5, i_3)(i_3, i_1) + ((i_5, i_4)(i_4, i_1) - (i_5, i_3)(i_3, i_2)(i_2, i_1) - (i_5, i_4)(i_4, i_2)(i_2, i_1) - (i_5, i_4)(i_4, i_3)(i_3, i_1) + (i_5, i_4)(i_4, i_3)(i_3, i_2)(i_2, i_1)] L_{i_5}$

Let $\mathbf{L}_{i_1 i_2 i_3 i_4 i_5} := d_{i_1} L_{i_1} + d_{i_2} L_{i_2} + d_{i_3} L_{i_3} + d_{i_4} L_{i_4} + d_{i_5} L_{i_5}$ $\diamondsuit$

2. Suppose $\alpha_{i_1} = \alpha_{i_2}$. Then we replace i_1 with i_2 and use $(i_2, i_1) = 2$ in the above expression to get

$L_{i_1 i_2 i_3 i_4 i_5} = L_{i_2} - 2L_{i_2} + [-(i_3, i_2) + 2(i_3, i_2)]L_{i_3} + [-(i_4, i_2) + 2(i_4, i_2) + (i_4, i_3)(i_3, i_2) - 2(i_4, i_3)(i_3, i_2)]L_{i_4} + [-(\mathrm{i}_5, i_2) + 2(i_5, i_2) + (i_5, i_3)(i_3, i_2) + (i_5, i_4)(i_4, i_2) - 2(i_5, i_3)(i_3, i_2) - 2(i_5, i_4)(i_4, i_2) - (i_5, i_4)(i_4, i_3)(i_3, i_2) + 2(i_5, i_4)(i_4, i_3)(i_3, i_2)L_{i_5}$

$= -L_{i_2} + (\mathrm{i}_3, i_2)L_{i_3} + [(i_4, i_2) - (i_4, i_3)(i_3, i_2)]L_{i_4} + [(i_5, i_2) - (i_5, i_3)(i_3, i_2) - (i_5, i_4)(i_4, i_2) + (i_5, i_4)(i_4, i_3)(i_3, i_2)]L_{i_5}$

Let $\mathbf{L}_{i_1 i_2 i_3 i_4 i_5} := c_{i_2} L_{i_2} + c_{i_3} L_{i_3} + c_{i_4} L_{i_4} + c_{i_5} L_{i_5}$ $\diamondsuit$

3. $\alpha^\vee_{i_1}$: $(\alpha^\vee_{i_1}, \alpha_{i_2}) = (\alpha_{i_2}, \alpha^\vee_{i_1}) = (i_2, i_1) = -d_{i_2}$

$\alpha^\vee_{i_1 i_2} = s_{\alpha_{i_2}}(\alpha^\vee_{i_1}) = \alpha^\vee_{i_1} - (\alpha^\vee_{i_1}, \alpha_{i_2})\alpha^\vee_{i_2}$:
$(\alpha^\vee_{i_1 i_2}, \alpha_{i_3}) = (i_3, i_1) - (i_3, i_2)(i_2, i_1) = -d_{i_3}$

$\alpha^\vee_{i_1 i_2 i_3} = s_{\alpha^\vee_{i_3}}(\alpha^\vee_{i_1 i_2}) = \alpha^\vee_{i_1} - (\alpha^\vee_{i_1}, \alpha_{i_2})\alpha^\vee_{i_2} - (\alpha^\vee_{i_1} - (\alpha^\vee_{i_1}, \alpha_{i_2})\alpha^\vee_{i_2}, \alpha_{i_3})\alpha^\vee_{i_3}$
$= \alpha^\vee_{i_1} - (\alpha^\vee_{i_1}, \alpha_{i_2})\alpha^\vee_{i_2} - (\alpha^\vee_{i_1}, \alpha_{i_3})\alpha^\vee_{i_3} + (\alpha^\vee_{i_1}, \alpha_{i_2})(\alpha^\vee_{i_2}, \alpha_{i_3})\alpha^\vee_{i_3}$:

$(\alpha^\vee_{i_1 i_2 i_3}, \alpha_{i_4}) = (i_4, i_1) - (i_4, i_2)(i_2, i_1) - (i_4, i_3)(i_3, i_1) + (i_4, i_3)(i_3, i_2)(i_2, i_1) = -d_{i_4}$

It is easy to check that

$(\alpha^\vee_{i_1 i_2 i_3 i_4}, \alpha_5) = -d_{i_5}$ and

$(\alpha^\vee_{i_2}, \alpha_3) = -c_{i_3}$, $(\alpha^\vee_{i_2 i_3}, \alpha_4) = -c_{i_4}$, $(\alpha^\vee_{i_2 i_3 i_4}, \alpha_5) = -c_{i_5}$

References

[Ba] Charles M. Barton, *Tensor Product of Ample Vector Bundles in Characteristic p* Amer. J. Math. 93 (1971) 429–438.

[C] P. Caldero, *Toric Degenerations of Schubert varieties*, Transformation Groups, Vol. 7, No. 1, (2002), 51-60.

[D] M. Demazure, *Desingularization des varietes de Schubert generalisees*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 7 (1974) 53-88.

[GL] N. Gonciulea and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*, Transformation Groups Vol. 1, No 3, 215-248 (1996).

[GK] M. Grossberg and Y. Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke Math. J. 76 (1994), no. 1, 23–58.

[H] R. Hartshorne, *Algebraic Geometry* Springer, 1977,

[J] J. C. Jantzen, *Representations of Algebraic groups*, Pure and Applied Mathematics, 9, Academic Press, 1972.

[KM] S. Keel, S. Mori, *Quotients by Groupoid*, Annals of Mathematics, 145, (1997) 193-213.

[K] G. R. Kempf, *Linear Systems of Homogeneous Spaces*, The Annals of Mathematics, Second Series, Vol 103, No. 3, (1976). 557-591.

[LT] N. Lauritzen, J. F. Thomsen, *Line bundles on Bott-Samelson varieties*, J. Algebraic Geom. 13 (2004), no. 3, 461–473.

[MFK] D. Mumford, J. Fogarty, F. Kirwan, *Goemetric Invariant Theory*, Springer Verlag 1994.

[O] T. Oda, *Convex Bodies and Algebraic Geometry*, Ergeb. Math, (15) Springer Verlag, 1988.

[S] T. A. Springer, *Linear algebraic groups, Second edition*, Progress in Mathematics, 9. Birkhauser Boston, Inc., Boston, MA, 1998. Birkhauser.

[W] J. A. Wisniewski, Toric Mori theory and Fano manifolds. Geometry of toric varieties, 249272, Semin. Congr. 6, Soc. Math. France, Paris, 2002.