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Cohomology of line bundles on Schubert varieties: The rank two case

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Abstract. In this paper we describe vanishing and non-vanishing of cohomology of 'most' line bundles over Schubert subvarieties of flag varieties for rank 2 semisimple algebraic groups.

Keywords. Semisimple algebraic group; root system; cohomology of line bundle; Schubert variety.

1. Introduction

Let G be a semisimple, simply connected algebraic group defined over an algebraically closed field k of characteristic zero. Fix a maximal torus T. Let $W = \mathrm{Nor}_G(T)/\mathrm{Cent}_G(T)$ be the Weyl group and $X(T) = \mathrm{Hom}(T, GL_1(k))$ be the group of characters. For a finite dimensional T-module V and $\chi \in X(T)$, set $V_\chi = \{v \in V \mid tv = \chi(t)v,\ \forall t \in T\}$. The finitely many $\chi \in X(T)$ such that $V_\chi \neq 0$ are called weights. Then V decomposes into a direct sum of weight spaces, i.e. $V = \bigoplus_{\chi_i} V_{\chi_i}$, where χ_i are weights.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and T respectively. Let T act on $\mathfrak{g}$ by the adjoint representation. The non-zero weights of this representation are called roots and we denote the set of all roots by Φ. The weight space corresponding to the zero weight is $\mathfrak{h}$. We identify $X(T)$ canonically as a subset of $X(T) \otimes \mathbb{R}$. Let $\triangle$ denote the set of simple roots, that is a subset of Φ such that (i) $\triangle$ is a basis of $X(T) \otimes \mathbb{R}$, (ii) for each root $\alpha \in \Phi$ there exist integers n_i of like sign such that $\alpha = \sum n_i \alpha_i,\ \alpha_i \in \triangle$. The set of all roots $\alpha \in \Phi$ for which the n_i are not negative is denoted by Φ^+ and an element of Φ^+ is called a positive root. $\Phi^- := \Phi - \Phi^+$ is the set of negative roots. The number of simple roots is called the *rank* of G. Let B be the Borel subgroup of G such that the Lie algebra $\mathfrak{b}$ of B is $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$.

There is a natural faithful W action on $X(T) \otimes \mathbb{R}$ and there exists a W invariant bilinear form $(,)$ on $X(T) \otimes \mathbb{R}$. For $\alpha \in \triangle$, $\alpha^\vee := 2\alpha/(\alpha, \alpha)$. For each $\alpha \in \Phi$ there is a reflection $s_\alpha \in W$ such that $s_\alpha(\psi) = \psi - (\psi, \alpha^\vee)\alpha$, $\forall \psi \in X(T) \otimes \mathbb{R}$. The set $\{s_\alpha, \alpha \in \triangle\}$ generates W. For each $\alpha_i \in \triangle$ there exists a weight $\omega_i \in X(T)$ such that $(\omega_i, \alpha_j^\vee) = \delta_{ij}$. Here δ is the Kronecker delta. These ω_i are called fundamental weights. An element $\lambda = \sum_i n_i \omega_i \in X(T)$ is called a dominant weight if $n_i \geq 0$, $\forall\, i$. Let $X(T)^+$ denote the set of all dominant weights. Note that the action of W permutes the roots. We define length of an element w of W to be the number of positive roots moved by w to the negative roots, and we denote it by $l(w)$.

The Schubert varieties are subvarieties of G/B, indexed by W. For $w \in W$ the corresponding Schubert variety $X(w)$ is defined to be $\overline{BwB}(\text{mod}B)$. The variety G/B itself is a Schubert variety for $w_0 \in W$, the longest element, that is the element which takes all positive roots to negative roots. The dimension of $X(w) = l(w)$.

Let $\lambda \in X(T)$. There is a unique extension of λ to a character of B; we denote this extension also by λ. We denote by L_λ the line bundle on G/B whose total space is the quotient of $G \times k$ by the equivalence relation $(g, x) \sim (gb, \lambda(b)^{-1}x)$ where $g \in G,\ b \in B$ and $x \in k$. For $\lambda \in X(T)$, we denote the restriction of the line bundle L_λ to a Schubert variety $X(w)$ also by L_λ.

Finally recall that the 'dot-action' of W on $X(T)$ is defined as $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is the half sum of positive roots. Note that under the usual action of W any weight can be moved to a dominant weight. But this is not true under dot-action. Those weights which cannot be moved to dominant weights are called singular weights. It can be easily seen that these are weights λ's such that $(\lambda + \rho, \alpha^\vee) = 0$ for some root $\alpha \in \Phi$.

1.1 *The main results*

We give explicitly the vanishing and non-vanishing of cohomology modules of 'most' line bundles over Schubert subvarieties of flag varieties of semisimple algebraic groups of rank 2 defined over algebraically closed fields of characteristic 0. The term 'most' is defined in the tables in §3 (there is one table each for A_2, B_2, and G_2) – the weight associated to the line bundle should belong to a Weyl group orbit of the dominant Weyl chamber C under the 'dot' action and further it should not lie along certain 'forbidden' lines (the tables specify the inequalities defining each $w \cdot C$ for $w \in W$ and also the equations of the forbidden lines).

The results are graphically expressed in terms of matrices in the figures in §7. For each type of group, there are d pictures (or matrices), named $H^0, H^1, \ldots, H^d$, of size $r \times r$ consisting of dark and light boxes, where $d = \dim(G/B)$ and $r = \#W$, and columns and rows are indexed by the Weyl group elements (to minimise the size of the table we have written, α for s_α, β for s_β, $\alpha\beta$ for $s_\alpha s_\beta$ and so on). If the weight λ belongs to $w' \cdot C$ and is not forbidden, then $H^i(X(w), L_\lambda)$ vanishes if and only if the (w, w')th entry, i.e. the intersection of wth row and the w'th column in the H^ith picture is not a dark box.

1.1.1. *Remarks*

(1) Besides being easily readable, by the method of depicting the results in this pictorial form we are able to observe some nice structures which are both illustrative and instructive. We remark that the symmetries in these pictures were the basis of some of the results proven in a more general context in [1]. These explicit computations also facilitate and provide evidence for certain conjectures made in [1]. The rank 2 set-up is the prototype for the general strategy followed in [1].

(2) To make the computation uniform with respect to the Weyl chamber (cf. Remark in 4.1), more importantly to make statements of vanishing and non-vanishing of cohomology modules with respect to Weyl chambers, we need to omit certain lines (we call them as forbidden lines). However we note that the techniques used in the computation continue to work even for a forbidden line bundle, i.e. line bundle corresponding to a weight lying on a forbidden line.

2. Preliminaries

In this section we fix some more notations and collect all the results which are frequently used in this paper. A standard reference for all this material is Jantzen's book [3].

1. *Definition of P_α, B_α, and $SL(2,\alpha)$.* We denote by U_α the root subgroup corresponding to α. We denote by P_α the minimal parabolic subgroup of G containing B and U_α. Let L_α denote the Levi subgroup of P_α containing T. We denote by B_α the intersection of L_α and B. Then L_α is the product of T and a homomorphic image of $SL(2)$ in G (cf. [3], II.1.3). We denote this copy of $SL(2)$ in G by $SL(2,\alpha)$.

2. We define the Bott–Samelson schemes which play an important role in our computation. Let $w = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_n}$ be a reduced expression for $w \in W$. Define

$$Z(w) = \frac{P_{\alpha_1}\times P_{\alpha_2}\times\cdots\times P_{\alpha_n}}{B\times\cdots\times B},$$

where the action of $B\times\cdots\times B$ on $P_{\alpha_1}\times P_{\alpha_2}\times\cdots\times P_{\alpha_n}$ is given by $(p_1,\dots,p_n)(b_1,\dots,b_n) = (p_1\cdot b_1, b_1^{-1}\cdot p_2\cdot b_2,\dots,b_{n-1}^{-1}\cdot p_n\cdot b_n)$, $p_i\in P_{\alpha_i}$, $b_i\in B$. Note that $Z(w)$ depends on the reduced expression chosen for w. It is well-known that $Z(w)$ is a smooth B-variety and is a resolution for $X(w)$.

We use the following non-trivial theorem (cf. [3]): The cohomology module $H^i(X(w),L_\lambda)$ is isomorphic to the cohomology module $H^i(Z(w),L_\lambda)$, for all Schubert varieties $X(w)$ and for all line bundles L_λ.

3. *New notations.* For $w\in W$ and $\lambda\in X(T)$, we denote $H^i(X(w),L_\lambda) = H^i(Z(w),L_\lambda)$ by $H^i(w,\lambda)$. When $w = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_r}\in W$, we denote $H^i(Z(w),L_\lambda)$ also by $H^i(\alpha_1\alpha_2\dots\alpha_r,\lambda)$.

4. We recall the following propositions (cf. [3], II.5.2) which are often used in the computations. Let $\alpha\in\triangle$ and $\lambda\in X(T)$. Then

1. if $(\lambda,\alpha^\vee) = -1$, then $H^i(\alpha,\lambda) = 0,\ \forall i$;
2. if $(\lambda,\alpha^\vee) = r\geq 0$, then $H^i(\alpha,\lambda) = 0,\ \forall i,\ i\neq 0$ and the module $H^0(\alpha,\lambda)$ has a basis $v_0,\ v_1,\dots,v_r$ such that $tv_i = (\lambda - i\alpha)(t)v_i,\ 0\leq i\leq r$;
3. if $(\lambda,\alpha^\vee)\leq -2$, then $H^i(\alpha,\lambda) = 0,\ \forall i,\ i\neq 1$ and the module $H^1(\alpha,\lambda)$ has basis $u_0,\ u_1\dots u_r$ where $r = -(\lambda,\alpha^\vee)-2$ such that $tu_j = (s_\alpha\cdot\lambda - j\alpha)(t)u_j,\ 0\leq j\leq r$.

5. Now we explain the steps of our computation. We compute the cohomology modules by induction on $l(w)$. Let $w\in W$ and $w = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_r}$ be a reduced decomposition of w.

$$\begin{array}{c} Z(s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_r}) \simeq \overline{Bs_{\alpha_1}B}\times^B Z(s_{\alpha_2}\dots s_{\alpha_r}) = P_{\alpha_1}\times^B Z(s_{\alpha_2}\dots s_{\alpha_r}) \\ \hfill \Big\downarrow {\scriptstyle Z(s_{\alpha_2}\dots s_{\alpha_r})} \\ \hfill P_{\alpha_1}/B\simeq\mathbb{P}^1. \end{array}$$

Let $w = s_\alpha\phi$ and $l(\phi) < l(w)$. By applying the Leray spectral sequence to the fibration

$$\begin{array}{ccc} X(w) & & Z(w) \\ \Big\downarrow {\scriptstyle X(\phi)} & \Longrightarrow & \Big\downarrow {\scriptstyle Z(\phi)} \\ X(\alpha) & & Z(\alpha) \end{array}$$

since the base is $\mathbb{P}^1$ we get short exact sequence of cohomology of B-modules for a given $\lambda \in X(T)$

$$0 \to H^1(\alpha, \widetilde{H^{i-1}(\phi, \lambda)}) \to H^i(w, \lambda) \to H^0(\alpha, \widetilde{H^i(\phi, \lambda)}) \to 0. \qquad (\kappa)$$

Note that we are using our new notation (3). Since we know the cohomology of the base space by (4), inductively, we can determine the cohomology of the fibre space; hence we can compute the cohomology of the total space using the above short exact sequence. More precisely, suppose we are to compute cohomology modules of $X(w)$. Let $w = s_{\alpha_1}s_{\alpha_2}, \ldots, s_{\alpha_r}$ be a reduced decomposition. Since we know the cohomology modules of $X(s_{\alpha_r})$ and $X(s_{\alpha_{r-1}})$ we can compute the cohomology modules of $X(s_{\alpha_{r-1}}s_{\alpha_r})$ using the short exact sequence with $w = s_{\alpha_{r-1}}s_{\alpha_r}$. Now we proceed with $w = s_{\alpha_{r-2}}s_{\alpha_{r-1}}s_{\alpha_r}$ and $\phi = s_{\alpha_{r-1}}s_{\alpha_r}$ in the short exact sequence (κ).

3. Genericity conditions on weights

Let G be a simple, simply connected algebraic group of rank 2. Then it will be one of the following type; A_2, B_2, G_2. Fix a maximal torus T. Then the weights, roots, etc are as defined in §1. Let α, β denote the simple roots and ω_α, ω_β denote the fundamental weights.

$$W \cdot X(T)^+ = \{\lambda \in X(T) | w \cdot \lambda \in X(T)^+, \quad \text{for some } w \in W\}.$$

For $w \in W$ we define 'w-chamber' as the set $\{\lambda \in X(T) \mid w \cdot \lambda \in X(T)^+\} \subset W \cdot X(T)$.

Clearly $W \cdot X(T)^+ = \bigcup_{w \in W} \{w\text{-chamber}\}$. We now write conditions on n and m for $\lambda = nw_\alpha + mw_\beta$ to be in a w-chamber. We will examine vanishing and non-vanishing of cohomology modules in a chamberwise manner, i.e. for $\lambda \in W \cdot X(T)^+$, and $w \in W$ we would like to make statements about vanishing and non-vanishing of cohomology modules $H^i(w, \lambda)$ only using the datum of chambers to which λ belongs. To have a consistent set of statements, we may need to omit certain hyperplanes from these chambers. We summarise all these computations in the following tables.

Table for A_2.

Weights	Identity	s_α	s_β	$s_\alpha s_\beta$	$s_\beta s_\alpha$	$s_\alpha s_\beta s_\alpha$
$\lambda = nw_\alpha + mw_\beta$	$n \geq 0$	$n \leq -2$	$n \geq -m-1$	$n \geq 0$	$n \leq -m-3$	$n \leq -2$
	$m \geq 0$	$m \geq -n-1$	$m \leq -2$	$m \leq -n-3$	$m \geq 0$	$m \leq -2$
†		$m = -n-1$	$n = -m-1$	$n = 0$	$m = 0$	

† = omitting the hyperplane from the chamber to make a general statement about vanishing and non-vanishing of cohomology of Schubert varieties with respect to Weyl chamber.

The table can be read as follows:

(i) $\lambda = n\omega_\alpha + m\omega_\beta$ belongs to identity chamber if $n \geq 0$ and $m \geq 0$. We need not omit any weights in this chamber to make general statements about vanishing and non-vanishing of cohomology modules.

(ii) $\lambda = n\omega_\alpha + m\omega_\beta$ belongs to s_α chamber if $n \leq -2$ and $m \geq -n-1$. To make general statements about vanishing and non-vanishing of cohomology modules we need to omit the hyperplane given by $m = -n-1$ and so on.

We make similar tables for B_2 and G_2, and because of space constraints, we omit the first column, i.e the weight column. In the following tables $\lambda = n\omega_\alpha + m\omega_\beta$ (we are using same notation for simple roots (fundamental weights) for A_2, B_2 and G_2).

Table for B_2. (β is the short root.)

Identity	s_α	s_β	$s_\alpha s_\beta$	$s_\beta s_\alpha$	$s_\alpha s_\beta s_\alpha$	$s_\beta s_\alpha s_\beta$	$s_\alpha s_\beta s_\alpha s_\beta$
$n \geq 0$	$n \leq -2$	$m \leq -2$	$n+m \leq -3$	$n+m \geq -1$	$n+m \leq -3$	$n \geq 0$	$n \leq -2$
$m \geq 0$	$m+2n \geq -2$	$m+n \geq -1$	$2n+m \geq -2$	$2n+m \leq -4$	$m \geq 0$	$2n+m \leq -4$	$m \leq -2$
	$m+2n = -2$	$m = -2$	$2n+m = -1$	$n+m = -1$	$m = 0$	$n = 0$	
	$m+2n = -1$		$2n+m = -2$	$2n+m = -4$	$m = 1$		

Table for G_2. (α is the short root.)

Identity	s_α	s_β	$s_\alpha s_\beta$	$s_\beta s_\alpha$	$s_\alpha s_\beta s_\alpha$
$n \geq 0$	$n \leq -2$	$n+3m \geq -3$	$n+3m \leq -5$	$2n+3m \geq -4$	$2n+3m \leq -6$
$m \geq 0$	$m+n \geq -1$	$m \leq -2$	$n+2m \geq -2$	$n+m \leq -3$	$n+2m \geq -2$
	$m+n = -1$	$n+3m = p$	$n+2m = -1$	$2n+3m = p$	$n+2m = -1$
		$p = -1, -2, -3$	$n+2m = -2$	$p = -1, -2, -3, -4$	$n+2m = -2$

$s_\beta s_\alpha s_\beta$	$s_\alpha s_\beta s_\alpha s_\beta$	$s_\beta s_\alpha s_\beta s_\alpha$	$s_\alpha s_\beta s_\alpha s_\beta s_\alpha$	$s_\beta s_\alpha s_\beta s_\alpha s_\beta$	$s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta$
$2n+3m \geq -4$	$2n+3m \leq -6$	$n+3m \geq -3$	$n+3m \leq -5$	$n \geq 0$	$n \leq -2$
$n+2m \leq -4$	$m+n \geq -1$	$n+2m \leq -4$	$m \geq 0$	$n+m \leq -3$	$m \leq -2$
$2n+3m = p$	$n+m = 0$	$n+3m = p$	$m = 0$	$n = p$	
$p = -1, -2, -3, -4$	$n+m = -1$	$p = 0, -1, -2, -3$		$p = 0, 1, 2$	

Remark. Sometimes it is convenient to have figures for Weyl chambers. We depict the above tables pictorially. In the above table we have considered the 'Weyl chambers' under 'dot'-action. In the following figures (figures 1–3) we draw the usual Weyl chamber with forbidden lines. As we stated in the introduction, the difference between the usual Weyl group action and the 'dot'-action is that the 'singular weights' cannot be moved to a dominant weight under dot-action but can be moved under usual action. For the convenience of the reader we have drawn the singular lines also.

Since there are too many forbidden lines in G_2, we have drawn two figures: one with forbidden lines and another one with singular lines.

4. Computations of cohomology

Since the computations are almost similar for all Schubert varieties and for all chambers, we do not write down all the computations. We give computation for some chambers in A_2 and G_2 (note that A_2 is simply laced but B_2 and G_2 are not) to illustrate how the computations are carried out. The final pictures give vanishing and non-vanishing of all cohomology modules for all Schubert varieties and for all chambers.

4.1 *A_2 type*

Even though in the simply laced types (in particular in A_n type) $\alpha^\vee = \alpha$, to avoid confusion we continue to write $\alpha^\vee$.

Let $\lambda = n\omega_\alpha + m\omega_\beta \in s_\alpha s_\beta$-chamber. Then $n \geq 0$, and $m \leq -n-3$.

$\mathbf{Z}(\mathbf{s}_\alpha)$: Since $(\lambda, \alpha^\vee) = n \geq 0$, by (2.4) $H^0(\alpha, \lambda) \neq 0$ and $H^r(\alpha, \lambda) = 0$, $\forall\, r \geq 1$.

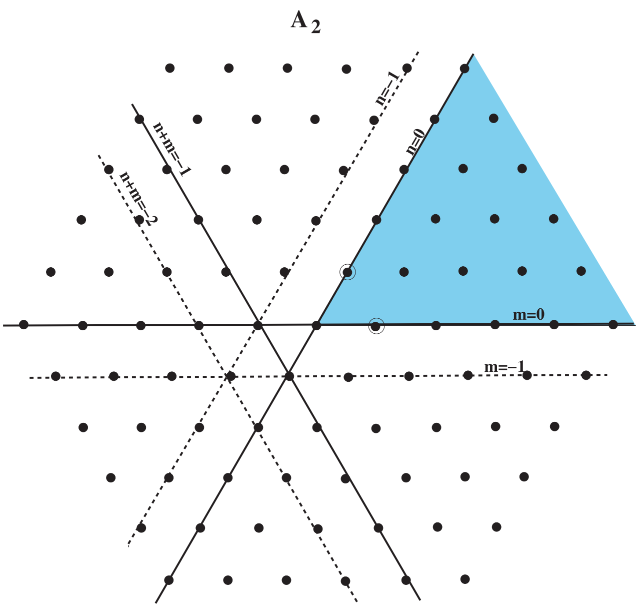


Figure 1. Pictorial representation of A_2. Thick lines stand for forbidden lines, dashed lines stand for singular lines, and the shaded region stands for the dominant chamber.

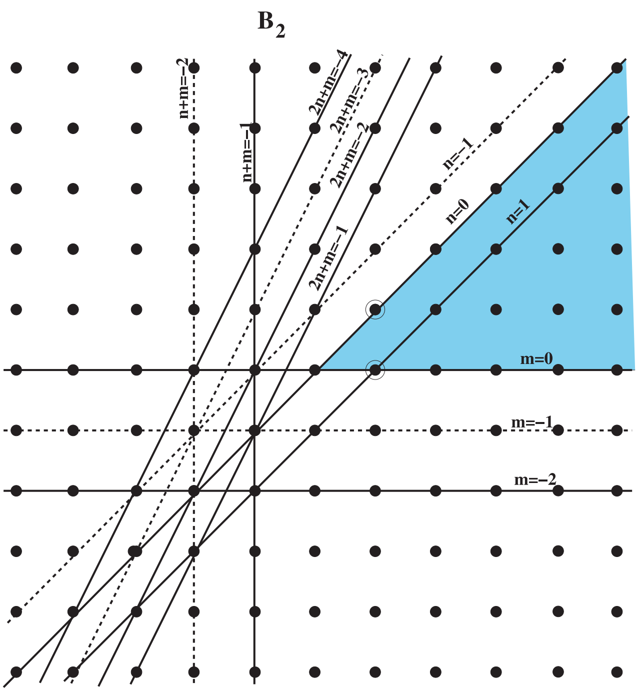


Figure 2. Pictorial representation of B_2. Thick lines stand for forbidden lines, dashed lines stand for singular lines, and the shaded region stands for the dominant chamber.

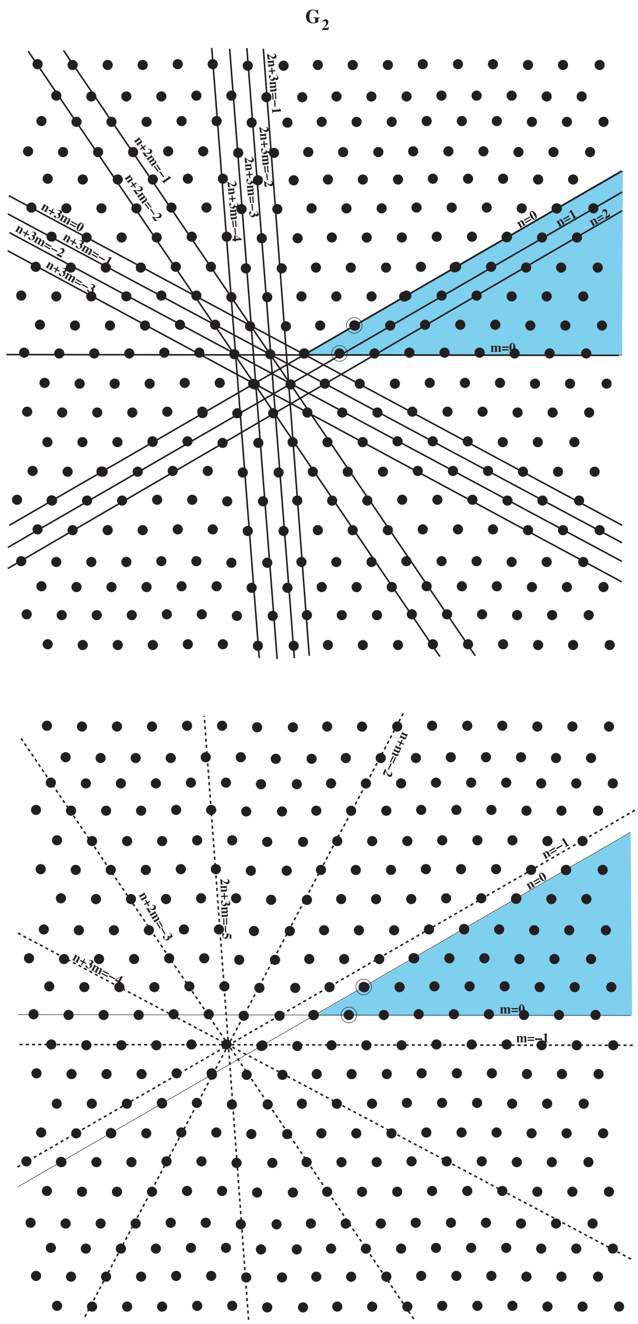


Figure 3. Pictorial representation of G_2. Thick lines stand for forbidden lines, dashed lines stand for singular lines, and the shaded region stands for the dominant chamber.

$\mathbf{Z}(\mathbf{s}_\beta)$: Since $(\lambda, \beta^\vee) = m \le -2$, by (2.4) $H^1(\beta, \lambda) \neq 0$ and $H^r(\beta, \lambda) = 0,\ \forall\, r,\ r \neq 1$.

$\mathbf{Z}(\mathbf{s}_\alpha \mathbf{s}_\beta)$: Since $H^0(\beta, \lambda) = 0$, we have $H^0(\alpha\beta, \lambda) \simeq H^0(\alpha, H^0(\beta, \lambda)) = H^0(\alpha, 0) = 0$. From ($\kappa$), we have $H^1(\alpha\beta, \lambda) \simeq H^0(\alpha, H^1(\beta, \lambda))$, because $H^0(\beta, \lambda) = 0$. By (2.4) $H^1(\beta, \lambda)$ has basis $u_0, u_1, \dots, u_{-m-2}$ such that the weight of $u_j = \lambda - (m+1+j)\beta,\ 0 \le j \le -m-2$.

Recall that $H^1(\beta, \lambda)$ is a P_β-module, and $B_\alpha \subset P_\beta$. Think of $H^1(\beta, \lambda)$ as a B_α-module. Then $H^1(\beta, \lambda)$ decomposes into one-dimensional B_α-submodules.

$$H^1(\alpha\beta, \lambda) = \bigoplus_{j=0}^{-m-2} H^0(\alpha, \lambda - (m+1+j)\beta) \text{ as } B_\alpha\text{-module}$$

$$(\lambda - (m+1+j)\beta, \alpha^\vee) = n + m + 1 + j$$

$$= \begin{cases} \le -2, & \text{if } 0 \le j \le -m-n-3 \\ = -1, & \text{if } j = -n-m-2 \\ \ge 0, & \text{if } -m-n-1 \le j \le -m-2. \text{ Assume } n > 0. \end{cases}$$

Remark. Here we could see why we need to avoid certain hyperplanes to make general statements about vanishing and non-vanishing. We need to know whether $n-1 \ge 0$ or $n-1 \le -2$ or $n-1 = -1$. Since $\lambda \in s_\alpha s_\beta$-chamber, it follows that $n \ge 0$. It is clear that $n-1 \ge 0,\ \forall\, n$ except $n = 0$. We want to avoid such odd cases. The vanishing and non-vanishing pattern of such λ could be the same as that of any other 'generic' λ, but for computational purposes we omit them.

Now by (2.4) and (κ) we have $H^1(\alpha\beta, \lambda) \neq 0$, $H^2(\alpha\beta, \lambda) \simeq H^1(\alpha, H^1(\beta, \lambda)) \neq 0$, and $H^r(\alpha\beta, \lambda) = 0,\ \forall\, r \ge 3$.

$\mathbf{Z}(\mathbf{s}_\beta \mathbf{s}_\alpha)$: Here $H^0(\beta\alpha, \lambda) \simeq H^0(\beta, H^0(\alpha, \lambda))$. By (2.4) $H^0(\alpha, \lambda)$ has a basis $v_0, v_1, \dots, v_n$ such that the weight of $v_i = \lambda - i\alpha,\ 0 \le i \le n$. $(\lambda - i\alpha, \beta) = m + i \le -3,\ 0 \le i \le n \Longrightarrow$ (by (2.4)) $H^0(\beta\alpha, \lambda) = 0$, and $H^1(\beta\alpha, \lambda) \simeq H^1(\beta, H^0(\alpha, \lambda)) \neq 0$. From ($\kappa$), we have $H^r(\beta\alpha, \lambda) = 0,\ \forall\, r \ge 2$.

$\mathbf{Z}(\mathbf{s}_\alpha \mathbf{s}_\beta \mathbf{s}_\alpha)$: We have $H^0(\alpha\beta\alpha, \lambda) \simeq H^0(\alpha, H^0(\beta\alpha, \lambda)) = H^0(\alpha, 0) = 0$, and $H^1(\alpha\beta\alpha, \lambda) \simeq H^0(\alpha, H^1(\beta\alpha, \lambda))$. We decompose the P_β-module $H^1(\beta\alpha, \lambda)$ as indecomposable B_α-modules. Let us write the weight diagram of the P_β-module $H^1(\beta\alpha, \lambda)$;

$$\begin{array}{llllllll}
\lambda - (m+1)\beta & \lambda - (m+2)\beta & \dots & \lambda - (m+n+1)\beta & \dots & \lambda - (-1)\beta \\
 & \lambda - \alpha - (m+2)\beta & \dots & \lambda - \alpha - (m+n+1)\beta & \dots & \lambda - \alpha - (-1)\beta \\
 & & & \cdot & & \cdot \\
 & & & \cdot & & \cdot \\
 & & & \cdot & & \cdot \\
 & & & \lambda - n\alpha - (m+n+1)\beta & \dots & \lambda - n\alpha - (-1)\beta
\end{array}$$

We will encounter diagrams of this nature quite often in our computation. So we study this diagram carefully and pin-point some facts which will be used frequently. The entries of this diagram are weights of a B_γ-module, γ a simple root (here it is the P_β-module $H^1(\beta\alpha, \lambda)$).

Corresponding to each weight there is a unique weight vector of that weight. Let Y_α denote the Chevalley operator for B_α. Recall that if v is a weight vector of weight λ then

$Y_\alpha v$ is either zero or a weight vector of weight $\lambda - \alpha$. This suggests that the weight vectors corresponding to the weights of a particular column is a B_α-module. We term these as *column modules*.

The diagram is drawn in such a way that entries of each column are weights of a B_α-submodule, namely, the column module defined above.

Claim. The column modules are indecomposable B_α-modules.

Proof of the Claim. We consider the following B-isomorphism given by Serre duality.

$$H^1(\beta, \widetilde{H^0(\alpha, \lambda)})^* \simeq H^0(\beta, \widetilde{H^0(\alpha, \lambda)^* \otimes L_{-\beta}}).$$

(We remark that the Serre duality isomorphism in general can have an ambiguity of a twist by a character (cf. [1], Remark 3.4; [4] §3.9). Since the weights of these two modules are the same, there cannot be any non-trivial twist by a character.)

Now let us write the weight diagram of this dual module:

$$\begin{array}{llllll}
-\lambda + n\alpha - \beta & \cdots & -\lambda + n\alpha + (m+n+1)\beta & & & \\
-\lambda + (n-1)\alpha - \beta & \cdots & -\lambda + (n-1)\alpha + (m+n+1)\beta & -\lambda + (n-1)\alpha + (m+n)\beta & & \\
\cdot & \cdots & \cdot & \cdot & & \\
\cdot & \cdots & \cdot & \cdot & & \\
\cdot & \cdots & \cdot & \cdot & & \\
-\lambda - \beta & \cdots & -\lambda + (m+n+1)\beta & -\lambda + (m+n)\beta & \cdots & -\lambda + (m+1)\beta
\end{array}$$

Let $\mathfrak{e}$ denote the evaluation map at the identity: $H^0(\beta, \widetilde{H^0(\alpha, \lambda)^* \otimes L_{-\beta}}) \longrightarrow H^0(\alpha, \lambda)^* \otimes k_{-\beta}$. Since $\mathfrak{e}$ is a B-map and $H^0(\alpha, \lambda)^* \otimes k_{-\beta}$ is a cyclic B_α-module, the first column module of the weight diagram of the dual module which is mapped isomorphically onto the image is also cyclic B_α-module and hence indecomposable. We have proved that the left most column module is an indecomposable B_α-module. To prove that the other column modules are indecomposable we use induction, i.e. we prove that if the ith column module is indecomposable then the $(i+1)$th column module is also indecomposable.

Let X_α, Y_α denote the Chevalley basis. Let v_{ij} denote the weight vectors of the respective weights in the dual weight diagram (we use matrix notation for the weight diagram). Our claim is that for a fixed j_0, the B_α-module generated by $\{v_{ij_0}$, as i varies$\}$ is indecomposable. For that we prove that $Y_\alpha v_{ij_0} \neq 0$, for all $i, i \neq n+1$; note that the $v_{i_{n+1}j}$ is the last weight in any column in our diagram. We have proved that the first column module is an indecomposable B_α-module. Assume the claim for the column j_0. We prove the claim for the column $j_0 + 1$. Suppose $Y_\alpha v_{i(j_0+1)} = 0$. We know that the 'row-modules' are $SL(2, \beta)$-modules. Now $X_\beta Y_\alpha v_{i(j_0+1)} = Y_\alpha X_\beta v_{i(j_0+1)} = Y_\alpha v_{ij_0} = 0$ which contradicts our assumption that the j_0th column module is indecomposable, if $i \neq n+1$. Now the claim follows from the following simple facts: (i) If V is indecomposable, then V^* is indecomposable. (ii) If $V \simeq \oplus_i V_i$, then $V^* \simeq \oplus_i V_i^*$.

Now we go back to the weight diagram of $H^1(\beta\alpha, \lambda)$. We denote the column modules corresponding to the first $n+1$ columns by V_i, $1 \leq i \leq n+1$, and the next $(-m-n-2)$ column modules by U_j, $n+2 \leq j \leq -m-1$. The highest weight of $V_i = \lambda - (m+i)\beta$, and the dimension of $V_i = i$. The highest weight of $U_j = \lambda - (m+j)\beta$ and its dimension is $n+1$ $\forall j$. The module V_i, $1 \leq i \leq n+1$, and U_j, $n+2 \leq j \leq -m-1$ are all indecomposable B_α-modules.

One knows that in general, the indecomposable B_α-modules are of the form $W \otimes \chi$, where W is an irreducible L_α module and χ is a character (cf. §9.3). Since we are in the rank 2 case, for our column modules we can write explicitly the χ's.

Let $\chi = -\lambda + (m+1)\beta$, and $\psi_j = (m+j)\beta$, $n+2 \leq j \leq -m-1$. Then $V_i \otimes \chi$, $2 \leq i \leq n+1$ and $U_j \otimes \psi_j$, $n+2 \leq j \leq -m-1$ are irreducible $SL_2(\alpha)$-module.

$$\begin{aligned}
&H^1(\alpha\beta\alpha, \lambda)\\
&\quad \simeq H^0(\alpha, \widetilde{H^1(\beta\alpha, \lambda)})\\
&\quad = H^0\left(\alpha, \bigoplus_{i=1}^{n+1} \mathcal{V}_i \oplus \bigoplus_{j=n+2}^{-m-1} \mathcal{U}_j\right)\\
&\quad \simeq H^0\left(\alpha, \bigoplus_{i=2}^{n+1} [(\mathcal{V}_i \otimes \chi) \otimes -\chi]\right) \oplus \bigoplus_{j=n+2}^{-m-1} H^0(\alpha, [(\mathcal{U}_j \otimes \psi_j) \otimes -\psi_j] \oplus \mathcal{V}_1)\\
&\quad \simeq \bigoplus_{i=2}^{n+1} \{[\mathcal{V}_i \otimes \chi] \otimes H^0(\alpha, -\chi)\} \oplus \bigoplus_{j=n+2}^{-m-1} \{[\mathcal{U}_j \otimes \psi_j] \otimes H^0(\alpha, -\psi_j)\} \oplus H^0(\alpha, \mathcal{V}_1)
\end{aligned}$$

(By the generalized tensor identity) ([3], I.4.8).

Now $(-\chi, \alpha^\vee) = (\lambda - (m+1)\beta, \alpha) = n + m + 1 \leq -2$, $\forall i$, $2 \leq i \leq n+1$ and $(-\psi_j, \alpha^\vee) = (-(m+j)\beta, \alpha) = m + j \leq -1$, $\forall j$, $n+2 \leq j \leq -m-1$. Note that the weight of $-\chi$ is the weight of V_1 also. Thus by Proposition 2.4, we have $H^1(\alpha\beta\alpha, \lambda) = 0$.

From (κ), we have $H^2(\alpha\beta\alpha, \lambda) \simeq H^1(\alpha, \widetilde{H^1(\beta\alpha, \lambda)})$. As before,

$$\begin{aligned}
H^2(\alpha\beta\alpha, \lambda) \simeq &\bigoplus_{i=2}^{n+1} \{[\mathcal{V}_i \otimes \chi] \otimes H^1(\alpha, -\chi)\}\\
&\oplus \bigoplus_{j=n+2}^{-m-1} \{[\mathcal{U}_j \otimes \psi_j] \otimes H^1(\alpha, -\psi_j)\} \oplus H^1(\alpha, \mathcal{V}_1).
\end{aligned}$$

Since $(-\chi, \alpha^\vee) \leq -2$, $2 \leq i \leq n+1$ and $(-\psi_j, \alpha^\vee) \leq -1$, $n+2 \leq j \leq -m-1$, by Proposition 2.4, $H^2(\alpha\beta\alpha, \lambda) \neq 0$.

Since $H^r(\beta\alpha, \lambda) = 0$, $\forall\, r \geq 2$ by (κ), we get $H^r(\alpha\beta\alpha, \lambda) = 0$, $\forall r$, $r \geq 3$.

4.2 *G_2 type*

Let α, β be simple roots with $(\alpha, \beta^\vee) = -1$ and $(\beta, \alpha^\vee) = -3$.

Let $\lambda = n\omega_\alpha + m\omega_\beta \in s_\alpha$-chamber. Then $n \leq -2$ and $n + m \geq -1$. We assume that λ does not lie in the hyperplane $n + m = -1$.

We examine the cohomology modules for the Schubert variety $Z(s_\alpha s_\beta s_\alpha s_\beta)$.

$\mathbf{Z}(\mathbf{s}_\beta)$: Since $(\lambda, \beta^\vee) = m \geq 0$, by Proposition 2.4, $H^0(\beta, \lambda) \neq 0$ and $H^r(\beta, \lambda) = 0$, $\forall r \geq 1$.

$\mathbf{Z}(\mathbf{s}_\alpha \mathbf{s}_\beta)$: From the short exact sequence (κ), we have $H^0(\alpha\beta, \lambda) \simeq H^0(\alpha, \widetilde{H^0(\beta, \lambda)})$. By Proposition 2.4 the P_β module $H^0(\beta, \lambda)$ has basis $v_0, v_1, \ldots, v_m$ and the weight of $v_i = \lambda - i\beta$, $0 \leq i \leq m$. If we think of $H^0(\beta, \lambda)$ as a B_α-module then $H^0(\beta, \lambda)$ decomposes into one-dimensional B_α-submodules.

$$H^0(\alpha\beta, \lambda) = \bigoplus_i H^0(\alpha, \lambda - i\beta) \text{ as } B_\alpha\text{-module}$$

$$(\lambda - i\beta, \alpha^\vee) = n + 3i = \begin{cases} \leq -2, & \text{if } 0 \leq i \leq -\frac{n}{3} - 1 \\ \geq 0, & \text{if } -\frac{n}{3} \leq i \leq m \end{cases}.$$

We assume that 3 divides $-n$. Otherwise 3 must divide either $-n-1$ or $-n-2$. If 3 divides $-n-1$, then we will have

$$(\lambda - i\beta, \alpha^\vee) = n + 3i = \begin{cases} \leq -2, & \text{if } 0 \leq i \leq -\frac{n+4}{3} \\ = -1, & \text{if } i = \frac{-n-1}{3} \\ \geq 0, & \text{if } \frac{-n+2}{3} \leq i \leq m \end{cases}.$$

If 3 divides $-n-2$, then we will have

$$(\lambda - i\beta, \alpha^\vee) = n + 3i = \begin{cases} \leq -2, & \text{if } 0 \leq i \leq -\frac{n+2}{3} \\ \geq 0, & \text{if } \frac{-n+1}{3} \leq i \leq m \end{cases}.$$

In any case, $H^0(\alpha\beta, \lambda) \simeq H^0(\alpha, \widetilde{H^0(\beta, \lambda)}) \simeq \bigoplus_{i=0}^m H^0(\alpha, \lambda - i\beta) \supset H^0(\alpha, \lambda - m\beta) \neq 0$. Similarly $H^1(\alpha\beta, \lambda) \simeq H^1(\alpha, \widetilde{H^0(\beta, \lambda)}) \simeq \bigoplus_{i=0}^m H^1(\alpha, \lambda - i\beta) \supset H^1(\alpha, \lambda) \neq 0$. Thus we have both $H^0(s_\alpha s_\beta, \lambda)$ and $H^1(s_\alpha s_\beta, \lambda)$ are non-zero.

Remark. Note that the weights of certain cohomology modules will change depending on whether 3 divides $-n$ or $-n-1$ or $-n-2$. But vanishing and non-vanishing of the cohomology module will not be affected. So we assume that 3 divides $-n$.

Let M denote the P_α-module $H^0(\alpha, \lambda) = \langle v_0, \ v_1, \ldots, v_n \rangle$, M_1 denote the B-submodule $\langle v_{-n/3}, \ldots, v_m \rangle$ and N denote the quotient module M/M_1.

$\mathbf{Z}(\mathbf{s}_\beta\mathbf{s}_\alpha\mathbf{s}_\beta)$: We write the weight diagram of the P_α-module $H^1(\alpha\beta, \lambda)$:

$$H^1(\alpha\beta, \lambda) \simeq H^1(\alpha, \widetilde{H^0(\beta, \lambda)}) \simeq \bigoplus_{i=0}^{\frac{-n}{3}-1} H^1(\alpha, \lambda - i\beta).$$

$$\begin{array}{llllll} \lambda-(n+1)\alpha & \lambda-(n+2)\alpha & \lambda-(n+3)\alpha & \lambda-(n+4)\alpha & \cdots\ \lambda+2\alpha & \lambda+\alpha \\ & & & \lambda-\beta-(n+4)\alpha & \cdots\ \lambda-\beta+2\alpha & \lambda-\beta+\alpha \\ & & & & \quad\ \vdots & \vdots \\ & & & & \quad\ \lambda-(\frac{-n}{3}-1)\beta+2\alpha & \lambda-(\frac{-n}{3}-1)\beta+\alpha \end{array}$$

First we prove that the column modules are indecomposable B_β-modules.

We have $H^1(\alpha\beta, \lambda) \simeq H^1(\alpha, \widetilde{H^0(\beta, \lambda)}) \simeq H^1(\alpha, \mathcal{N})$.

Now $H^1(\alpha, \widetilde{H^0(\beta, \lambda)})^* \simeq H^1(\alpha, \mathcal{N})^* \simeq H^0(\alpha, \mathcal{N}^* \otimes L_{-\alpha})$.

Let $\mathfrak{e}$ denote the evaluation map at the identity: $H^0(\alpha, \mathcal{N}^* \otimes L_{-\alpha}) \longrightarrow N^* \otimes k_{-\alpha}$. Note that $\mathfrak{e}$ is a B-map and $N^* \otimes k_{-\alpha}$ is a cyclic B_α-module. Now the rest of the proof is similar to that of §4.1.

As we explained earlier each column is a B_β-module. We denote the column modules by V_i, $1 \leq i \leq -n-1$. Let $\mathcal{V}_i$ be the associated vector bundle of V_i,

$$H^1(\beta\alpha\beta, \lambda) \simeq \bigoplus_{i=1}^{-n-1} H^1(\beta, \mathcal{V}_i).$$

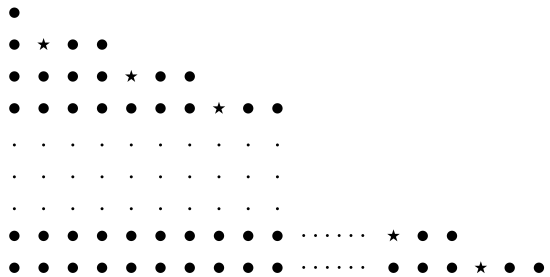


Figure 4. Pictorial representation of the weights of the B-module $H^0(\alpha\beta, \lambda)$.

Let μ_i be the highest weight of V_i and d_i be the dimension of V_i. As we explained earlier, the cohomology $H^1(\beta, \mathcal{V}_i)$ is zero or non-zero depends on whether $(\mu_i, \beta^\vee) + 1 - d_i \geq 0$ or not. (If $(\mu_i, \beta^\vee) + 1 - d_i \leq -2$ then $H^2(\beta, \mathcal{V}_i) \neq 0$. If $(\mu_i, \beta^\vee) + 1 - d_i = -1$ then $H^k(\beta, \mathcal{V}_i) = 0,\ \forall\, k$.)

Now $(\mu_1, \beta^\vee) + 1 - d_1 = (\lambda - (n+1)\alpha, \beta^\vee) + 1 - 1 = m + n + 1 \geq 0$. Therefore $H^1(\beta, \mathcal{V}_1) \neq 0$. Note that$(\mu_i, \beta^\vee)$ increases by 1 when i increases but the dimension of V_i increases either by 0 or by 1, hence $(\mu_i, \beta^\vee) + 1 - d_i \geq 0,\ \forall\, i$.

This implies that the cohomology module $H^1(\beta\alpha\beta, \lambda) \neq 0$.

Now we examine $H^0(\beta\alpha\beta, \lambda)$. We have $H^0(\beta\alpha\beta, \lambda) \simeq H^0(\beta, \widetilde{H^0(\alpha\beta, \lambda)})$. We decompose the P_α-module $H^0(\alpha\beta, \lambda)$ into indecomposable B_β-modules. Let us write the weight diagram of the B-module $H^0(\alpha\beta, \lambda)$. Note that

$$H^0(\alpha\beta, \lambda) = \bigoplus_{i=0}^{m} H^0(\alpha, \lambda - i\beta) = \bigoplus_{i=-\frac{n}{3}}^{m} H^0(\alpha, \lambda - i\beta).$$

$$\begin{array}{llllll}
\lambda + \frac{n}{3}\beta & & & & & \\
\lambda + (\frac{n}{3} - 1)\beta & \lambda + (\frac{n}{3} - 1)\beta - \alpha & \lambda + (\frac{n}{3} - 1)\beta - 2\alpha & \lambda + (\frac{n}{3} - 1)\beta - 3\alpha & & \\
\cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & & \\
\lambda - m\beta & \lambda - m\beta - \alpha & \lambda - m\beta - 2\alpha & \lambda - m\beta - 3\alpha & \cdots & \lambda - m\beta - (n+3m)\alpha
\end{array}$$

To prove that column modules are indecomposable B_β-modules, consider $H^0(\alpha\beta, \lambda) = H^0(\alpha, \mathcal{M}_1) \xrightarrow{\mathfrak{e}} M_1$. The rest of the proof is the same as that of §4.1.

The pictorial representation of the above weight diagram is as shown in figure 4.

In figure 4, $\star$ does not have any special meaning. For the computational purpose we want to consider the column modules starting with $\star$.

Let $U_j,\ 0 \leq j \leq n + 3m$ denote the column modules and ψ_j denote their highest weight. Let d_j denote the dimension of U_j. Let $\mathcal{U}_j$ be the associated bundle of U_j,

$$H^0(\beta\alpha\beta, \lambda) \simeq \bigoplus_{j=0}^{n+3m} H^0(\beta, \mathcal{U}_j).$$

The cohomology module $H^0(\beta, \mathcal{U}_j)$ is zero or non-zero depending on whether $(\psi_j, \beta^\vee) + 1 - d_j \geq 0$ or not. Now $(\psi_{n+3m}, \beta^\vee) + 1 - d_{n+3m} = (\lambda - m\beta - (n +$

$3m)\alpha, \beta^\vee) + 1 - 1 = m - 2m + n + 3m = n + 2m \geq 0$. Therefore $H^0(\beta, \mathcal{U}_{n+3m}) \neq 0$, which implies that $H^0(\beta\alpha\beta, \lambda) \neq 0$.

We want to determine the U_j's which contribute to H^0, i.e. $H^0(\beta, \mathcal{U}_j) \neq 0$. It depends on whether $(\psi_j, \beta^\vee) + 1 - d_j \geq 0$ or not. We need this information to compute $H^0(\alpha\beta\alpha\beta, \lambda)$. Now $(\psi_0, \beta^\vee) + 1 - d_0 = m + (2n/3) + 1 - (m + (n/3) + 1) = \frac{n}{3}$. Suppose $n = -3$, then $n/3 = -1$. In this case $(\psi_j, \beta^\vee) + 1 - d_j \geq 0,\ \forall\, j > 0$, so all the column modules, except the first one will contribute to H^0. We consider the other case i.e. $n \ll 0$.

The column modules starting with $\star$, i.e. $U_{3i-2},\ 1 \leq i \leq m + (n/3)$, have $\lambda + ((n/3) - i)\beta - (3i - 2)\alpha$ as their highest weight, and dimension $m + (n/3) + 1 - i$. Now $(\lambda + ((n/3) - i)\beta - (3i - 2)\alpha, \beta^\vee) = m + (2n/3) - 2i + 3i - 2 = m + (2n/3) + i - 2$. Now we do the desired computation (highest weight, $\beta^\vee$) + 1-dimension $= m + (2n/3) + i - 2 - (m + (n/3) - i) = (n/3) - 2 + 2j$. Without loss of generality, we may assume 6 divides n. Observe that the computation (highest weight, $\beta^\vee$) + 1-dimension is the same for the $\star$ column module and the previous column module. Now it is clear that the column module with highest weight $\lambda + (n/2)\beta + (n/2)\alpha$ onwards (including this) contributes to H^0, i.e. the modules $H^0(\beta, U_j) \neq 0, \forall j,\ (-n/6) \leq j \leq n + 3m$. Therefore the weight diagram of the column modules of $H^0(\alpha\beta, \lambda)$ contributing to $H^0(\beta\alpha\beta, \lambda)$ is:

$$\begin{array}{lllll}
\lambda + \frac{n}{2}\beta + \frac{n}{2}\alpha & & & & \\
\lambda + (\frac{n}{2} - 1)\beta + \frac{n}{2}\alpha & \lambda + (\frac{n}{2} - 1)\beta + (\frac{n}{2} - 1)\alpha & \lambda + (\frac{n}{2} - 1)\beta + (\frac{n}{2} - 2)\alpha & \lambda + (\frac{n}{2} - 1)\beta + (\frac{n}{2} - 3)\alpha & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\lambda - m\beta + \frac{n}{2}\alpha & \lambda - m\beta + (\frac{n}{2} - 1)\alpha & \lambda - m\beta + (\frac{n}{2} - 2)\alpha & \lambda - m\beta + (\frac{n}{2} - 3)\alpha & \cdots\ \lambda - m\beta - (n + 3m)\alpha
\end{array}$$

The shape of the weight diagram of $H^0(\beta\alpha\beta, \lambda)$ will be as shown in figure 5 (we omit the entries except the last one).

The last column module is one-dimensional and its weight is $\mu = \lambda - (n + 3m)\beta - (n + 3m)\alpha$. $(\lambda - (n + 3m)\beta - (n + 3m)\alpha, \alpha^\vee) = 2n + 3m \geq 0$, which implies that $H^0(\alpha\beta\alpha\beta, \lambda) \neq 0$.

Now we look at the weights of $H^1(\beta\alpha\beta, \lambda)$. From ($\kappa$), we have

$$0 \longrightarrow H^1(\beta, \widetilde{H^0(\alpha\beta, \lambda)}) \longrightarrow H^1(\beta\alpha\beta, \lambda) \longrightarrow H^0(\beta, \widetilde{H^1(\alpha\beta, \lambda)}) \longrightarrow 0.$$

Note that the above exact sequence is not a split exact sequence as a B-module exact sequence. If we think of this as a T-module then it splits. In particular we know all the

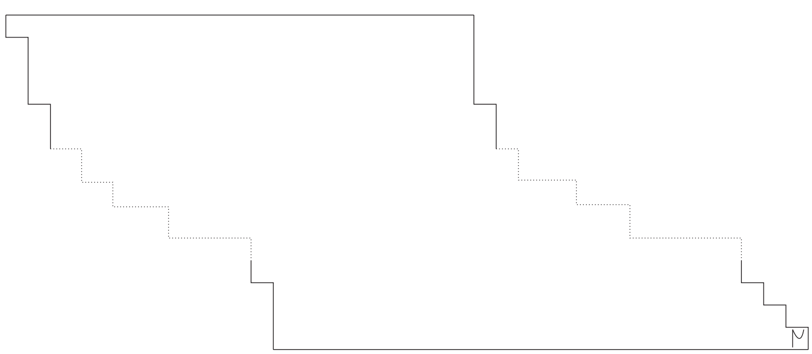


Figure 5. The shape of the weight diagram of $H^0(\beta\alpha\beta,\ \lambda)$.

weights of the B-module $H^1(\beta\alpha\beta, \lambda)$. If we look at its weights carefully, we will have a column B_α-module consisting of the weight vectors $\lambda-(n+1)\alpha, \lambda-(n+2)\alpha, \lambda-(n+3)\alpha, \ldots, \lambda-(-1)\alpha$. In fact it is an $SL_2(\alpha)$-module. Let us denote it by V. Since $H^0(\alpha, \mathcal{V}) \neq 0$ we have $H^0(\alpha, \widetilde{H^1(\beta\alpha\beta, \lambda)}) \neq 0$ hence $H^1(\alpha\beta\alpha\beta, \lambda) \neq 0$.

Thus we have proved that for $Z(\alpha\beta\alpha\beta)$ we have both $H^0(\alpha\beta\alpha\beta, \lambda) \neq 0$, and $H^1(\alpha\beta\alpha\beta, \lambda) \neq 0$. Other higher cohomology modules $H^r(\alpha\beta\alpha\beta, \lambda) = 0,\ \forall r,\ r \geq 2$ can be seen from the computation (i.e. by observing all the column modules of $H^1(\beta\alpha\beta, \lambda)$ contributes only to $H^0(\alpha,\ )$) or from the following result [1].

If λ belongs to w'-chamber and $l(w') = l_0$, then for any $w \in W$ the cohomology module $H^r(X(w), L_\lambda) = 0,\ \forall r,\ r > l_0$.

5. Remarks about the computation

For $\lambda \in W \cdot X(T)$ and $w \in W$, we are interested in knowing whether $H^k(w, \lambda)$ is zero or not. This is comparatively easier than actually having the full description of the cohomology module $H^k(w, \lambda)$. Here we highlight some points which we used to find out the vanishing and non-vanishing of cohomology modules.

(1) Suppose $w = s_\alpha w_1$ such that $l(w) = l(w_1) + 1$. We have

$$0 \longrightarrow H^1(\alpha, \widetilde{H^{i-1}(w_1, \lambda)}) \longrightarrow H^i(w, \lambda) \longrightarrow H^0(\alpha, \widetilde{H^i(w, \lambda)}) \longrightarrow 0.$$

From the above exact sequence $H^i(w, \lambda) \neq 0$ if $H^1(\alpha, \widetilde{H^{i-1}(w_1, \lambda)}) \neq 0$, or $H^0(\alpha, \widetilde{H^i(w, \lambda)}) \neq 0$.

(2) Let w be as in 1 above. We decompose the B-module $H^i(w_1, \lambda)$ into indecomposable B_α-submodules, V_i. If $H^1(\alpha, \mathcal{V}_i) \neq 0$, for some i, then $H^{i+1}(w, \lambda) \neq 0$. Similarly if, $H^0(\alpha, \mathcal{V}_i) \neq 0$, for some i, then $H^i(w, \lambda) \neq 0$.

(3) In the case of A_2 and B_2 we do not use any theorem other than the spectral sequence argument and Proposition 2.4. We can do the same thing for G_2 also. But it is tedious. If we use Demazure trick (given below) half of the work will be reduced. The following is from Demazure's work [2].

At this point we would like to mention that Demazure's work is for G/B only. Note that $G/B = X(w_0)$ and for any simple root α, w_0 has a representation $w_0 = w's_\alpha$. We use Demazure's trick for $X(\tau)$, $\tau \in W$, under some special condition, which will be clear below.

Let $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. If $(\lambda, \alpha^\vee) \geq 0$, then $H^i(\tau, \lambda) = H^{i+1}(\tau, s_\alpha \cdot \lambda)$ and if $(\lambda, \alpha^\vee) \leq -2$, then $H^i(\tau, \lambda) = H^{i-1}(\tau, s_\alpha \cdot \lambda)$.

6. Observations from the computations

We would like to mention that we have verified the following conjecture for A_2 and B_2 and certain cases in G_2.

(1) Let $l(s_\alpha w) = l(w) + 1$ and $H^i(w, \lambda) \neq 0$. We consider this B-module $H^i(w, \lambda)$ as a B_α-module and decompose into indecomposable B_α-modules, and then list them one after another so that the highest weights are in the decreasing order. Let us denote them

as $V_1, V_2, \ldots, V_r$. The observation is that there exists an i_0, $1 \leq i_0 \leq r$ such that $H^1(\alpha, \mathcal{V}_i) \neq 0$, $\forall i,\ i \leq i_0$, and $H^0(\alpha, \mathcal{V}_i) \neq 0$, $\forall i,\ i \geq i_0+1$ or, there exist a j_0, $1 \leq j_0 \leq r$ such that $H^1(\alpha, \mathcal{V}_j) \neq 0$, $\forall j,\ j \leq j_0-1$, and $H^0(\alpha, \mathcal{V}_j) \neq 0$, $\forall j,\ j \geq j_0+1$, and both $H^0(\alpha, \mathcal{V}_{j_0}) = 0$ and $H^1(\alpha, \mathcal{V}_{j_0}) = 0$.

(2) For $w \in W$ and $\lambda \in X(T)$, we define a positive integer n_0 to be an upper bound for the non-vanishing index of cohomology modules if $H^k(w, \lambda) = 0$ for all $k > n_0$. Similarly we define a lower bound. We explain an algorithm to find (inductively) these bounds.

Let G be any semisimple simply connected algebraic group, not necessarily rank two. Let $w = s_1 s_2 \ldots s_n$, where $s_i = s_{\alpha_i}$'s are simple reflections (not necessarily distinct) and $l(w) = n$. Let μ_1 be the highest weight and ψ_1 be the lowest weight of $H^r(s_n, \lambda) \neq 0$ ($r = 0$ or 1). For the Schubert variety $X(s_n)$ we have upper bound = lower bound = r. For the Schubert variety $X(s_{n-1}s_n)$ we compute the bounds as follows:

We compute the bounds by the following rules. Upper bound for $X(s_{n-1}s_n) = n_2 = r+1$, if $(\mu_1, \alpha_{n-1}^\vee) \leq -2$, otherwise r.

Lower bound for $X(s_{n-1}s_n) = m_2 = r+1$, if $(\psi_1, \alpha_{n-1}^\vee) \leq -2$, otherwise r.

Let us decompose the B-module $H^{n_2}(s_{n-1}s_n, \lambda)$ into indecomposable $B_{\alpha_{n-2}}$-module. Pick the set of highest weights of these $B_{\alpha_{n-2}}$-modules, and denote it by $\mathfrak{F}_{n-2}$. Let μ_2 be the highest weight in $\mathfrak{F}_{n-2}$ and a_2 be the dimension of the corresponding $B_{\alpha_{n-2}}$-module.

Let us decompose the B-module $H^{m_2}(s_{n-1}s_n, \lambda)$ into indecomposable $B_{\alpha_{n-2}}$-module. Pick the set of highest weights of these $B_{\alpha_{n-2}}$-modules, and denote it by $\mathfrak{K}_{n-2}$. Let ψ_2 be the lowest weight in $\mathfrak{K}_{n-2}$ and b_2 be the dimension of the corresponding $B_{\alpha_{n-2}}$-module.

Upper bound for $X(s_{n-2}s_{n-1}s_n) = n_3 = n_2 + 1$ if $(\mu_2, \alpha_{n-2}) + 1 - a_2 \leq -2$, otherwise $n_3 = n_2$. Lower bound for $X(s_{n-2}s_{n-1}s_n) = m_3 = m_2 + 1$ if $(\psi_2, \alpha_{n-2}) + 1 - b_2 \leq -2$, otherwise $m_3 = m_2$.

Proceeding this way we get the bounds for $X(s_1 s_2 \ldots s_n)$. There are some theorems regarding these bounds in [1].

7. Pictures of vanishing and non-vanishing of cohomology

We give explicitly the vanishing and non-vanishing of Schubert cohomology modules $H^i(X(w), L_\lambda)$ for all the rank two groups, and for all non-singular weights which are not forbidden. We give the complete results in the form of pictures (figures 6–8). Note that there are three types (Dynkin classification) of rank two groups, namely, A_2, B_2, and G_2. For each type, we have d pictures (or matrices), named $H^0, H^1, \ldots, H^d$, of size $r \times r$ consisting of dark and light boxes, where $d = \dim(G/B)$ and $r = \#W$, and columns and rows are indexed by identity, α, β, $\alpha\beta$, $\beta\alpha$, $\alpha\beta\alpha$, $\beta\alpha\beta$, $\alpha\beta\alpha\beta$, $\beta\alpha\beta\alpha$, $\alpha\beta\alpha\beta\alpha$, $\beta\alpha\beta\alpha\beta$, $\alpha\beta\alpha\beta\alpha\beta$. In columns they stand for identity, s_α, s_β, $s_\alpha s_\beta$, $s_\beta s_\alpha$, $s_\alpha s_\beta s_\alpha$, $s_\beta s_\alpha s_\beta$, $s_\alpha s_\beta s_\alpha s_\beta$, $s_\beta s_\alpha s_\beta s_\alpha$, $s_\alpha s_\beta s_\alpha s_\beta s_\alpha$, $s_\beta s_\alpha s_\beta s_\alpha s_\beta$, $s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta$. In rows they stand for a point, $X(\alpha)$, $X(\beta)$, $X(\alpha\beta)$, $X(\beta\alpha)$, $X(\alpha\beta\alpha)$, $X(\beta\alpha\beta)$, $X(\alpha\beta\alpha\beta)$, $X(\beta\alpha\beta\alpha)$, $X(\alpha\beta\alpha\beta\alpha)$, $X(\beta\alpha\beta\alpha\beta)$, $X(\alpha\beta\alpha\beta\alpha\beta)$.

Now we explain how to read the pictures. Let $w,\ w' \in W$ and $\lambda \in w'$-chamber (but not in the forbidden hyperplanes as we have mentioned in §3), i.e. $w' \cdot \lambda \in X(T)^+$. If we want to know whether $H^i(X(w), L_\lambda)$ is zero or not, we look at the (w, w')th entry, i.e. the intersection of wth row and the w'th column in the H^ith picture, if the entry is a dark box, then it is non-zero, otherwise it is zero.

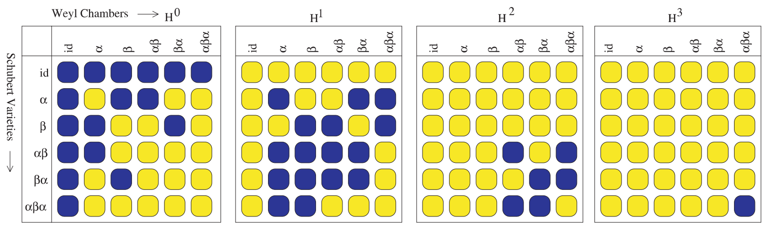


Figure 6. The vanishing and non-vanishing of cohomology of Schubert varieties in A_2.

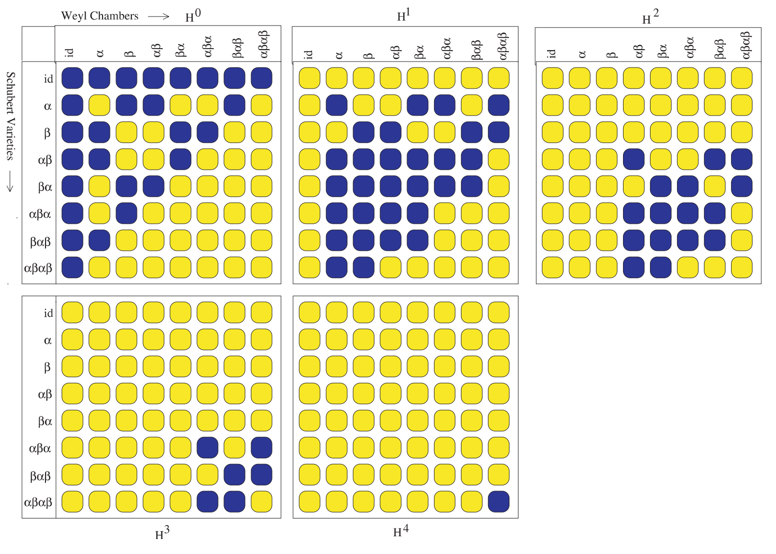


Figure 7. The vanishing and non-vanishing of cohomology of Schubert varieties in B_2.

7.1 *Remark*

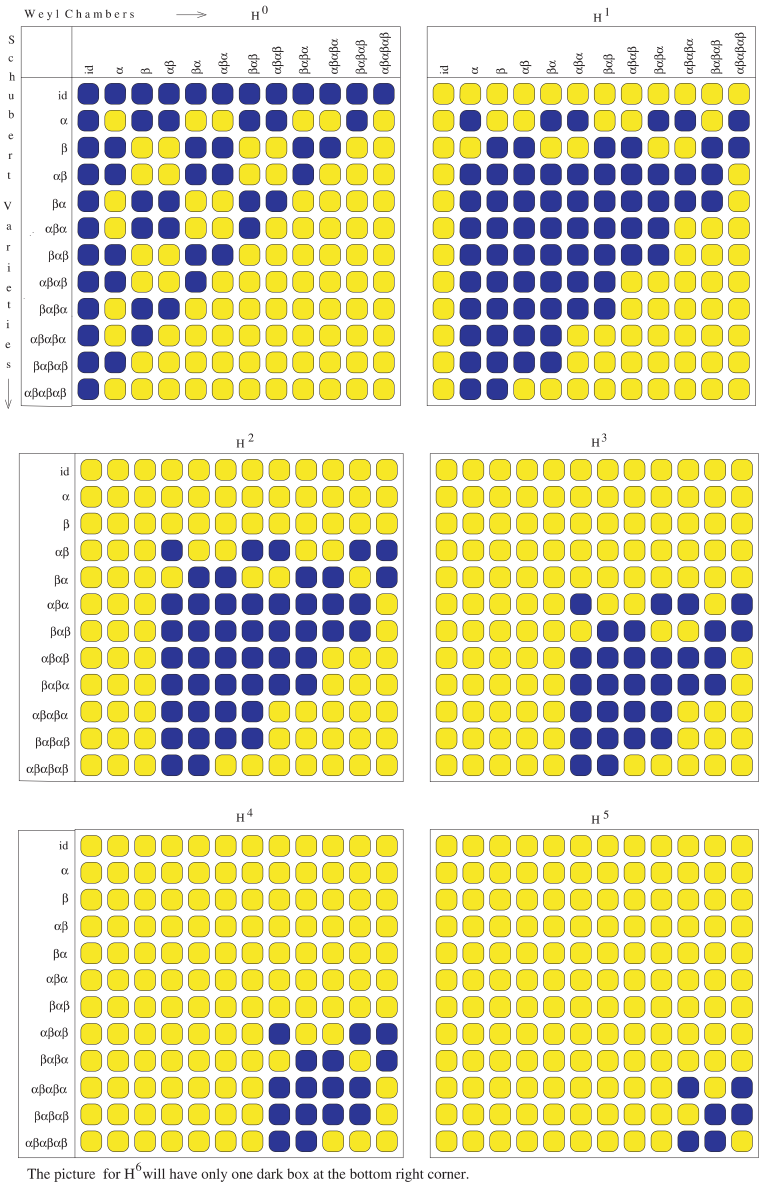
In the thesis [4], we have computed vanishing and non-vanishing of cohomology modules for singular weights for A_2 and B_2. There we have also shown that the conjecture 9.3 is not true for singular weights.

8. Observations from the pictures

We define matrices corresponding to the pictures. Let T_{ij}^k denote the $(i,\ j)$th entry of the kth matrix, i.e. the matrix corresponding to the H^kth cohomology picture. We assign values to T_{ij}^k's as follows: $T_{ij}^k = 1$, if $(i,\ j)$th box is dark, otherwise zero.

Then we can observe the following:

(1) The pictures H^0, H^{d-1} and H^d are symmetric, where d is the dimension of G/B.
(2) *Continuity of non-zeroness of the Schubert cohomology modules*. Suppose $m \leq n$, $H^m(X(w), \lambda) \neq 0$ and $H^n(X(w), \lambda) \neq 0$, then $H^i(X(w), \lambda) \neq 0,\ \forall i,\ m \leq i \leq n$.



The picture for H^6 will have only one dark box at the bottom right corner.

Figure 8. The vanishing and non-vanishing of cohomology of Schubert varieties in G_2.

(3) For $0 \leq k \leq \dim(G/B)$, $\sum_i T_{ij}^k$ depends only on $l(j) =$ length of j (note that j corresponds to a Weyl group element). We are saying if $l(j_1) = l(j_2)$, then $\sum_i T_{ij_1}^k = \sum_i T_{ij_2}^k$. Similarly $\sum_j T_{ij}^k$ depends only on $l(i)$.

(4) $\sum_{i,k} T_{i1}^k = \sum_{j,k} T_{1j}^k = \sum_{i,k} T_{i|W|}^k = \sum_{j,k} T_{|W|j}^k = |W|$. This reflects the Borel–Weil–Bott theorem and the fact that if $\lambda \in w_0$-chamber (recall that w_0 is the longest element of W), then only one cohomology survives for each Schubert variety, more

precisely if $\lambda \in w_0$-chamber then $H^k(X(w), \lambda) \neq 0$ only for $k = l(w) =$ the dimension of $X(w)$.

8.1 *Remark*

The pictures have more nice structures which are yet to be formulated. For example the diagonal of H^0 in B_2 and G_2 have a nice pattern.

9. Conjectures

Let G be semisimple, simply connected algebraic group over an algebraically closed field. Here we are not assuming rank of G to be two.

9.1 *Chain condition or continuity of non-zeroness of Schubert cohomology modules*

If $m \leq n$, $H^m(X(w), \lambda) \neq 0$ and $H^n(X(w), \lambda) \neq 0$ then $H^i(X(w), \lambda) \neq 0$, $\forall i$, $m \leq i \leq n$.

9.2 *Cohomological non-triviality of Schubert cohomology modules*

The following conjecture is stated in [1]. Let $w \in W$ be an element of the Weyl group and let $\alpha \in \triangle$ be a simple root such that $l(s_\alpha w) = l(w) + 1$. Let λ be any *generic* weight. If the cohomology module $H^i(w, \lambda)$ is non-zero then it is *cohomologically non-trivial* when considered as a B_α-module. More precisely, if $H^i(w, \lambda)$ is non-zero then both $H^0(\alpha, \widetilde{H^i(w, \lambda)})$ and $H^1(\alpha, \widetilde{H^i(w, \lambda)})$ cannot simultaneously vanish.

9.3 *Conjecture* 3

Let $l(s_\alpha w) = l(w) + 1$. Let $H^i(w, \lambda) \neq 0$. As we have done in the rank 2 cases we think of this B-module as a B_α-module and decompose it into indecomposable B_α-modules. Let us denote these indecomposable B_α-modules by V_i, $1 \leq i \leq n$. We can make these modules as $SL_2(\alpha)$-modules by tensoring with a suitable one-dimensional module $-\psi_i$. The conjecture is that $(\psi_i, \alpha^\vee) \neq -1$, $\forall i$.

In connection with the above conjecture, we recall the following results [1, 5] for the convenience of the reader.

Lemma. If V is a finite dimensional B_α-module then V is a direct sum of cyclic B_α-modules each of them generated by weight vectors.

COROLLARY

Let V be an indecomposable B_α-module. Then, there exists a character $\chi\colon B_\alpha \longrightarrow \mathbf{G}_m$ such that $V \simeq W \otimes \chi$, with W an irreducible L_α-module.

9.4 *Remarks*

(1) It is easy to see that the cohomological non-triviality conjecture and Conjecture 3 are equivalent.

(2) For the rank two cases the Conjecture 9.1 can be verified from the pictures of vanishing and non-vanishing of cohomology modules. Conjecture 9.2 can not be seen from the picture (reason can be seen in the next remark). But we remark that the second conjecture is verified in all chambers in A_2 and B_2 and verified in some cases in G_2.

(3) Immediately we cannot say whether the first two conjectures are equivalent or one is stronger than the other. The conjecture (cohomological non-triviality) implies that if $H^i(w,\lambda) \neq 0$ and $l(s_\alpha w) = l(w) + 1$ then $H^i(s_\alpha w,\lambda) \neq 0$, or $H^{i+1}(s_\alpha w,\lambda) \neq 0$. Now let $w = s_1 s_2 s_3$, where s_i's are simple reflections and $l(w) = 3$. The following could happen. $H^1(s_3,\lambda) \neq 0$ and $H^0(s_2, \widetilde{H^1(s_3,\lambda)}) \neq 0$ and $H^1(s_2, \widetilde{H^1(s_3,\lambda)}) \neq 0$. Then we will have $H^1(s_2 s_3,\lambda) \neq 0$ and $H^2(s_2 s_3,\lambda) \neq 0$. For the sake of argument we assume that $H^1(s_1, \widetilde{H^2(s_2 s_3,\lambda)}) \neq 0$ and $H^0(s_1, \widetilde{H^2(s_2 s_3,\lambda)}) = 0$. Also assume that $H^0(s_1, \widetilde{H^1(s_2 s_3,\lambda)}) \neq 0$ and $H^1(s_1, \widetilde{H^1(s_2 s_3,\lambda)}) = 0$. Then we will have $H^3(s_1 s_2 s_3,\lambda) \neq 0$ and $H^1(s_1 s_2 s_3,\lambda) \neq 0$ but $H^2(s_1 s_2 s_3,\lambda) = 0$. This implies that the cohomological non-triviality conjecture need not imply the continuity of cohomology conjecture.

Similarly, the conjecture 'continuity of non-zeroness of Schubert cohomology modules' need not imply cohomological non-triviality conjecture.

$$0 \longrightarrow H^1(\alpha, \widetilde{H^{i-2}(w,\lambda)}) \longrightarrow H^{i-1}(s_\alpha w,\lambda) \longrightarrow H^0(\alpha, \widetilde{H^{i-1}(w,\lambda)}) \longrightarrow 0.$$

$$0 \longrightarrow H^1(\alpha, \widetilde{H^{i-1}(w,\lambda)}) \longrightarrow H^{i}(s_\alpha w,\lambda) \longrightarrow H^0(\alpha, \widetilde{H^{i}(w,\lambda)}) \longrightarrow 0.$$

$$0 \longrightarrow H^1(\alpha, \widetilde{H^{i}(w,\lambda)}) \longrightarrow H^{i+1}(s_\alpha w,\lambda) \longrightarrow H^0(\alpha, \widetilde{H^{i+1}(w,\lambda)}) \longrightarrow 0.$$

From the above exact sequences we can see that $H^0(\alpha, \widetilde{H^i(w,\lambda)}) = 0$ and $H^1(s_\alpha, \widetilde{H^i(w,\lambda)}) = 0$ will not create any problem to have $H^{i+1}(s_\alpha w,\lambda) \neq 0$, $H^i(s_\alpha w,\lambda) \neq 0$, $H^{i+1}(s_\alpha w,\lambda) \neq 0$ simultaneously.

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