

REMARKS ON THE ABG INDUCTION THEOREM

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ABSTRACT. A key result in a 2004 paper by S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg [ABG] compares the bounded derived category $D^b\text{block}(\mathbb{U})$ of modules for the principal block of a Lusztig quantum enveloping algebra \mathbb{U} at an ℓ th root of unity with a subcategory $D_{\text{triv}}(\mathbb{B})$ of the derived category of integrable type 1 modules for a Borel part $\mathbb{B} \subset \mathbb{U}$. Specifically, according to this “Induction Theorem” [ABG, Theorem 3.5.5] the right derived functor of induction $\text{Ind}_{\mathbb{B}}^{\mathbb{U}}$ yields an equivalence of categories $\text{RInd}_{\mathbb{B}}^{\mathbb{U}} : D_{\text{triv}}(\mathbb{B}) \xrightarrow{\sim} D^b\text{block}(\mathbb{U})$ (under appropriate hypotheses on ℓ). The authors of [ABG] suggest a similar result holds for algebraic groups in positive characteristic p , and this paper provides a statement with proof for such a modular induction theorem. Our argument uses the philosophy of [ABG] as well as new ingredients. A secondary goal of this paper has been to put the original characteristic zero quantum result on firmer ground, and we provide arguments as needed to give a complete proof of that result also. Finally, using the modular result, we have been able in [HKS] to introduce truncation functors, associated to finite weight posets, which effectively commute with the modular induction equivalence, assuming $p > 2h - 2$, with h the Coxeter number. This enables interpreting the equivalence at the level of derived categories of modules for suitable finite dimensional quasi-hereditary algebras. We expect similar results to hold in the original quantum setting, assuming $\ell > 2h - 2$.

1. INTRODUCTION

If G is a semisimple algebraic group and B a Borel subgroup, it is well known that the category of rational G -modules fully embeds via the restriction functor into the category of rational B -modules. Explicitly describing the objects in the image of restriction is a difficult problem, unsolved in general. However, as we will see here, it is possible to make progress at the derived category level. Our starting point is a result [ABG, Theorem 3.5.5] by S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg in the world of quantum groups. The result establishes a natural equivalence between the bounded derived category of modules for the principal block of a Lusztig quantum enveloping algebra at a root of unity with an *explicit* subcategory of the bounded derived category of integrable modules for a Borel part of this quantum algebra. We will refer to this result as “the induction theorem.” We begin this paper with its explicit statement.

Suppose \mathbb{U} is a Lusztig quantum algebra, associated to a root datum $\mathfrak{R} = (\Pi, \chi, \Pi^\vee, \chi^\vee)$, and specialized to a characteristic 0 field K with ℓ^{th} root of unity $q \in K$, as defined in section 2. In particular, we assume q is defined by $q^\ell = 1$ with ℓ odd, and not divisible by 3 if the root system corresponding to \mathfrak{R} has a component of type G_2 . We denote the root system in general by R . Moreover, we assume $\ell > h$, where h is the Coxeter number of R , unless otherwise noted. Suppose $\mathbb{B} = \mathbb{U}^- \otimes_K \mathbb{U}^0 \subset \mathbb{U}$ is a ‘Borel part’ of \mathbb{U} arising from a triangular decomposition of \mathbb{U} as in Section 2. Denote by $D^b\text{block}(\mathbb{U})$ the

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bounded derived category of the abelian category of type 1 integrable modules in the “principal block” of \mathbb{U} .¹ For $D^b(\mathbb{B})$ the usual bounded derived category of the module category for \mathbb{B} , let $D_{triv}(\mathbb{B})$ be the full triangulated subcategory of $D^b(\mathbb{B})$ whose objects are complexes representable by

$$M = \cdots \rightarrow M_{i-1} \rightarrow M_i \rightarrow M_{i+1} \rightarrow \cdots \quad i \in \mathbb{Z}$$

so that for all $i \in \mathbb{Z}$,

- (i) M_i is an integrable \mathbb{B} -module;
- (ii) M_i has a grading $M_i = \bigoplus_{\nu \in \mathbb{Y}} M_i(\nu)$ by the root lattice \mathbb{Y} of the root datum \mathfrak{R} ;
- (iii) for any $m \in M_i(\nu)$ and $u \in \mathbb{U}^0$, $um = \nu(u) \cdot m$;
- (iv) the total cohomology module $H^\bullet(M) = \bigoplus_{i \in \mathbb{Z}} H^i(M)$ has a finite composition series, all of whose successive quotients are of the form $K_{\mathbb{B}}(\ell\lambda)$, $\lambda \in \mathbb{Y}$.

Here $K_{\mathbb{B}}(\ell\lambda)$ denotes a 1-dimensional \mathbb{B} -module associated to $\ell\lambda$. Further details on notation may be found in Section 2.

Theorem 1. [*Induction Theorem*, Theorem 3.5.5 [ABG]] For an appropriately defined induction functor $\text{Ind}_{\mathbb{B}}^{\mathbb{U}}$, its right derived functor $R\text{Ind}_{\mathbb{B}}^{\mathbb{U}}$ yields an **equivalence** of triangulated categories

$$D_{triv}(\mathbb{B}) \xrightarrow{\sim} D^b\text{block}(\mathbb{U}).$$

A precise definition for $\text{Ind}_{\mathbb{B}}^{\mathbb{U}}$ appears in Section 2. It is an analog of induction (right adjoint to restriction) in the theory of representations of algebraic groups, and has similar properties. To “induce” a module, one applies induction in the sense of associative rings and algebras, then passes to the largest type 1 integrable submodule.

The present paper contains, as a secondary feature, a complete proof of the above result, along the lines of [ABG], though with some variations and a number of corrections. We are grateful to Pramod Achar for alerting us to possible issues (first observed by his collaborator, Simon Riche) in the proof of [ABG, Lemma 4.1.1(ii)]. The argument we eventually found (the proof of our Lemma 3.2(ii)) is quite substantial, spanning two appendices and improving a theorem of Rickard [R94]. Other corrections we make are more minor, often rooted in inadequacies in the quantum literature. The “variations” mentioned often occur from our desire to give a proof that “carries over” to the characteristic p algebraic groups case. Indeed, the latter has been the central aim of our work here.

The existence of a modular analog of the induction theorem was suggested by the assertion [ABG, p. 616]: “An analogue of Theorem 3.5.5 holds also for the principal block of complex representations of the algebraic group $G(F)$ over an algebraically closed field of characteristic $p > 0$. Our proof of the theorem applies to the latter case as well.” Replacing the term “complex representations” in the quote above with “rational representations” (likely intended) yields the statement below, which this paper confirms is indeed a theorem. However, while the philosophy and some ingredients of the proof we present may be found in the [ABG] treatment of the quantum case, additional critical ingredients are also required. See, for example, Corollary 2.15(1) and Lemma 3.6.

To set the notation, $\text{block}(G)$ is the principal block of finite-dimensional rational G -modules, and $D_{triv}(B)$ is defined analogously to $D_{triv}(\mathbb{B})$ in the quantum case. That is, rational B -modules replace (Type 1) integrable \mathbb{B} modules, the distribution algebra of a maximal split torus $T \subseteq B$ is used for \mathbb{U}^0 , and $k_B(p\lambda)$, with k as below, replaces $K_{\mathbb{B}}(\ell\lambda)$ above (in the definition of $D_{triv}(\mathbb{B})$, which becomes $D_{triv}(B)$). As before h denotes the Coxeter number of the underlying root system, now regarded as associated to G .

¹More precisely, the principal block of \mathbb{U} is the full subcategory of finite-dimensional integrable type 1 \mathbb{U} -modules whose composition factors are all “linked” to the trivial module. Equivalently, the highest weights of these composition factors all belong to the “dot” orbit of 0 under the affine Weyl group. A precise definition of the “dot” action is given in Section 2.

Theorem 2. Let G be a semisimple algebraic group over an algebraically closed field k of positive characteristic $p > h$. Let B be a Borel subgroup of G . Then the functor RInd_B^G yields an **equivalence** of triangulated categories

$$D_{\mathrm{triv}}(B) \rightarrow D^b \mathrm{block}(G).$$

Generally, we use the “quantum case” to refer to the context of Theorem 1, and the “algebraic groups case” (or “positive characteristic case,” or “modular case”) when referring to the context of Theorem 2. Of course, some discussions in a given “case” do not require the full hypotheses of these theorems. (We sometimes keep track of such situations.)

Theorem 2 is a starting point for yet another result, proved in [HKS]. It shows, for $p > 2h - 2$, that RInd_B^G in Theorem 2 induces an equivalence between certain natural full triangulated subcategories

$$D_{\mathrm{triv}}(\mathrm{Dist}(B)_{\Lambda_m}) \rightarrow D^b(\mathrm{block}(G)_{\Gamma_m}),$$

depending on p and indexed by an integer $m > 0$. Λ_m is a finite subposet in a variation of van der Kallen’s “excellent order” on weights [vdK1], and Γ_m is a finite subposet of dominant weights in the usual dominance order. The arguments $\mathrm{Dist}(B)_{\Lambda_m}$ and $\mathrm{block}(G)_{\Gamma_m}$ of the constructions in the display refer to finite-dimensional quasi-hereditary algebra quotients of the distribution algebra $\mathrm{Dist}(B)$ and $\mathrm{Dist}(G)$, respectively, the latter associated to G —a mild abuse of notation.

For further details, see [HKS]. Collectively, these more “finite” equivalences can be used to reconstruct the full equivalence given by RInd_B^G in Theorem 2, thereby deepening our understanding of it.

This paper is organized as follows. Section 2 collects notation and some needed background material. Section 3 proves Theorems 1 and 2. The statements above these theorems contain the start of a dictionary for going back and forth between the characteristic 0 quantum root of unity case and the positive characteristic algebraic group case. Indeed, there is nothing to stop us from using the same names as in Theorem 1 for parallel objects in Theorem 2, putting $\mathbb{U} = \mathrm{Dist}(G)$, $\mathbb{B} = \mathrm{Dist}(B)$, $\mathrm{block}(\mathbb{U}) = \mathrm{block}(G)$, $D_{\mathrm{triv}}(\mathbb{B}) = D_{\mathrm{triv}}(B)$, and even writing $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$ for RInd_B^G . We can then restate

THEOREM 2.1 (Cosmetic variation on Theorem 2) *Let \mathbb{U} be the distribution algebra of a semisimple algebraic group over an algebraically closed field of prime characteristic $\ell = p > h$. Let \mathbb{B} be the distribution algebra of a Borel subgroup. Then the induction functor $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$ induces an equivalence*

$$D_{\mathrm{triv}}(\mathbb{B}) \rightarrow D^b \mathrm{block}(\mathbb{U}).$$

In Section 3 we give a simultaneous proof of both Theorem 1 and the above version of Theorem 2.

Some of the rationale for the overall approach is discussed in subsection 3.5. Sections 4, 5, and 6 present three appendices, labeled A, B, C, respectively. The first two are used to prove Lemma 3.2(ii), which restates [ABG, Lem. 4.1.1(ii)], asserting that it holds in both the quantum and modular algebraic groups cases. The modular case, at least, is of independent interest of categorifying a theorem [R94, Thm. 2.1] of Rickard in the regular weight case, and the quantum case of the lemma may be viewed as giving an analogous quantum result. Also, Appendix C, independent of the rest of this paper, corrects the statement and proof of [ABG, Lem. 9.10.5] as a service to the reader. Appendix C was previously labeled and quoted as “Appendix,” in previous versions of this paper. Finally, a few acknowledgements and thanks are collected in the final section.

Theorem 2 was first announced in [HKS], though the proof underwent several corrections after that, the last in August, 2015, when a proof of Lemma 3.2(ii) was written down. This was done in the modular case, and completed our proof of the modular induction theorem. The proof actually also works in the quantum case, thus proving [ABG, Lem. 4.1.1(ii)], though we found it necessary to work through some foundational issues regarding quantum induction (Remarks 2.11(d),(e)). We also found it necessary to fill in other details in the quantum literature to complete our simultaneous treatment of the quantum and modular induction theorems. Another proof of the modular induction theorem, as part of a larger geometric program, has recently been posted by Achar and Riche [AR].

2. BACKGROUND

Generally, we follow Lusztig [L5], [L3] for basic material on quantum enveloping algebras, and Andersen’s paper [A] for many additional results on their representation theory, especially results on induced representations that parallel those found in Jantzen [J] in the case of semisimple algebraic groups. These results on induced representations have their origin in an earlier paper of Andersen-Polo-Wen [APW], as supplemented by [AW]. For the study of (characteristic zero) quantum groups at ℓ^{th} roots of unity with ℓ a prime power, the [APW] paper is generally sufficient, while the context of [AW] allows all values ℓ (orders of roots of unity) used in (the main results of) this paper. (It does restrict ℓ to be odd, and not divisible by 3 in case the root system has a component of type G2.) The context of [A] is even more general, though it references an argument from [AW], and there are a number of references of convenience (which could be avoided) to arguments in [APW].

We are interested in the semisimple algebraic groups case as much or more so than in the quantum case, but focus now on giving notation below as befits the quantum case, where there is much less uniformity in the literature than in the algebraic groups case. All the notation and results have analogs in [J], however. In later parts of this paper, excluding the appendices, we will try to treat both the algebraic groups and quantum cases simultaneously and with the same notation. Some of our quantum group notation has been chosen to maintain consistency with these later discussions.² Some important background on induction and cohomology is given in subsection 2.5, modifying and completing a number of references given by [ABG] to the literature on quantum group representations. Many of the results we discuss in that subsection are well-known in the algebraic groups case, and we generally do not track their analogs there in detail. Starting with subsection 2.6 and continuing in the rest of Section 2 and all of Section 3, we use the “uniform” notation for both the quantum and positive characteristic cases, though some differentiation of the two cases is sometimes required for proofs. Appendices A and B, used to prove Lemma 3.2, are given in algebraic groups notation, with the quantum case treated in remarks. A compact quantum group reference written in the spirit of comparing general results in the quantum and algebraic groups cases may be found [J, Appendix H] in summary form.

2.1. Quantum Enveloping Algebras and Algebraic Groups.

2.1.1. Root Datum. Assume $\mathfrak{g}_{\mathbb{C}}$ is a complex semisimple Lie algebra of rank n , with Cartan matrix $C = (c_{ij})_{1 \leq i, j \leq n}$, and Killing form $\kappa : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \rightarrow \mathbb{C}$. Then from a choice of Cartan subalgebra $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ one obtains a root-datum realization $\mathfrak{R} = (\Pi, X, \Pi^{\vee}, X^{\vee})$ of C from the following data.

- $R \subset \mathfrak{h}_{\mathbb{C}}^*$ denotes the set of roots arising from the Cartan decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\mathbb{C}, \alpha}$ into $\mathfrak{h}_{\mathbb{C}}^*$ -weight spaces under the restriction of the adjoint action of $\mathfrak{g}_{\mathbb{C}}$ to $\mathfrak{h}_{\mathbb{C}}$, with corresponding elements $t_{\alpha} \in \mathfrak{h}_{\mathbb{C}}$, $t_{\alpha} \leftrightarrow \alpha \in R$ arising from the identification of $\mathfrak{h}_{\mathbb{C}}$ with $\mathfrak{h}_{\mathbb{C}}^*$ obtained from the nondegeneracy of the Killing form by setting, for any $\phi \in \mathfrak{h}_{\mathbb{C}}^*$, $t_{\phi} \in \mathfrak{h}_{\mathbb{C}}$ to be the unique element such that $\phi(h) = \kappa(t_{\phi}, h)$ for all $h \in \mathfrak{h}_{\mathbb{C}}$.
- Take as the set of coroots $R^{\vee} := \{\alpha^{\vee} := \frac{2\alpha}{(\alpha, \alpha)} \mid \alpha \in R\}$. For $h_{\alpha} := \frac{2t_{\alpha}}{\kappa(t_{\alpha}, t_{\alpha})}$, there is a correspondence $h_{\alpha} \leftrightarrow \alpha^{\vee}$ under the identification of $\mathfrak{h}_{\mathbb{C}}$ with $\mathfrak{h}_{\mathbb{C}}^*$.
- Take $E = E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$, for $E_{\mathbb{Q}}$ the \mathbb{Q} -span of the roots R in $\mathfrak{h}_{\mathbb{C}}^*$. The rational space $E_{\mathbb{Q}}$ has a nondegenerate bilinear form obtained from restriction of $(\lambda, \mu) = (t_{\lambda}, t_{\mu})$ on $\mathfrak{h}_{\mathbb{C}}^*$; this extends uniquely to a positive definite form $(-, -)$ on E , making E into an n -dimensional Euclidean space. Both R and R^{\vee} are root systems in E ; in particular they both span E . Observe that, for $\langle \zeta, \eta \rangle := \frac{2(\zeta, \eta)}{(\eta, \eta)} \forall \zeta, \eta \in E$, one has $\langle \beta, \alpha \rangle = (\beta, \alpha^{\vee})$ for $\alpha, \beta \in R$.
- The Weyl group W is the subgroup of $GL(E)$ generated by reflections $s_{\alpha}(v) = v - (v, \alpha^{\vee})\alpha$, $\alpha \in R$.

²In some cases, our notation differs from [ABG]. In particular we use a “simply connected” set-up, which allows modules with weights in \mathbb{X} (defined below), rather than the “adjoint” set-up of [ABG], which restricts attention to modules with weights in \mathbb{Y} below (the root lattice). This does not affect the triangulated categories (up to natural equivalences) entering into the statements of Theorems 1, 2 and 2.1. Also, we always induce from Borel subgroups associated to negative roots, and correspondingly use a different (but more standard) affine Weyl group “dot” action, defined later in subsection 2.4.2.

- $\Pi = \{\alpha_1, \dots, \alpha_n\}$ to be a set of simple roots (i.e., basis for E such that any $\alpha \in R$ satisfies $\alpha = \sum m_i \alpha_i \in \bigoplus_{1 \leq i \leq n} \mathbb{Z} \alpha_i$ with all $m_i \geq 0$ or all $m_i \leq 0$). Then W is generated by the $s_i := s_{\alpha_i}, 1 \leq i \leq n$.
- $\Pi^\vee := \{\alpha_i^\vee \mid 1 \leq i \leq n\}$ to be the corresponding set of simple coroots
- $X^\vee = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i^\vee$ coweight lattice
- $X = \{\lambda \in E \mid \langle \lambda, \alpha \rangle = (\lambda, \alpha^\vee) \in \mathbb{Z} \quad \forall \alpha \in R\} = \{\lambda \in E \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z} \quad \forall \alpha_i \in \Pi\} \cong \text{Hom}_{\mathbb{Z}}(X^\vee, \mathbb{Z})$ weight lattice
- Y is the subgroup of X generated by R ; $Y = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i$ root lattice
- $X^+ := \{\lambda \in X \mid (\lambda, \alpha_i^\vee) \geq 0 \quad \forall 1 \leq i \leq n\}$ dominant weights; for $\varpi_1, \dots, \varpi_n$ the dual basis defined by $(\varpi_i, \alpha_j^\vee) = \delta_{i,j}$, we get $X = \bigoplus_{i=1}^n \mathbb{Z} \varpi_i$.
- In our notation the Cartan matrix $C = (c_{i,j})$ is given by $c_{i,j} = (\alpha_j, \alpha_i^\vee), 1 \leq i, j \leq n$.

Furthermore, for any $\alpha \in R$, set $d_\alpha = \frac{(\alpha, \alpha)}{2}$, so then $d_\alpha \in \{1, 2, 3\}$ and $(\lambda, \alpha) = d_\alpha (\lambda, \alpha^\vee) = d_\alpha \langle \lambda, \alpha \rangle \in \mathbb{Z}$. Writing d_i for d_{α_i} , and $D = \text{diag}(d_1, \dots, d_n)$, one has that $DC = (d_i c_{i,j})$ is symmetric.

2.1.2. *Quantum Enveloping Algebra.* Our description of quantum enveloping algebras here follows Lusztig [L3], with similar notation, especially for generators. There are some differences in the notational names of algebras and subalgebras, and the labeling of relations. We make no distinction in the terms “quantum enveloping algebra,” “quantum algebra,” and “quantum group.”

Take v to be an indeterminate, and consider the following expressions in the ring $\mathbb{Q}(v)$:

$$(2.0.1) \quad [n]_d := \frac{v^{nd} - v^{-nd}}{v^d - v^{-d}}, \text{ for } d, n \in \mathbb{N}; \text{ when } d = 1, \text{ have } [n] := [n]_1 = \frac{v^n - v^{-n}}{v - v^{-1}}$$

$$(2.0.2) \quad \begin{aligned} [n]_d! &:= \prod_{s=1}^n \frac{v^{d \cdot s} - v^{-d \cdot s}}{v^d - v^{-d}} = \prod_{s=1}^n [s]_d, \text{ for } n, d \in \mathbb{N} \\ \left[\begin{matrix} n \\ t \end{matrix} \right]_d &:= \prod_{s=1}^t \frac{v^{d(n-s+1)} - v^{-d(n-s+1)}}{v^{ds} - v^{-ds}} \text{ for } n \in \mathbb{Z}, d, t \in \mathbb{N}; \text{ when } d = 1, \text{ we set } \left[\begin{matrix} n \\ t \end{matrix} \right] := \left[\begin{matrix} n \\ t \end{matrix} \right]_1. \end{aligned}$$

The simply connected quantum enveloping algebra³ $U'_v = U'_v(\mathfrak{R})$ is the $\mathbb{Q}(v)$ -algebra generated by the symbols $E_i, F_i, K_i^{\pm 1}, 1 \leq i \leq n$, subject to the five sets of relations below, as found in [L3, p.90]. We take this opportunity to warn the reader that we will sometimes also need to refer to Lusztig’s book [L], where the symbols K_i here (and in [L3]) correspond to symbols \tilde{K}_i there.⁴

- (a1) $K_i K_j = K_j K_i, K_i K_i^{-1} = 1 = K_i^{-1} K_i, \text{ for } 1 \leq i, j \leq n, i \neq j;$
- (a2) $K_i E_j = v^{d_i c_{i,j}} E_j K_i, \text{ and } K_i F_j = v^{-d_i c_{i,j}} F_j K_i, \text{ for } 1 \leq i, j \leq n;$
- (a3) $E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}, \text{ for } 1 \leq i \leq n.$
- (a4) $\sum_{s+t=1-c_{i,j}} (-1)^s \left[\begin{matrix} 1-c_{i,j} \\ s \end{matrix} \right] E_i^s E_j E_i^t = 0, \text{ for } 1 \leq i, j \leq n, \quad i \neq j;$
- (a5) $\sum_{s+t=1-c_{i,j}} (-1)^s \left[\begin{matrix} 1-c_{i,j} \\ s \end{matrix} \right] F_i^s F_j F_i^t = 0, \text{ for } 1 \leq i, j \leq n, \quad i \neq j;$

Note that it is also common to let $v_i = v^{d_i}$, so e.g., the first part of (a2) can be rewritten as $K_i E_j K_i^{-1} = v^{(\alpha_i, \alpha_j)} E_j$ for $1 \leq i, j \leq n$, and similarly for the second part of (a2).

The algebra U'_v is also a Hopf algebra, with comultiplication Δ , antipode S , and counit ϵ given as below [ibid]. These formulas hold for all indices i with $1 \leq i \leq n$.

$$(b) \quad \Delta E_i = E_i \otimes 1 \oplus K_i \otimes E_i, \quad \Delta F_i = F_i \otimes K_i^{-1} \oplus 1 \otimes F_i,$$

³Over $\mathbb{Q}(v)$ Lusztig [L3, p.90] and in [L5] credits this form to Drinfeld and Jimbo. Lusztig himself considers in these papers more general rings as coefficients, especially $\mathbb{Z}[v, v^{-1}]$. The \prime notation, convenient for us here (freeing the unprimed U for other uses) is not used in [L3]. Our usage of it is similar to that of [L5], suggesting the use of a quotient field, such as $\mathbb{Q}(v)$, in the coefficient system.

⁴In [L] larger quantum algebras are built, which we do not need. There are (new) elements K_i in these larger algebras, which serve as d_i^{th} roots for the elements \tilde{K}_i .

$$\begin{aligned} \Delta K_i &= K_i \otimes K_i, \\ (c1) \quad SE_i &= -K_i^{-1}E_i, \quad SF_i = -F_iK_i, \quad SK_i = K_i^{-1}, \\ (c2) \quad \epsilon E_i &= \epsilon F_i = 0, \quad \epsilon K_i = 1. \end{aligned}$$

We next give a brief discussion of the Lusztig integral form of this Hopf algebra [L3].

Starting with $E_i, F_i, K_i \in \mathcal{U}_v$, $1 \leq i \leq n$, and $s, t \in \mathbb{N}$, $c \in \mathbb{Z}$, define the divided powers $E_i^{(s)}, F_i^{(s)}, [K_i^{c;c}]$ by

$$(2.0.3) \quad \begin{aligned} E_i^{(s)} &:= \frac{E_i^s}{[s]_{d_i}!}, \\ F_i^{(s)} &:= \frac{F_i^s}{[s]_{d_i}!}, \\ [K_i^{c;c}] &:= \prod_{j=1}^t \frac{K_i v^{d_i(c-j+1)} - K_i^{-1} v^{d_i(-c+j-1)}}{v^{d_i j} - v^{-d_i j}}. \end{aligned}$$

Each term on the left above with $s = 0$ or $t = 0$ is defined to be 1.

Set $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$. Keeping the hypotheses that \mathfrak{R} is the root datum for a semisimple complex Lie algebra, with $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$, then the Lusztig integral form⁵ $\mathcal{U}_{\mathcal{Z}} = \mathcal{U}_{\mathcal{Z}}(\mathfrak{R})$ of \mathcal{U}'_v is the \mathcal{Z} -subalgebra of \mathcal{U}'_v generated by $E_i^{(s)}, F_i^{(s)}, K_i^{\pm 1}$ and $[K_i^{c;c}]$, for all i with $1 \leq i \leq n$, $s, t \in \mathbb{N}$, $c \in \mathbb{Z}$; equivalently (it turns out) $\mathcal{U}_{\mathcal{Z}}$ is generated by all $E_i^{(s)}, F_i^{(s)}, K_i^{\pm 1}$ and $[K_i^{c;0}]$, $1 \leq i \leq n$, $s, t \in \mathbb{N}$. Corresponding to either set of generators, both \mathcal{U}'_v and $\mathcal{U}_{\mathcal{Z}}$ have (compatibly generated) triangular decompositions $\mathcal{U}'_v = \mathcal{U}'_v{}^- \otimes \mathcal{U}'_v{}^0 \otimes \mathcal{U}'_v{}^+$, resp., $\mathcal{U}_{\mathcal{Z}} = \mathcal{U}_{\mathcal{Z}}^- \otimes \mathcal{U}_{\mathcal{Z}}^0 \otimes \mathcal{U}_{\mathcal{Z}}^+$. Either set of generators, with the relations (a1), ..., (a5), define \mathcal{U}'_v over $\mathbb{Q}(v)$. These relations are often sufficient to work with $\mathcal{U}_{\mathcal{Z}}$. However, there are many additional useful relations [L3, §6] on the elements $[K_i^{c;0}]$ and their interactions with the “divided powers” $E_i^{(s)}, F_i^{(s)}$. Also, there are analogs of the latter elements for all positive roots. All of these elements belong to $\mathcal{U}_{\mathcal{Z}}$, and may be used to define the latter by generators and relations in its own right, and to construct for it a monomial basis [L3].

Finally, the \mathcal{Z} -algebra $\mathcal{U}_{\mathcal{Z}}$ is a Hopf algebra, inheriting its Hopf algebra structure from \mathcal{U}'_v [L3, 8.11]. (All the Hopf algebras mentioned in this paragraph have bijective antipodes, with clearly invertible squares.) Similar statements apply for $\mathcal{U}_{\mathcal{Z}}^0$, $\mathcal{U}_{\mathcal{Z}}^- \otimes \mathcal{U}_{\mathcal{Z}}^0$, and $\mathcal{U}_{\mathcal{Z}}^0 \otimes \mathcal{U}_{\mathcal{Z}}^+$ [ibid]. Also, the root of unity specializations discussed in the next section inherit Hopf algebra structures from $\mathcal{U}_{\mathcal{Z}}$, as do the “small” quantum groups u, u^-, u^0, u^+ [ibid]. The (“Frobenius”) homomorphism discussed later in section 2.7 is a homomorphism of Hopf algebras.

2.2. Quantum specializations at roots of unity–notation. For any commutative ring K and invertible element $q \in K$, with unique accompanying ‘evaluation morphism’ $\epsilon_q : \mathcal{Z} \rightarrow K$ satisfying $v \mapsto q$, define the **specialization** of $\mathcal{U}_{\mathcal{Z}} = \mathcal{U}_{\mathcal{Z}}(\mathfrak{R})$ by

$$(2.0.4) \quad \mathcal{U}_{q,K} = \mathcal{U}_{\mathcal{Z}} \otimes_{\mathcal{Z}} K,$$

where the tensor product is formed by using the \mathcal{Z} -module structure on K given by ϵ_q . For this paper we will be interested in specializations where

- K is a field of characteristic zero
- q is a primitive ℓ^{th} -root of unity in K with ℓ odd, and $\ell \neq 3$ if the root system of \mathfrak{g} has a component of type G_2 .

⁵This is the form employed in [A] (and also [J, Appendix H]). In [ABG, 2.4], a version of Lusztig’s integral form is given very loosely, but apparently intended to be defined by an “adjoint type” version of the relations we use here. The latter relations, however, appear to be consistent with the alternate “simply connected” development suggested [ABG, Remark 2.6], a point of view we have used throughout this paper.

⁶These additional expressions in the K_i are redundant—see the brief discussion [A, p.3, bottom]—but are needed for the integral triangular decomposition.

We henceforth fix this meaning for K, ℓ, q , unless otherwise noted. We also take $\ell > h$, the Coxeter number⁷, from Corollary 2.10 forward. We also now introduce further notation that will be used in this quantum root of unity setting, and also used in a parallel setting from algebraic groups, discussed below. Relatively abbreviated notations are chosen to facilitate later parallel discussions. In the present *quantum context*, we let U denote the specialization $U_{q,K}$ as in (2.0.4). Similar conventions are adapted for U^+, U^0, U^- and $B = U^0 \cdot U^-$. Lusztig's finite dimensional Hopf algebra [L3, §8.2] (the "small" quantum group) is denoted u , with components of its triangular decomposition denoted u^-, u^0, u^+ . For example, u^0 is generated by all K_i^\pm , and u^- is generated by all the F_i [L3, pp.107-108]. Imitating the notation in [ABG] we set $b := u^- \cdot u^0$ and $p := b \cdot U^0$.

Overall, our notation here is quite similar to that used for quantum groups at a root of unity in [ABG], with the exception that our characteristic 0 field K (which may be compared with k in [ABG]) is not assumed to be algebraically closed. Also, our B is $U^0 \cdot U^-$, whereas in [ABG] the same symbol B is used to denote $U^0 \cdot U^+$.

2.3. Some parallel algebraic groups notation. Let G be a simply connected semisimple algebraic group, with root datum \mathfrak{R} , over an algebraically closed field k of characteristic $p > h$. We assume G is defined and split over the prime field F_p . In particular there is a Borel subgroup $B = TU$, with U the unipotent radical of B , and T a maximal torus, all defined over F_p , with T isomorphic (over the same field) to a direct product of copies of k^\times . The root groups in B are viewed as negative. We refer to this set-up as the *algebraic groups context* or, even more loosely, as the *algebraic groups case*. The distribution algebras $\text{Dist}(G)$, $\text{Dist}(B)$, $\text{Dist}(T)$, $\text{Dist}(U)$, and $\text{Dist}(G_1)$ (the restricted enveloping algebra) parallel U , B , U^0 , U^- , and u , respectively. We will use the latter symbol set in place of the former, when the context is clear, or if both the algebraic groups and quantum contexts have been explicitly allowed. In either of these circumstances, additional notational substitutions in the same spirit may also be made, such as p for $\text{Dist}(B_1T)$.

2.4. Affine Weyl Groups. Affine Weyl groups W_ℓ are used to index modules in both the quantum and algebraic groups context, with p used for ℓ in the latter. Our main references for affine Weyl groups are [J] and [A]. To clarify discussions and differences in these references, we temporarily allow ℓ to be any positive integer.

2.4.1. *Affine Weyl Groups, as in [J].* Following e.g., the conventions and notation in [J, §6.1], for $\beta \in R$ and $m \in \mathbb{Z}$, define the affine reflection on X by

$$s_{\beta,m}(\lambda) = \lambda - (\lambda, \beta^\vee) \beta, \quad \forall \lambda \in X;$$

one could take $X \otimes_{\mathbb{Z}} R$ in place of X . Thus, for the reflections s_β given by $s_\beta(\lambda) = \lambda - (\lambda, \beta^\vee) \beta$ one has

$$s_{\beta,m}(\lambda) = s_\beta(\lambda) + m\beta \quad \forall \lambda.$$

For any positive integer ℓ , the affine Weyl group W_ℓ is the group

$$W_\ell := \langle s_{\beta,n\ell} \mid \beta \in R, n \in \mathbb{Z} \rangle.$$

In the notation used in [J, §6.1], there is a (largely formal) isomorphism $W_\ell \cong W_a(R^\vee)$, for $W_a(R^\vee) := \langle s_{\beta,m} \mid \beta \in R, m \in \mathbb{Z} \rangle$, as defined by Bourbaki [B, ch. VI, §2]. When $\ell = 1$ this isomorphism is an equality. The Bourbaki reference makes a good case for the labelling with R^\vee , though it is common in algebraic group theory to associate both W_ℓ and $W_a(R^\vee)$ with the root system R . A familiar semidirect product description is obtained by regarding $\ell\mathbb{Z}R$ as a group of translations on $X \otimes_{\mathbb{Z}} R$, namely, $W_\ell \cong \ell\mathbb{Z}R \rtimes W = \ell Y \rtimes W$ ([J] references [B, ch. VI§2 prop. 1] for a proof). Here W is the usual Weyl group associated to R .

⁷All these restrictions agree in substance with those in [ABG], though h there denotes the dual Coxeter number. Also, the literature differs as to whether q is chosen to be the image of v or the image of v^2 , the latter fitting somewhat better with Hecke algebra notation. This makes little difference when the order of the image of v is odd, as is the case here.

2.4.2. *Dot Action.* Set $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. For $w \in W$, $\lambda \in X$, one sets,

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

More generally, the affine Weyl group $W_\ell \cong \ell Y \rtimes W$ acts, for $w_a := \ell \tau \rtimes w \in \ell Y \rtimes W$ by

$$w_a \cdot \lambda := w(\lambda + \rho) - \rho + \ell \tau, \quad \text{for all } \lambda \in X.$$

Our “dot” action \cdot , which is often used, is not given by the same formula as the action \bullet defined in [ABG], possibly intending some variation on [ABG, Lem. 3.5.1] (which is incorrect as stated for “positive” Borel subalgebras). A simpler approach, it seems to us, is to use “negative” Borel subalgebras and the usual “dot” action.

2.4.3. *Affine Weyl Groups as in [A].* Take $q \in K$ to be a root of unity. (In [A] the field K can have any characteristic, though that is not relevant to our discussions here, and we may keep our assumption that K has characteristic 0.) Set ℓ to be the order of q^2 , so that q is a primitive ℓ^{th} or $2\ell^{\text{th}}$ root of unity. For the Cartan matrix C of \mathfrak{R} and symmetrization DC (as at the end of our Section 2.1.1), set $\ell_i = \frac{\ell}{\gcd(\ell, d_i)}$. For each $\beta \in R$, there is some i , $1 \leq i \leq n$ so that β is conjugate under the Weyl group W of R to α_i . Set $\ell_\beta = \ell_i$ (well-defined). For each $\beta \in R$ and $m \in \mathbb{Z}$, as in Section 2.4.1, we have the affine reflection $s_{\beta, m\ell_\beta}$ with

$$s_{\beta, m\ell_\beta} \cdot \lambda = s_\beta \cdot \lambda + m\ell_\beta \beta \quad \forall \lambda \in X.$$

Here, s_β and $s_\beta \cdot \lambda$ are defined just as in Sections 2.4.1 and 2.4.2. Now define a new group of affine reflections

$$W_{D, \ell} := \langle s_{\beta, m\ell_\beta} \mid \beta \in R, m \in \mathbb{Z} \rangle.$$

Take W_ℓ^\vee to be the group

$$W_\ell^\vee := \langle s_{\beta^\vee, n\ell} \mid \beta \in R^+, n \in \mathbb{Z} \rangle,$$

generated by reflections as in Section 2.4.1, but utilizing coroots in place of roots. The following proposition relates the three groups W_ℓ , $W_{D, \ell}$, and W_ℓ^\vee .

Proposition 2.1. *Assume R is indecomposable. There are identifications giving inclusions*

$$W_\ell \subseteq W_{D, \ell} \subseteq W_\ell^\vee$$

so that

- (i) If $\gcd(d_i, \ell) = 1$ for all $1 \leq i \leq n$, then $W_\ell = W_{D, \ell}$.
- (ii) On the other hand, if $\gcd(d_i, \ell) \neq 1$ for some $1 \leq i \leq n$, then $W_{D, \ell} = W_\ell^\vee$.

Proof. Without loss, some $d_i \neq 1$. Since R is assumed to be indecomposable, all $d_i \neq 1$ take the same value $d \in \{1, 2, 3\}$. If d does not divide ℓ , $\ell_i = \ell$ for all indices i , and it follows that $\ell = \ell_\beta$ for all $\beta \in R$. Consequently, $W_{D, \ell} = W_\ell$. On the other hand, if d does divide ℓ , then $d\beta^\vee = \beta$ and $d\ell_\beta = \ell$ for all long $\beta \in R$, and $\beta^\vee = \beta$ for all short roots β . It follows that $W_{D, \ell} = W_\ell^\vee$ in this case. This proves the proposition. Q.E.D.

Remark 2.2. We have included Proposition 2.1 in part to address possible confusion that a reader casually comparing [J] and [A] may encounter. [A, p.6] says “Note that if ℓ is prime to all entries of the Cartan matrix, then the group $W_{D, \ell}$ (denoted W_ℓ in [A]!) is the ‘usual’ affine Weyl group of R . However, in general $W_{D, \ell}$ is the affine Weyl group of the dual root system”. As we have pointed out above, the “‘usual’ affine Weyl group” in algebraic groups discussions is W_ℓ as defined in [J] and Section 2.4.1 above, and that “the affine Weyl group on the dual root system” referred to by [A] is $W_\ell^\vee \cong W_a(R)$, rather than $W_a(R^\vee)$. The proposition and our previous discussion perhaps make precise what Andersen intended. In any case, in this paper, under the assumption below (2.0.4) that ℓ be odd, and not divisible by 3 in case the root system has a component of type G_2 , it is clear from the proposition that $W_\ell = W_{D, \ell}$.

2.5. Induction and Cohomology. We continue the notation of Section 2.4.3 above, appropriate for Andersen’s paper [A]. This is somewhat more general than our standard assumptions stated below (2.0.4). Those more special assumptions are all that we need for this paper, and are explicitly used as hypotheses in [AW]. The latter paper, along with some arguments of [APW] could possibly be used as an alternate source for some of the results of this subsection, with the standard assumptions below (2.0.4) as hypotheses. Unfortunately, it is not possible to quote [APW] directly, since its standing assumptions effectively require ℓ to be a prime power. (See [APW, Lem. 6.6], [AW, p.35].) On the other hand, [A] contains explicit statements (with weaker hypotheses) of most of the results we need, with the exceptions tractable with modest effort.

Accordingly, we follow [A] using the notation for K, q, ℓ in the previous subsection. In addition, we use the notation $\mathbb{U}_{q,K}$ as in (2.0.4), though with more general assumptions than those below (2.0.4). We will define induction functors $\text{Ind}_{\mathbb{B}_{q,K}}^{\mathbb{U}_{q,K}}$ from the category of integrable $\mathbb{B}_{q,K}$ -modules of Type 1 to the category of integrable $\mathbb{U}_{q,K}$ -modules of Type 1 (both categories defined below). For the moment, we will not use our preferred $\mathbb{U}, \mathbb{B}, \dots$ notation, to help remind the reader of our slightly different context, with weaker hypotheses. Of course, we will obtain from this construction the induction functors $\text{Ind}_{\mathbb{B}}^{\mathbb{U}}$ whose right derived functors are the focus of this paper.

We begin as in [A, §1]. First, we coordinate the notation X in our section 2.1.1 with the “weights” Z^n , given in [A, p.3]. The correspondence is simply to let $\lambda \in X$ correspond to the n -tuple with i^{th} coordinate $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle$. Then, as in *loc. cit.*, λ defines a 1-dimensional representation of (what we call here) $\mathbb{U}_{q,K}^0$ via the homomorphism $\chi_\lambda : \mathbb{U}_{q,K}^0 \rightarrow K$ sending K_i^\pm to $q^{\pm d_i \lambda_i}$ and $\begin{bmatrix} K_{i;c} \\ t \end{bmatrix}$ to $\begin{bmatrix} \lambda_i + c \\ t \end{bmatrix}_{d_i}$. Here $1 \leq i \leq n$, $c \in Z$, and $t \in N$. For any $\mathbb{U}_{q,K}^0$ -module M , let M_λ denote the sum of all 1-dimensional submodules on which $\mathbb{U}_{q,K}^0$ acts via the homomorphism χ_λ . We will call M_λ the “weight space” for M associated to λ . If M is the sum (necessarily direct) of its weight spaces M_λ , $\lambda \in X$, we say that M is integrable of Type 1 as a $\mathbb{U}_{q,K}^0$ -module. If we start with M a \mathbb{B} -module (resp., a \mathbb{U} -module), we say that M is integrable of Type 1 as a \mathbb{B} -module (resp., as a \mathbb{U} -module) if it is integrable of Type 1 as a $\mathbb{U}_{q,K}^0$ -module, and each vector $v \in M$ is, for each index i , killed by all $F_i^{(s)}$ for s sufficiently large (resp., killed by $F_i^{(s)}$ and $E_i^{(s)}$ for s sufficiently large).

Next, suppose that V is any $\mathbb{U}_{q,K}$ -module. Define

$$(2.2.1) \quad \mathcal{F}(V) := \{v \in \bigoplus_{\lambda \in X} M_\lambda \mid E_i^{(r)} v = F_i^{(r)} v = 0 \ \forall i = 1, \dots, n \text{ and } \forall r \gg 0\}.$$

According to [A, p.5], the submodule $\mathcal{F}(V)$ is a Type 1 integrable $\mathbb{U}_{q,K}$ -module.⁸ We can now define

$$(2.2.2) \quad H_q^0(M) := \mathcal{F}(\text{Hom}_{\mathbb{B}_{q,K}}(\mathbb{U}_{q,K}, M)),$$

for any Type 1 integrable $\mathbb{B}_{q,K}$ -module M . This yields a Type 1 integrable $\mathbb{U}_{q,K}$ -module which we call the induced module $\text{Ind}_{\mathbb{B}_{q,K}}^{\mathbb{U}_{q,K}}(M)$, later to be written in this paper as $\text{Ind}_{\mathbb{B}}^{\mathbb{U}}(M)$. (Andersen uses the word “induction,” but does not use our notation for the induced module, preferring instead $H_q^0(M)$.) In the definition of $H_q^0(M)$ above, left multiplication of $\mathbb{B}_{q,K}$ on $\mathbb{U}_{q,K}$ provides the $\mathbb{B}_{q,K}$ -module structure on $\mathbb{U}_{q,K}$, and a $\mathbb{U}_{q,K}$ -module structure on $\text{Hom}_{\mathbb{B}_{q,K}}(\mathbb{U}_{q,K}, M)$ is given by $uf(x) = f(xu)$ for all $u, x \in \mathbb{U}_{q,K}$ and $f \in \text{Hom}_{\mathbb{B}_{q,K}}(\mathbb{U}_{q,K}, M)$. The categories of Type I integrable $\mathbb{B}_{q,K}$ -modules and $\mathbb{U}_{q,K}$ -modules have enough injectives (as may be seen from the ring cases, applying the “largest Type 1 integrable submodule functors,” such as \mathcal{F} above) and, hence, the left exact functor $\text{Ind}_{\mathbb{B}_{q,K}}^{\mathbb{U}_{q,K}} = H_q^0$ has right derived functors $R^n \text{Ind}_{\mathbb{B}_{q,K}}^{\mathbb{U}_{q,K}} = H_q^n$.

⁸No argument is given in [A], noting the property is “not hard to check.” Perhaps this is true, once one knows how to do it. An argument for the case of (positive or negative Borel) subalgebras may be obtained with the method of [L, proof of Lem. 3.5.3], but using the generalized quantum Serre relations (through their corollary [L, Cor. 7.1.7]) in place of the quantum Serre relations. The case of the full quantum enveloping algebra then reduces to the rank 1 case, which can be handled with the formulas [L, 3.14(b),(c)].

Definition 2.3. For \leq the usual order on X determined by the positive roots R^+ , set $\mu, \lambda \in X$ to be **linked** if $\mu = w \cdot \lambda$ for some $w \in W_{D,\ell}$. If there is a chain $\lambda = \lambda_1, \dots, \lambda_s = \mu$ and a sequence $s_{\beta_1, m_1 \ell_{\beta_1}}, \dots, s_{\beta_{s-1}, m_{s-1} \ell_{\beta_{s-1}}}$ for which $\lambda_i \geq \lambda_{i+1} := s_{\beta_i, m_i \ell_{\beta_i}} \cdot \lambda_i$, $i = 1, \dots, s-1$, then μ is **strongly linked** to λ , denoted $\mu \uparrow_{D,\ell} \lambda$.

Remarks 2.4. (1) The relationship of strong linkage for weights X refines that of the usual ordering \leq . That is, $\mu \uparrow_{D,\ell} \lambda$ implies $\mu \leq \lambda$.

(2) In the analogous $\text{char}(k) = p > 0$ representation theory of algebraic groups, one defines $\mu \uparrow \lambda$ for weights $\mu, \lambda \in X$ by using the affine Weyl group $W_\ell, \ell = p$, in place of $W_{D,\ell}$. In this circumstance, under mild restrictions on the prime p relative to the root system R , one has $W_p = W_{D,\ell}$, by Proposition 2.1(1).

Let \mathcal{C}_q denote the category of Type 1 integrable $U_{q,K}$ -modules. The following fundamental result ultimately yields a splitting of \mathcal{C}_q into a direct sum of blocks associated to orbits of an appropriate affine Weyl group. For application to the Induction Theorem 1, we will just need the version \mathcal{U} of $U_{q,K}$ described below (2.0.4) in which case $W_{D,\ell} = W_\ell$. We will then focus on $\text{block}(\mathcal{U})$, the **principal block** of \mathcal{C}_q , corresponding to the orbit $W_{D,\ell} \cdot 0$. (Composition factors $L_q(\mu)$ of modules in the block are indexed by dominant weights μ in the orbit.) However, the results below hold more generally. They are claimed in [A] under the standing hypotheses of this subsection on q, K, ℓ , even with K allowed to have positive characteristic. However, it should be pointed out that the only reference given in support of one key auxiliary result [A, Thm. 2.1], a Grothendieck vanishing theorem needed in the proofs, is to the paper [AW]. The latter has as one of its standing assumptions that ℓ be odd, and not divisible by 3 in case the root system has a component of type G2. This assumption on ℓ is, of course, satisfied by our \mathcal{U} , so we have not pursued the issue further. Possibly, it was the intent of Andersen to claim that the argument in [AW] worked in the more general set-up of [A], though there is no explicit comment to that effect.

Theorem 3. (1) (Strong Linkage Principle [A, Theorem 3.1, Theorem 3.13]) Let $\lambda \in X^+ - \rho$. (Thus, $\lambda + \rho \in X^+$.) Let $\mu \in X^+$. If $L_q(\mu)$ is a composition factor of some $H_q^i(w \cdot \lambda)$ with $w \in W$ and $i \in \mathbb{N}$, then $\mu \uparrow_{D,\ell} \lambda$.

(2) (Linkage Principle [A, Thm. 4.3, Cor. 4.4]). Let $\lambda, \mu \in X^+$. If $\text{Ext}_{\mathcal{C}_q}^1(L_q(\lambda), L_q(\mu)) \neq 0$, then λ is linked, but not equal to, μ . Consequently, if $M \in \mathcal{C}_q$ is indecomposable, then the highest weights of all composition factors of M are linked, and the category \mathcal{C}_q splits into blocks corresponding to the orbits for the dot action of $W_{D,\ell}$ on X^+ .

Proof. We refer the reader to [A] for the proofs, on which we make several remarks which may be helpful. First, note that there appears to be a serious misprint, an expression apparently carried over unintentionally to one result from a previous one, in the statement of the auxiliary result [A, Prop. 3.6]: In the expression “ $\langle \lambda, \alpha_i^\vee \rangle = -1$ ”, the subexpression “ $= -1$ ” should be replaced with “ ≥ 0 ”.

Next, note that the exact sequences labeled (3) and (4) of [A, p.8] exist (and are later used) in the case $s = 1$ of the discussion there, with all terms in both exact sequences equal to zero. (The reader might have been led by the wording to think these sequences were defined only for $s > 1$.)

Next, there is an organizational issue on [A, p.10]. The first three lines of the proof of [A, Cor. 3.8] do not use the “minimality” hypothesis of that corollary, and are implicitly quoted later on the same page, in the proof of [A, Thm. 3.8], where it is claimed “we have already checked the result for $w = 1$.”

There are further minor points which occur on the same page [A, p.10]. In one repeated case, the vanishing of H_q^0 on 1-dimensional nondominant B -modules is given without proof, or hint. One approach that works is to use the version proved in the rank 1 case, then use induction from a corresponding parabolic subalgebra (and a Grothendieck spectral sequence).

At another place on the same page, [APW] is quoted to help determine, using a Weyl group action, the highest weight of a module $H_q^0(k_\lambda)$. However, an alternate argument may be given directly from the induced module definition. Quoting [APW] in this context is undesirable, because of the (implicit) restrictive set-up of that paper regarding ℓ . A similar issue, which we already noted above, before the

statement of theorem, regards the reference to [AW] for a proof of [A, Thm. 2.1]. As noted above, the generality of the [AW] set-up is sufficient for applications in this paper.

Finally, the “splitting into blocks” is justified in [A] by corollary [A, Cor. 4.4]. Both the corollary and the splitting are made a straightforward consequence of [A, Thm. 4.3] by the local finiteness of Type 1 integrable modules, for which we refer ahead to Proposition 2.8 below. Q.E.D.

The paper [A] gives some history of the Theorem 3, most of which had been proved piecemeal previously by Andersen and his students and collaborators. There is, of course, a completely corresponding theorem—first proved in full generality by Andersen— for semisimple algebraic groups, as discussed in Jantzen’s book [J]. With a certain amount of hindsight, some conceptual similarities can be imposed on the proofs and statements of supporting results. In particular, the presentation in [J] of Strong Linkage for the algebraic groups case ([J, II 6.13]), working with X^+ rather than $X^+ - \rho$, breaks the proof down into a lemma and two propositions [J, II 6.15, 6.16]. We have combined these propositions into an $X^+ - \rho$ quantum analogue stated below. We need this extra detail (for X^+) in order to provide more precise information about the appearance of irreducible modules $L_q(\mu)$ as composition factors in appropriate cohomology modules. $H_q^i(\nu)$. The Theorem [A, Thm. 2.1], discussed above, is needed in the proof (beyond the use of Strong Linkage).

Proposition 2.5. (1) *Let $i \in \mathbb{N}$ and $w \in W$. If $L_q(\mu)$, $\mu \in X^+$ is a composition factor of $H_q^i(w \cdot \lambda)$ with $\lambda \in X^+ - \rho$, then $\mu \uparrow_{D,\ell} \lambda$. If $\ell(w) \neq i$, then $\mu < \lambda$.*
 (2) *Suppose $\lambda \in X^+$. Then $L_q(\lambda)$ is a composition factor with multiplicity one of each $H_q^{\ell(w)}(w \cdot \lambda)$ with $w \in W$.*

Proof. The first part of item (1) just repeats Strong Linkage. The second part of item (1), and item (2), can be deduced from the approach in the proof of [A, Thm. 3.9]. Note that any weight μ in that proof which arises from the application of [A, Lem. 3.7] is strictly less than λ . As a consequence, the argument shows, in the presence of Strong Linkage, that the lemma holds for i and w if and only if it holds for $i + 1$ and sw (assuming $sw >$). This property can be applied repeatedly, moving i up or down. Using it, as in the first three lines of the proof of [A, Cor. 3.8], we obtain item (1) of the proposition. (This uses the discussed [A, Thm. 2.9].) Similarly, item (2) is reduced to the case $i = 0$ and $w = 1$. Here it follows by showing, directly from the definition, that λ has a 1-dimensional weight space in $H_q^0(\lambda)$. This completes the proof of the proposition. Q.E.D.

We remark that the discussions above of results in [A] corrects the proof of [ABG, Lem. 3.5.1]. This is with our choice of “negative” Borel subalgebras and our (standard) “dot” action. Both [A] and [APW] use “negative” Borel subalgebras as we do. The statement of the lemma given by [ABG] is apparently an attempt to use [APW] in a “positive” Borel subalgebra context, but the lemma is still incorrectly stated for that context. Also, they quote [APW] for the proof, though the latter paper does not contain as strong a result [ABG, Lem. 3.5.1] Instead, the main result [APW, Thm. 6.7] of its Borel-Weil-Bott section is a “lowest ℓ alcove” version, and explicitly requires that ℓ be a prime power.

The next corollary restates the main conclusions (those dealing with X^+) of the proposition above in a form handy for later use. As already indicated, the (completely analogous) algebraic groups version is the combination of the two propositions [J, II 6.15, 6.16].

Corollary 2.6. *If $\mu \in X$ and $\lambda = w \cdot \mu \in X(T)^+$ (i.e., λ is the dominant weight in the W -orbit of μ), then $L_q(\lambda)$ occurs just once as a composition factor of any of the modules $H_q^i(\mu)$, i running over all nonnegative integers. Precisely, one has $[H_q^i(\mu) : L_q(\lambda)] \neq 0$ only for $i = \ell(w)$, and then $[H_q^{\ell(w)}(\mu) : L_q(\lambda)] = 1$. If $\eta \in X^+$ with $\eta \neq \lambda$, then for all $i \in \mathbb{N}$, $[H_q^i(\mu) : L_q(\eta)] \neq 0$ implies $\eta < \lambda$ and that η is strongly linked to λ*

The proposition below is quite important for applications, especially in the next subsection. There is a completely analogous result for induction from Borel subgroups in reductive algebraic groups, a special case of [J, II,4.2].

Proposition 2.7. ([A, Thm. 3.9, Thm. 2.1]) *Let $\mu \in X$. Then $H_q^i(\mu)$ has finite dimension over K , and vanishes for $i > N$, the number of positive roots.*

Proof. We ask the reader again to read [A] for proofs, after first reviewing our comments on the proof of Theorem 3 above. Q.E.D.

The final proposition in this section, also useful in the next section, is an analogue in the generality of [A] of [APW, Cor. 1.28] and of [L, Prop. 32.1.2]. It does not appear to be implied by either of these latter results, however.

Proposition 2.8. *Let M be any Type 1 integrable $U_{q,K}$ -module. Then M is locally finite, in the sense that each vector $v \in M$ generates a finite dimensional $U_{q,K}$ -module. Similarly, Type 1 integrable $B_{q,K}$ -modules are locally finite.*

Proof. For this proof, let U_j^+ denote, for each index j , the \mathcal{Z} span (a subalgebra) in $U_{\mathcal{Z}}$ of all the elements $E_j^{(s)}$, $s \in \mathbb{N}$, with a similar notation for U_j^- . Lusztig constructs his PBW-type basis [L3, Thm. 6.7] for the quantum enveloping algebra $U_{\mathcal{Z}}$ using (finitely many) compositions of his explicit braid group automorphisms T_i , applied to the various U_j^{\pm} . This process yields, for each positive root α , \mathcal{Z} -subalgebras U_{α}^{\pm} , and the whole quantum algebra $U_{\mathcal{Z}}$ is a (ring-theoretic) product of finitely many of these, together with $U_{\mathcal{Z}}^0$.

In several formulas listed in [L, 37.1.3] Lusztig gives explicit formulas for several similar automorphisms, including their action on basis elements of each U_j^{\pm} . The setting for the action of these automorphisms is a $\mathbb{Q}(v)$ -algebra \mathbf{U} containing the algebra we have called U'_v ; moreover, the action of these automorphisms on the various elements K_i (in the notation of [L]) shows that all these automorphisms act bijectively on U'_v . It is easy to pick out the braid group automorphism T_i defined in [L3] in this context, as (the restriction to U'_v of) $T''_{i,1}$ in [L, 37.1.2]). Accordingly, we learn, for each index j , that $T_i(U_j^{\pm})$ is contained in a product of $U_{\mathcal{Z}}^0$ and at most three \mathcal{Z} subalgebras, each of the latter having the form $U_{j''}^{\pm}$, for some index j'' (in $1, \dots, n$). To this information we add the fact that T_i stabilizes U_0 , which may be deduced from [L3, Thm.3.3, Thm.6.6(ii), Thm.6.7(c)].

It follows now that $U_{\mathcal{Z}}$ is a product of finitely many of the various subalgebras U_j^{\pm} , together with $U_{\mathcal{Z}}^0$. However, it is obvious that, if V is any finite-dimensional subspace of M , then any (ring-theoretic) product $U_j^{\pm}V$ is finite-dimensional. Repeated application of this fact completes the proof of the proposition. Q.E.D.

2.6. Some derived category considerations. We finally begin to use the assumptions and notation first given below (2.0.4, which the reader should review at this point). The notations include a common notation U for a quantum enveloping algebra, specialized at an ℓ^{th} root of unity, and the distribution algebra of a simply connected semisimple algebraic group G . There are similar common notations associated to various subalgebras of U , and distribution algebras associated to subgroups of G , such as the (negative) Borel subgroup B . Both $p > h$ and $\ell > h$ are required, and there are further conditions on ℓ . (It must be odd, and not divisible by 3 when the root system of U has a component of type G2).

In addition, we introduce here the notations $\mathcal{C}_U, \mathcal{C}_B, \dots$ for the categories of Type 1 integrable U, B, \dots -modules, respectively, in the quantum case. In the algebraic groups case, the same notations reference the categories of rational G, B, \dots -modules, respectively. These latter categories may be rewritten, according to our conventions for naming distribution algebras, as the categories of rational U, B, \dots -modules. Here, “locally finite” would be a more accurate term than “rational,” but we will use either term in unambiguous contexts.

This section provides a starting point for the proof of the Induction Theorems 1 and 2, the latter as reformulated in Theorem 2.1. The result below, a corollary of the those in the previous subsection, is the starting point. The statement and proof work in both the quantum and algebraic groups context, in the notation discussed above.

By $block(\mathcal{U})$ we mean the category of finite dimensional modules in the principal block of $\mathcal{C}_{\mathcal{U}}$; equivalently, it is the full subcategory of all finite-dimensional modules whose composition factors have highest weights in $W_{\ell} \cdot 0$ (taking $\ell = p$ in the algebraic groups case).

By $D^b block(\mathcal{U})$ we mean the bounded derived category of the abelian category $block(\mathcal{U})$, as defined by Verdier—see, for example, [Ha, Chapter I]. Let $D_{block(\mathcal{U})}^b(\mathcal{C}_{\mathcal{U}})$ denote the full subcategory of $D^b(\mathcal{C}_{\mathcal{U}})$ consisting of objects which have each of their (finitely many) cohomology groups in $block(\mathcal{U})$. Using the local finiteness of rational modules (see Proposition 2.8 in the quantum case), we observe the following lemma.

Lemma 2.9. *The natural map $D^b block(\mathcal{U}) \rightarrow D^b(\mathcal{C}_{\mathcal{U}})$, arising from the inclusion functor at the abelian category level, induces an equivalence*

$$D^b block(\mathcal{U}) \cong D_{block(\mathcal{U})}^b(\mathcal{C}_{\mathcal{U}})$$

Proof. Let K^{\bullet} be a bounded complex of objects in $\mathcal{C}_{\mathcal{U}}$ with each cohomology group belonging to $block(\mathcal{U})$. We claim there is a bounded subcomplex F^{\bullet} of K^{\bullet} , with finite dimensional objects in $\mathcal{C}_{\mathcal{U}}$ in each degree, such that the inclusion map $F^{\bullet} \subseteq K^{\bullet}$ is a quasi-isomorphism. To construct the subcomplex F^{\bullet} , we may assume, inductively, its terms in all degrees $\geq i$ are constructed, so that they form a subcomplex $F^{\geq i}$. In addition, we require, inductively, that inclusion of this complex into K^{\bullet} induces an isomorphism on cohomology in grades $> i$ and an epimorphism on the i^{th} cohomology groups. Then, we wish to construct $F^{i-1} \subseteq K^{i-1}$ so that the resulting complex $F^{\geq i-1}$ has the analogous properties for $i-1$ in place of i . Let δ denote the differential $K^{i-1} \rightarrow K^i$. Then $\delta(K^{i-1}) \cap F^i$ is, of course, both finite and contained in the image of δ . Choose a finite dimensional subspace E of K^{i-1} such that $\delta(E) = \delta(K^{i-1}) \cap F^i$. Also choose a finite dimensional subspace E' of the Kernel of δ such that the image of E' in the natural surjection $Ker \delta \rightarrow H^{i-1}(K^{\bullet})$ is all of $H^{i-1}(K^{\bullet})$. Take F^{i-1} to be the \mathcal{U} -module generated by $E + E'$. Then $\delta(F^{i-1}) = \delta(K^{i-1}) \cap F^i$. Consequently, inclusion induces a monomorphism $H^i(F^{\geq(i-1)}) \rightarrow H^i(K^{\bullet})$. The (downward) induction hypothesis implies the same map is a surjection, so it must be an isomorphism. Our construction of F^{i-1} gives a surjection of H^{i-1} for the same inclusion of complexes. The inductive step may be repeated, eventually reaching cohomological degrees j where K^j and all lower degree terms are 0. At that point we may take F^j also zero, and zero in lower degrees. This gives that $F^{\bullet} \subseteq K^{\bullet}$ induces an isomorphism on all cohomology groups. That is, it induces a quasi-isomorphism, as required in the claim.

We remark further, that, by taking block projections, the complex F^{\bullet} may be assumed to consist in each degree of objects in $block(\mathcal{U})$.

The lemma proposes that the natural map induces an equivalence. It follows from the claim and remark that every object on the right-hand side of the proposed equivalence is, indeed, in the strict image of the left hand side. It remains to show the natural map induces a full embedding at the derived category morphism level. For this, observe the claim above can be strengthened so that the constructed complex F^{\bullet} contains any given finite dimensional subcomplex N^{\bullet} of K^{\bullet} . (Strengthen the induction hypothesis in the proof by adding the assertion $N^{\geq i} \subseteq F^{\geq i}$. Then, at the inductive step, replace E by $E + N^{i-1}$; this does not effect $\delta(E)$, since $\delta(N^{i-1}) \subseteq \delta(K^{i-1}) \cap F^i$.) As before, if N^{\bullet} is a complex of objects in the principal block, we may assume the complex F^{\bullet} constructed is also a complex of objects in the principal block.

Taking the same idea yet another step further, we can even assume F^{\bullet} contains any given finite number of subcomplexes like N^{\bullet} , since the sum of any number of finite subcomplexes of K^{\bullet} is again a finite subcomplex.

Now use the standard direct limit constructions (in the second variable) of derived category morphisms. Here we mean the Verdier localization construction of derived categories (and bounded derived categories), which proceeds (first) by localization of homotopy categories of complexes. (See [Ha, pp.32,37].) In particular any morphism on the right hand side from an object M_1^{\bullet} to M_2^{\bullet} is represented by an object K^{\bullet} and two morphisms $M_1^{\bullet} \rightarrow K^{\bullet}$ and $M_2^{\bullet} \rightarrow K^{\bullet}$, the latter a quasi-isomorphism. Now

assume M_1^\bullet and M_2^\bullet are finite dimensional complexes of objects in $\text{block}(\mathcal{U})$, and let N_1^\bullet and N_2^\bullet be their respective images in K^\bullet . Applying the strengthened versions of the claim, above, we may construct a finite dimensional complex F^\bullet containing both N_1^\bullet, N_2^\bullet , and contained in K^\bullet . Moreover, the latter inclusion is constructed to be a quasi-isomorphism. It follows that the pair of morphisms $M_1^\bullet \rightarrow F^\bullet$ and $M_2^\bullet \rightarrow F^\bullet$ represent a derived category morphism on the left hand side of the display in the lemma. This proves the surjectivity required in the full embedding property at the morphism level.

It remains to prove injectivity. Suppose we are given a morphism on the left hand side of the display which becomes zero on the right hand side. The morphism on the left may be represented by the following configuration: We are given M_1^\bullet, M_2^\bullet and J^\bullet , all finite dimensional complexes of objects in $\text{block}(\mathcal{U})$, and a pair of morphisms $M_1^\bullet \rightarrow J^\bullet$ and $M_2^\bullet \rightarrow J^\bullet$, the latter a quasi-isomorphism. To say that the derived category morphism represented by this configuration becomes zero, when considered on the right hand side, means the following: There is a complex K^\bullet of objects in $\mathcal{C}_{\mathcal{U}}$ and a quasi-isomorphism $J^\bullet \rightarrow K^\bullet$ such that the composite map of complexes $M_1^\bullet \rightarrow J^\bullet \rightarrow K^\bullet$ is homotopy equivalent to zero. Let $h = \{h_i\}_{i \in \mathbb{Z}}$ be a family of maps defining the homotopy in question. That is, each $h_i : M_1^i \rightarrow K^{i-1}$ is a morphism in $\mathcal{C}_{\mathcal{U}}$, and $\delta_{K^\bullet} \circ h + h \circ \delta_{M_1^\bullet}$ is the given map $M_1^\bullet \rightarrow J^\bullet \rightarrow K^\bullet$. Here the subscripted symbols δ denote the evident families of differentials. Observe that the sum L^\bullet over i of all $\delta_{K^{i-1}} \circ h_i(M_1^i) + h_i(M_1^i)$ is a finite dimensional subcomplex of K^\bullet , with all of its objects and differentials in $\text{block}(\mathcal{U})$. Using the extended claims above, we can construct a finite dimensional $\text{block}(\mathcal{U})$ -complex F^\bullet , contained in K^\bullet as a $\mathcal{C}_{\mathcal{U}}$ -subcomplex, and itself containing each of L^\bullet , the image of J^\bullet in K^\bullet , and the image of M_1^\bullet in K^\bullet (already in L^\bullet , actually). In addition, the above constructions allow us to assume that the inclusion $F^\bullet \subseteq K^\bullet$ is a quasi-isomorphism. It follows that $J^\bullet \rightarrow F^\bullet$ is a quasi-isomorphism. Consequently, the derived category morphism (viewed as a direct limit) represented by the original configuration is also represented by the pair of maps $M_1^\bullet \rightarrow J^\bullet \rightarrow F^\bullet$ and $M_2^\bullet \rightarrow J^\bullet \rightarrow F^\bullet$. However, the map $M_1^\bullet \rightarrow J^\bullet \rightarrow F^\bullet$ is visibly homotopic to zero, using the same function h to define the required homotopy. (By construction $h_i(M_1^i) \subseteq F^{i-1}$ for each i .) Thus, the morphism represented by the original configuration is zero in its associated direct limit. This proves the required injectivity and completes the proof of the lemma.

Q.E.D.

It is suggested below [A, Defn. 3.5.6] that, in the quantum case, $\text{block}(\mathcal{U})$ is “known” to have enough injectives. There is such a result about injectives in [APW]. But the contest, while possibly too restrictive, ostensibly applies only to the cases where ℓ is a prime power, as do the discussions in [APW2]. Our argument above does not depend on such a property, and, indeed, applies to the algebraic groups case, where there are no finite-dimensional injectives.

Though it is somewhat informal, we henceforth identify $D^b\text{block}(\mathcal{U})$ and $D^b_{\text{block}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}})$ through the isomorphism above. This language is used in the result below.

Corollary 2.10. (i) For any $\lambda \in \mathcal{Y}$, $R\text{Ind}_{\mathbb{B}}^{\mathcal{U}}(l\lambda) \in D^b\text{block}(\mathcal{U})$.

(ii) The category $D^b\text{block}(\mathcal{U})$ is generated, as a triangulated category, by the family of objects $\{R\text{Ind}_{\mathbb{B}}^{\mathcal{U}}(l\lambda)\}_{\lambda \in \mathcal{Y}}$.

Proof. The linkage principle, Corollary 2.6 and Proposition 2.7 imply item (i) above. Corollary 2.6, used with induction on weights and standard cohomological degree truncation operators [BBD, p.29], implies item (ii). Q.E.D.

Remarks 2.11. (a) The result above is stated as Cor. 3.5.2 in [ABG] in their quantum enveloping algebra set-up. Their proof, overall, relies on similar considerations, though some of the references supplied to [APW] for their preparatory lemma [ABG, Lem. 3.5.1] are inaccurate. It is not clear if their proof applies to the case where ℓ is not a prime power.

(b) This is perhaps a good point to mention that Kempf’s vanishing theorem, well-known in the algebraic groups case [J, II, §4], also holds [AW, Thm. 5.3] in the quantum case under the hypotheses of this section. Thus, the higher derived functors $R^n\text{Ind}_{\mathbb{B}}^{\mathcal{U}}(\mu)$ are zero for μ dominant and $n > 0$. The

generalized tensor identities [J, I, Prop.4.8] also work here [AW, Prop.4.7]. These results are stated using individual higher derived functors R^n in each degree $n \geq 0$, but their proofs show that there are isomorphisms $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M \otimes N|_{\mathbb{B}}) \cong \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M) \otimes N$ whenever $M \in \mathcal{C}_{\mathbb{B}}$ and $N \in \mathcal{C}_{\mathbb{U}}$. These isomorphisms may also be deduced from the natural maps in (d) below, applied with M replaced by an injective resolution.

(c) The roles of left and right in the tensor identity may be reversed. (See (d) below. The first argument for such a reversal is probably that for [APW, Prop. 2.7].) Also, although the quantum algebras we deal with are not generally cocommutative (\mathbb{U}^0 being an exception), the orders of tensor products of integrable modules we deal with can often be interchanged (up to isomorphism). This holds in particular for tensor products of finite dimensional modules in $\mathcal{C}_{\mathbb{U}}$. See [L, 32.16]. We have, however, not investigated the naturality properties this reversal may or may not have. The reversal is natural in the tensor identity case, as can be seen from (d) below. If M there is then replaced by a complex of injective modules, a natural reversal is obtained in the generalized tensor identity case.

(d) It is sometimes useful to have explicit natural isomorphisms

$$\begin{aligned} \alpha_{M,N} : \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(M \otimes N|_{\mathbb{B}}) &\longrightarrow \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(M) \otimes N; \text{ and} \\ \gamma_{N,M} : \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(N|_{\mathbb{B}} \otimes M) &\longrightarrow N \otimes \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(M) \end{aligned}$$

where $M \in \mathcal{C}_{\mathbb{B}}, N \in \mathcal{C}_{\mathbb{U}}$. We give such natural isomorphisms for the convenience of the reader:

Drop the subscripts M, N and regard both modules in the top row as a contained in $\mathrm{Hom}_k(\mathbb{U}, M \otimes N)$.

We have, for $f \in \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(M \otimes N|_{\mathbb{B}})$, $x \in \mathbb{U}$,

$$\alpha(f)(x) = (1 \otimes S(x_2))f(x_1)$$

in (implicit sum) Sweedler notation. (Thus $\Delta(x) = x_1 \otimes x_2$, a sum over an invisible implicit index shared by x_1, x_2 .) To check B-equivariance of $\alpha(f)$, let $h \in \mathbb{B}$. Then

$$\begin{aligned} \alpha(f)(hx) &= (1 \otimes S((hx)_2))f((hx)_1) \\ &= (1 \otimes S(x_2)S(h_2))f(h_1x_1), \end{aligned}$$

where the last line is a sum over two implicit and independent indices, one for the x 's and one for the h 's. Continuing, we obtain further similar expressions

$$\begin{aligned} &= (1 \otimes S(x_2)S(h_3))(h_1 \otimes h_2)f(x_1) \\ &= (1 \otimes S(x_2))(h_1 \otimes S(h_3h_2))f(x_1) \\ &= (1 \otimes S(x_2))(h_1 \otimes S\epsilon(h_2))f(x_1) \\ &= (1 \otimes S(x_2))(h_1 \otimes 1)f(x_1) \\ &= (h \otimes 1)(1 \otimes S(x_2))f(x_1) \\ &= (h \otimes 1)\alpha(x) \end{aligned}$$

which is the desired equivariance. Notice the top line is, for any fixed x_1, x_2 , the image of $h \in \mathbb{B}$ under a linear map $\mathbb{B} \xrightarrow{\Delta} \mathbb{B} \otimes \mathbb{B} \otimes \mathbb{B} \longrightarrow M$. Here Δ denotes, with some abuse of notation, the map usually denoted $(1 \otimes \Delta) \circ \Delta$ or $(\Delta \otimes 1) \circ \Delta$, with $\Delta : \mathbb{B} \longrightarrow \mathbb{B} \otimes \mathbb{B}$ the comultiplication. We write both $\Delta(h) = h_1 \otimes h_2$ and $\Delta(h) = h_1 \otimes h_2 \otimes h_3$, depending on context. The inverse β of α is given, for $g \in \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(M) \otimes N \subseteq \mathrm{Hom}_k(\mathbb{U}, M \otimes N)$, $x \in \mathbb{U}$, by

$$\beta(g)(x) = (1 \otimes x_2)g(x_1).$$

We leave it to the reader to check that $\beta(g)$ satisfies the appropriate B-equivariance (by an argument similar in spirit to that for α), and that β is inverse to α . The formula for γ , is for $f \in \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(N|_{\mathbb{B}} \otimes M) \subseteq \mathrm{Hom}_{\mathit{mathbb{b}k}}(\mathbb{U}, N \otimes M)$, $x \in \mathbb{U}$,

$$\gamma(f)(x) = (S(x_1) \otimes 1)f(x_2)$$

For the inverse δ of γ it is, for $g \in N \otimes \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(M) \subseteq \mathrm{Hom}_{\mathit{mathbb{b}k}}(\mathbb{U}, N \otimes M)$, $x \in \mathbb{U}$,

$$\delta(g) = (x_1 \otimes 1)g(x_2)$$

Again, the reader may check, with arguments similar in spirit to those illustrated, that γ and δ satisfy the appropriate equivariance properties and are inverse to each other.

(e) We remark that $Ind_{\mathbb{B}}^{\mathbb{U}}(M)$ is equipped with a natural “counit” $\epsilon_M : Ind_{\mathbb{B}}^{\mathbb{U}}(M)|_{\mathbb{B}} \rightarrow M$ which, as is well-known (see Wikipedia) may be used its property of being right adjoint to restriction. In the full module categories for \mathbb{U} and \mathbb{B} ϵ_M may be given as evaluation at 1 on the right adjoint $\text{Hom}_{\mathbb{B}}(\mathbb{U}, M)$, and it follows that ϵ_M may be similarly interpreted for $Ind_{\mathbb{B}}^{\mathbb{U}}(M)$ when dealing with integrable modules. We only want to observe here that there is a similar “evaluation at 1” counit for each of the modules $Ind_{\mathbb{B}}^{\mathbb{U}}(M \otimes N|_{\mathbb{B}})$, $Ind_{\mathbb{B}}^{\mathbb{U}}(M) \otimes N$, $Ind_{\mathbb{B}}^{\mathbb{U}}(N|_{\mathbb{B}} \otimes M)$, $N \otimes Ind_{\mathbb{B}}^{\mathbb{U}}(M)$ above, providing each of these constructions with the structure of a right adjoint to restriction. The proof is easy, noting the isomorphisms in (d) commute with evaluation at 1 on the ambient $\text{Hom}_{\mathbb{B}}(\mathbb{U}, -)$ module. Rewriting this fact in the ϵ notation, we have that $\epsilon_M \otimes N|_{\mathbb{B}}$ and $N|_{\mathbb{B}} \otimes \epsilon_M$ are counits (that is, provide a right adjoint structure) for $Ind_{\mathbb{B}}^{\mathbb{U}}(M) \otimes N$, $N \otimes Ind_{\mathbb{B}}^{\mathbb{U}}(M)$, respectively. We will use this fact in Appendix B.

(f) Finally, we explain briefly how the induction functors we have used above, based on the formalism in [A] and compatible with [APW], fit with the algebraic groups formalism in [J, I,3.3]. Actually, the original definition [CPS77, §1] of induction $Ind_{\mathbb{B}}^G(M)$ in the algebraic groups case, for a finite-dimensional rational G -module M , was the set $\text{Morph}_B(G, M)$ of B -equivariant morphisms from G to M , with an evident direct union used for a general rational module M . This definition is formally quite close to the [A] definition in the quantum case. Using Sullivan’s theorem [CPS80, Thm. 6.8], that all locally finite $\text{Dist}(G)$ modules are rational, it is easy to see that this definition coincides with the definition of [A] used above, with $\text{Dist}(G)$ in the role of \mathbb{U} and $\text{Dist}(G)$ in the role of \mathbb{B} . Finally, to connect the [CPS77] definition with that of [J], simply replace the $B \times G^{op}$ action $(b, g)x = bxg$ on G with the isomorphic action $(b, g)x = g^{-1}xb^{-1}$, where $b \in B$ and $x, g \in G$.

2.7. Some Special Twisted Induced Modules. In this subsection and the next, we will adapt the “uniform” notation for the quantum and positive characteristic cases introduced in the discussion of Theorem 2.1 and elaborated in subsection 2.2. We will presume and utilize the definitions for the quantum Frobenius morphism as in, e.g., [L3, Thm. 8.10], [L]. We use a similar notation in positive characteristic, where the Frobenius morphism originated and is well-known.

Remarks 2.12. In the quantum case, the Frobenius morphism is a homomorphism $\varphi : \mathbb{U} \rightarrow \text{Dist}(G')$, where $\text{Dist}(G')$ is the distribution algebra (over K) of an algebraic group G' (semisimple, simply connected, and defined and split over K , with the same root datum as \mathbb{U}). If M is a rational G' -module over K , we may twist it through φ and obtain an integral \mathbb{U} module ${}^{\varphi}M$, trivial on \mathfrak{u} . We will use the notation $M^{[1]} := {}^{\varphi}M$, and the same notation for twisting a module through the Frobenius in the corresponding characteristic p algebraic groups situation (where $\mathbb{U} = \text{Dist}(G)$ is both the domain and the target of the Frobenius homomorphism).

Returning to the quantum case, we remind the reader of our notation $\mathfrak{p} = \mathfrak{b} \cdot \mathbb{U}^0$ (and that this notation is similar to that in [ABG], except that our \mathfrak{b} is associated to negative roots). The Frobenius homomorphism is compatible with triangular decompositions of its domain and target; see [L3, Thm. 8.10]. So, the above $M^{[1]}$ notation also makes sense, if M is (in the obvious analogous notation) a rational B' or T' -module. This results, respectively, in an integral \mathbb{B} or \mathfrak{p} module $M^{[1]}$, trivial as a \mathfrak{b} -module. (There is some ambiguity of notation here: if M is not obviously a G' -module, we deliberately do not include all of \mathfrak{u} as part of the domain of definition of $M^{[1]}$ without explicit mention otherwise.) Conversely, we claim any integrable module N for \mathbb{B} or \mathfrak{p} , which is trivial for \mathfrak{b} , has, respectively, this form, and in a unique way. The corresponding assertion for \mathbb{U}^- for modules trivial for \mathfrak{u}^- follows from the (negative root analogs of) [L3, Lem.s 8.8, 8.9]. (Note also from these results that the Kernel ideal of the Frobenius homomorphism on \mathbb{U}^- is the left \mathbb{U}^- ideal generated by the augmentation ideal of \mathfrak{u}^- . Similarly, the Kernel must be the right ideal generated by this augmentation ideal.) At the level of \mathbb{U}^0 for modules trivial on \mathfrak{u}^0 it follows from the explicit form of the Frobenius homomorphism on \mathbb{U}^0 in [*op cit*, p.110, bottom] and the monomial bases [*op cit*, Thm. 6.7(c), Thm.8.3(ii)] for \mathbb{U}^0 and \mathfrak{u}^0 , respectively. The claim follows.

We remark that the parenthetic note above shows an interesting additional property: If N is any integrable \mathbb{B} module, with unspecified action of \mathfrak{b} , the largest \mathfrak{b} -module quotient of N with trivial \mathfrak{b} -module action is naturally a (twisted) \mathbb{B} -module, call it $M^{[1]}$. Consequently, if E is any T' -module, then, using rational induction for algebraic groups, $\mathrm{Hom}_{\mathbb{B}}(N, \mathrm{Ind}_{T'}^{B'}(E)^{[1]}) \cong \mathrm{Hom}_{\mathbb{B}}(M^{[1]}, \mathrm{Ind}_{T'}^{B'}(E)^{[1]}) \cong \mathrm{Hom}_{B'}(M, \mathrm{Ind}_{T'}^{B'}(E)) \cong \mathrm{Hom}_{T'}(M|_{\mathfrak{p}'}, E) \cong \mathrm{Hom}_{\mathfrak{p}}(N|_{\mathfrak{p}}, E^{[1]})$. This shows the functor sending $E^{[1]}$ to $\mathrm{Ind}_{T'}^{B'}(E)^{[1]}$ serves, on appropriate categories of twisted integrable modules, as a right adjoint to restriction on corresponding integrable (but not necessarily twisted) categories. In fact, a right adjoint $\mathrm{Ind}_{\mathfrak{p}}^{\mathbb{B}}$ on the full integrable categories is constructed in [ABG, §2.7]. (There is a similar construction in [APW2], but the set-up there ostensibly requires ℓ to be a prime power.) The adjointness properties observed above in this paragraph imply

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathbb{B}}(E^{[1]}) \cong \mathrm{Ind}_{T'}^{B'}(E)^{[1]},$$

for E any rational T' -module.

A completely adequate version of the isomorphism is noted without proof in [ABG, (2.8.2)], presumably based on [ABG, Lem. 2.6.5], which discusses also details of the Frobenius homomorphism at the \mathbb{B} module level. The main application in both [ABG] and this paper occurs with $E^{[1]}$ a 1-dimensional \mathfrak{p} module $k(\ell\lambda) := k_{\mathfrak{p}}(\ell\lambda)$, with $\lambda \in \mathcal{Y}$.

We note further that, once an induction $\mathrm{Ind}_{\mathfrak{p}}^{\mathbb{B}}$ is available (as a right adjoint to restriction from $\mathcal{C}_{\mathbb{B}}$ to $\mathcal{C}_{\mathfrak{p}}$), an induction functor $\mathrm{Ind}_{\mathfrak{p}}^{\mathbb{U}}$ can be constructed as the composition $\mathrm{Ind}_{\mathfrak{p}}^{\mathbb{U}} \circ \mathrm{Ind}_{\mathfrak{p}}^{\mathbb{B}}$.

We conclude these remarks by noting that all features of the above paragraphs have obvious parallels that hold in the (characteristic p) algebraic groups case, with $G = G'$, etc. We continue this dual use of the notations G', \dots below.

Adapting [ABG, §4.3] to our negative Borel framework, set

$$(2.12.1) \quad I_{\mu} := \mathrm{Ind}_{T'}^{B'}(\mu) \cong \varinjlim_{\nu \in \mathcal{Y}^{++}} (V_{\nu}|_{B'} \otimes k_{B'}(\mu + \nu))$$

where V_{ν} denotes the ‘‘costandard’’ G' -module $\mathrm{Ind}_{B'}^{G'}(-w_0\nu) = H^0(-w_0\nu)$ with highest weight $-w_0\nu$. We take μ to be any weight in \mathcal{X} , though we will only use the case $\mu \in \mathcal{Y}$. In the quantum case, G' is a semisimple algebraic group in characteristic 0, so V_{ν} is irreducible, though we will not need that to explain the isomorphism in 2.12.1, which we do now:

In general, the lowest weight of V_{ν} is $-\nu$, appearing with multiplicity 1, and $k_{B'}(-\nu)$ is the B' -socle of V_{ν} . Let ω be in \mathcal{Y}^{++} , so that $\nu + \omega$ is also in \mathcal{Y}^{++} , and is ‘‘larger’’ than ν if $\omega \neq 0$. This defines an evident directed system of weights. There is a natural homomorphism of G' -modules $V_{\nu} \otimes V_{\omega} \rightarrow V_{\nu+\omega}$ which is an isomorphism on highest (and, applying w_0 , on lowest) weight spaces. In particular, the induced homomorphism of B' -modules $V_{\nu} \otimes k_{B'}(-\omega) \rightarrow V_{\nu+\omega}$ is an isomorphism on B' -socles, hence injective. Tensoring on the right with 1-dimensional modules $k_{B'}(\nu + \omega + \mu)$ gives injections

$$V_{\nu} \otimes k_{B'}(\nu + \mu) \rightarrow V_{\nu+\omega} \otimes k_{B'}(\nu + \omega + \mu).$$

For fixed μ these describe the directed system underlying the direct limit in 2.12.1, and shows it is a directed system of injections, all with a common socle $k_{B'}(\mu)$ and with the weight μ appearing with multiplicity 1. In particular, the direct limit exists as a rational B' module I and has the same socle $k_{B'}(\mu)$. Consequently, there is a map $I \rightarrow \mathrm{Ind}_{T'}^{B'}(k(\mu)) = I_{\mu}$ which is an isomorphism on socles. (The induced module definition of I_{μ} shows its only 1-dimensional submodule is $k_{B'}(\mu)$.) Thus, $I \subseteq I_{\mu}$. To get equality, it is enough to show that, for any weight τ of I_{μ} , there is a $\nu \in \mathcal{Y}^{++}$ such that the τ weight space of $V_{\nu} \otimes k_{B'}(\nu + \mu)$ has dimension equal to that of the τ weight space of I_{μ} . The (weight space by weight space) linear dual I_{μ}^* of I_{μ} is (after conversion to a left B' -module) generated by its (1-dimensional) $-\mu$ weight space, call it kv . Thus, $I_{\mu}^* = \mathrm{Dist}(U'^{-})v$ may be viewed as the homomorphic image of $M(\nu)|_{B'} \otimes k_{B'}(-\nu - \mu)$, where $M(\nu)$ denotes the Verma module for $\mathrm{Dist}(G')$ with highest weight ν . The linear dual of the injection $V_{\nu} \otimes k_{B'}(\nu + \mu) \rightarrow I_{\mu}$ gives a surjection $I_{\mu}^* \rightarrow \Delta(\nu)|_{B'} \otimes k_{B'}(-\nu - \mu)$,

where $\Delta(\nu)$ is the Weyl module of highest weight ν . Thus, we have a composition of surjections

$$M(\nu)|_{B'} \otimes \mathfrak{k}_{B'}(-\nu - \mu) \rightarrow I_\mu^* \rightarrow \Delta(\nu)|_{B'} \otimes \mathfrak{k}_{B'}(-\nu - \mu).$$

We want to show that the $-\tau$ weight space dimensions on the left and right (and, thus, also in the middle) are the same for some choice of ν . This is equivalent to showing that the $\nu + \mu - \tau$ weight spaces of the Verma and Weyl modules with highest weight ν are the same for some ν , given μ and τ . This dimension is obviously independent of the base field \mathfrak{k} in the Verma module case, and the same independence is true in the Weyl module case by [J, II,8.3(3)]. Over the complex numbers, the Kernel of the map $M(\nu) \rightarrow \Delta(\nu)$ is generated as a B' -module, by the elements $F_i^{N_i+1}v^+$, where v^+ is a highest weight vector, N_i is the coefficient of μ at the i^{th} fundamental weight, and F_i is a Chevalley basis root vector associated with the negative of the i^{th} fundamental root α_i . (All observed in [ABG, §4.3].) It is easy to choose all ν so that each coefficient of $\tau - \mu$ at α_i is smaller than $N_i + 1$. In this case the Kernel has a zero weight space for weight $\nu + \mu - \tau$. Thus, the dimensions of the weight spaces for this weight are the same in both the Verma and Weyl module, as desired. This proves $I = I_\mu$.

By applying Frobenius twists, one has

$$(2.12.2) \quad \text{Ind}_{\mathbb{P}}^{\mathbb{B}}(\ell\mu) \cong I_\mu^{[1]} = \text{Ind}_{T'}^{B'}(\mu)^{[1]} \cong \varinjlim_{\nu \in Y^{++}} (V_\nu^{[1]}|_{\mathbb{B}} \otimes \mathfrak{k}_{\mathbb{B}}(\ell\mu + \ell\nu)).$$

Definition 2.13. For any dominant weight σ define $J_{\sigma,\mu} = V_\sigma^{[1]}|_{\mathbb{B}} \otimes \mathfrak{k}_{\mathbb{B}}(\ell\mu + \ell\sigma)$.

In the lemma below, and elsewhere in this paper, we freely use “Ext n ” for a derived category “Hom n .”

Lemma 2.14. *Let Y be any finite dimensional \mathbb{B} -module, and μ any weight in Y (or X). Then, for sufficiently large σ , we have*

- (a) $\text{Ext}_{\mathbb{B}}^n(Y, J_{\sigma,\mu}) \cong \text{Ext}_{\mathbb{B}}^n(Y, I_\mu^{[1]})$, and
- (b) $\text{Ext}_{\mathbb{B}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), J_{\sigma,\mu}) \cong \text{Ext}_{\mathbb{B}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), I_\mu^{[1]})$

for all nonnegative integers n . In addition we have (independently of σ)

- (c) $\text{Ext}_{\mathbb{B}}^n(Y, I_\mu^{[1]}) \cong \text{Ext}_{\mathbb{P}}^n(Y, \mathfrak{k}_{\mathbb{B}}(\ell\mu))$, and
- (d) $\text{Ext}_{\mathbb{B}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), I_\mu^{[1]}) \cong \text{Ext}_{\mathbb{U}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]}))$ for all nonnegative integers n .

Proof. For part a) observe that $\text{Ext}_{\mathbb{B}}^n(Y, \mathfrak{k}_{\mathbb{B}}(\omega)) = 0$ unless ω is dominated by some weight ν with the weight space $Y_\nu \neq 0$. There is no such ν , if the height of ω is sufficiently large, depending only on the finite dimensional module Y . For σ sufficiently large, all the nonzero weight spaces in $J'_{\sigma,\mu} := I_\mu^{[1]}/J_{\sigma,\mu}$ occur for weights with a large height, determined by μ and the choice of σ . For such a σ and μ , we have $\text{Ext}_{\mathbb{B}}^n(Y, J'_{\sigma,\mu}) = 0$ for all nonnegative integers n . (This can be easily seen with direct limit arguments.) Part a) follows, and part b) may be obtained by using (modules in a finite complex representing) $\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y)$ for Y in part a) to give part b). Parts c) and d) are standard reciprocity results. (For part (c), note that $I_\mu^{[1]} \cong \text{Ind}_{\mathbb{P}}^{\mathbb{B}}(\mathfrak{k}_{\mathbb{B}}(\ell\mu)) \cong \text{RInd}_{\mathbb{P}}^{\mathbb{B}}(\mathfrak{k}_{\mathbb{B}}(\ell\mu))$.) This completes the proof of the lemma. Q.E.D.

Corollary 2.15. *Let Y be a finite-dimensional \mathbb{B} -module. Assume all composition factors of Y have weight $\ell\omega$ for some weight $\omega \in Y$, and fix $\mu \in Y$.*

- (1) *If n is an odd nonnegative integer, then both $\text{Ext}_{\mathbb{B}}^n(Y, I_\mu^{[1]})$ and $\text{Ext}_{\mathbb{U}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]}))$ are zero.*
- (2) *For any nonnegative integer n , if the Y -composition factor weights $\ell\omega$ all have ω of sufficiently large height, depending only on n and μ , then both $\text{Ext}_{\mathbb{B}}^n(Y, I_\mu^{[1]})$ and $\text{Ext}_{\mathbb{U}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]}))$ are zero. [We refer ahead to (3.4.2) in the proof of this part.]*

Proof. The \mathfrak{b} -cohomology of the trivial module is, as a \mathbb{B} -module, the symmetric algebra $S^*(\mathfrak{n}^{*[1]})$. Here \mathfrak{n} is the Lie algebra of the “unipotent radical” of B' , with $\mathfrak{n}^{*[1]}$ the linear dual of that Lie algebra twisted by the Frobenius, and given cohomological degree 2. See [AJ, Prop. 2.3] in characteristic $p > h$ and

[GK, Thm. 3]⁹ for the analogous characteristic 0 quantum group result. This gives part (1), using Lemma 2.14(c).

Part (2) also follows (in both the quantum and algebraic group cases), using Lemma 2.14, and (3.4.2) below. In more detail, observe first that it is sufficient to take Y of dimension 1. Then (2.14(c) clearly gives the required vanishing of $\text{Ext}_{\mathbb{B}}^n(Y, I_{\mu}^{[1]})$, provided the weight of Y , call it $\ell\lambda$, is of sufficient large height. Note this also gives vanishing of $\text{Ext}_{\mathbb{B}}^n(Y, J_{\sigma, \mu})$ for all sufficiently large σ . Next, regarding $I_{\mu}^{[1]}$ as the direct union of the various modules $J_{\sigma, \mu}$, we obtain $\text{Ext}_{\mathbb{U}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_{\mu}^{[1]}))$ as a direct limit of the various $\text{Ext}_{\mathbb{U}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(J_{\sigma, \mu})) = \text{Ext}_{\mathbb{U}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(V_{\sigma}^{[1]} \otimes \kappa_{\mathbb{B}}(\ell\mu + \ell\sigma))) \cong \text{Ext}_{\mathbb{U}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), V_{\sigma}^{[1]} \otimes \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(\kappa_{\mathbb{B}}(\ell\mu + \ell\sigma)))$. Applying 3.4.2 with $\kappa_{\mathbb{B}}(\ell\lambda) = Y$ and N in 3.4.2 equal to V_{σ} , we obtain that the dimension of $\text{Ext}_{\mathbb{U}}^n(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), V_{\sigma}^{[1]} \otimes \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(\kappa_{\mathbb{B}}(\ell\mu + \ell\sigma)))$ is $\dim \text{Ext}_{\mathbb{B}}^n(Y, V_{\sigma}^{[1]} \otimes \kappa_{\mathbb{B}}(\ell\mu + \ell\sigma)) = \dim \text{Ext}_{\mathbb{B}}^n(Y, J_{\sigma, \mu}) = 0$ for large σ . This proves part (2), and the proof of the corollary is complete. Q.E.D.

3. A PROOF OF THE INDUCTION THEOREMS

After providing an initial framework and brief outline of the steps to be used, this section proceeds step-by-step to complete the proof of Theorem 1 and Theorem 2, the latter treated in its equivalent formulation, Theorem 2.1. Indeed, at this stage the proofs can be given simultaneously, largely thanks to results in the previous section, such as Corollary 2.15. In those results, the statements make sense and are correct in both the quantum and positive characteristic cases, though some attention to differences—at least to different sources—are sometimes required in their proofs. Such differentiation is no longer necessary in the wording of proofs in this section. Once the proof has been completed, subsection 3.5 summarizes some of the similarities and differences between our approach and that taken in [ABG].

3.1. Sketch of the Proof of the Induction Theorems. The proof of the induction theorems begins, as is implicit in [ABG], with the idea of utilizing the following ‘general nonsense’ result on categorical equivalences. The proof is left to the reader.

Lemma 3.1. *Let \mathcal{A} and \mathcal{B} be two triangulated categories, and let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism of triangulated categories, that is, \mathcal{F} sends triangles in \mathcal{A} to triangles in \mathcal{B} , and commutes with the respective translation functors on \mathcal{A} and \mathcal{B} . Then \mathcal{F} is an equivalence of triangulated categories if there is a set of objects S in \mathcal{A} , such that the following two conditions hold:*

- (A) *the minimal full triangulated subcategory of \mathcal{A} containing S is, up to isomorphic objects, the whole category \mathcal{A} , and likewise for \mathcal{B} in place of \mathcal{A} and $\mathcal{F}(S) := \{\mathcal{F}(a) \mid a \in S\}$ in place of S ;*
- (B) *for any objects $a, a' \in S$, the functor \mathcal{F} induces isomorphisms*

$$\text{Hom}_{\mathcal{A}}(a, a'[k]) \mapsto \text{Hom}_{\mathcal{B}}(\mathcal{F}(a), \mathcal{F}(a')[k]) \quad \text{for all } k \in \mathbb{Z}.$$

This lemma will be applied with $\mathcal{A} = D_{\text{triv}}(\mathbb{B})$, $\mathcal{B} = D^{\text{block}}(\mathbb{U})$ ¹⁰ $S = \kappa_{\mathbb{B}}(\ell\lambda)$, $\lambda \in \mathbb{Y}$. Standard arguments with (distinguished triangles arising from) homological degree truncations (see [BBD, Exemples 1.3.2]) show that S does, indeed, generate \mathcal{A} as a triangulated category.

The functor $\text{Ind}_{\mathbb{B}}^{\mathbb{U}}$ has been discussed in Section 2. It is an additive left exact functor, so, from general principles, its right derived functor $\text{RInd}_{\mathbb{B}}^{\mathbb{U}}$, which exists, is a morphism of triangulated categories.

⁹While several references are made in the proof of this theorem to [APW] and [APW2], they are of a general formal nature, similar to those of [A] we discuss above of 2.3, and do not requiring that ℓ be a prime power. It is, however, necessary in [GK] to quote a case of Kempf’s theorem, but there [GK] gives a (correct) reference to [AW, Thm. 5.3].

¹⁰Throughout this paper we use quantum and algebraic groups in a “simply connected” setting. In particular this means that all of our \mathbb{B} -modules, always assumed to be a direct sum of their weight spaces, can have associated weights which are in \mathbb{X} , not just \mathbb{Y} , as in the [ABG] “adjoint group” setting. However, the more general \mathbb{B} -modules are obviously the natural direct sum of submodules whose associated weights belong to a fixed coset (one for each summand) of \mathbb{Y} in \mathbb{X} . This has the consequence that the two versions of $D_{\text{triv}}(\mathbb{B})$ in these respective \mathbb{B} -module contexts are naturally equivalent. Similar considerations apply to \mathbb{U} -modules and $\text{block}(\mathbb{U})$.

Corollary 2.10 shows that $\mathrm{RInd}_{\mathbb{B}}^{\cup}(S)$ generates $D^b\mathrm{block}(\mathbb{U})$. Consequently, the starting hypotheses of Lemma 3.1 and its condition (A) are met, with \mathcal{F} the restriction of $\mathrm{RInd}_{\mathbb{B}}^{\cup}$ to $D_{\mathrm{triv}}(\mathbb{B})$.

So, to establish the induction theorems, it suffices to prove condition (B) of Lemma 3.1 holds, that is, $\mathrm{RInd}_{\mathbb{B}}^{\cup}$ gives isomorphisms

$$\mathrm{Hom}_{D_{\mathrm{triv}}(\mathbb{B})}(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu)[n]) \cong \mathrm{Hom}_{D^b\mathrm{block}(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\mu)[n])$$

for any $\lambda, \mu \in \mathbb{Y}$ and $n \in \mathbb{Z}$. That is, it suffices prove for all $\lambda, \mu \in \mathbb{Y}$ and $n \geq 0$ that applying $\mathrm{RInd}_{\mathbb{B}}^{\cup}$ produces isomorphisms

$$(3.1.1) \quad \mathrm{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu)) \cong \mathrm{Ext}_{\mathrm{block}(\mathbb{U})}^n(\mathrm{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\mu)),$$

where the right hand side is an ‘Ext’ computed in the sense of hypercohomology. To establish (3.1.1) we will proceed as follows, with the first step the same as in [ABG]:

STEP 1: Show

$$(3.1.2) \quad \dim(\mathrm{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu))) = \dim(\mathrm{Ext}_{\mathrm{block}(\mathbb{U})}^n(\mathrm{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\mu)))$$

for any $\lambda, \mu \in \mathbb{Y}$ and $n \geq 0$.

STEP 2 : Employing the equalities (3.1.2), show, for the twisted \mathbb{B} -modules $I_{\mu}^{[1]}$ defined in (2.12.2), that there are isomorphisms

$$(3.1.3) \quad \mathrm{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), I_{\mu}^{[1]}) \cong \mathrm{Ext}_{\mathrm{block}(\mathbb{U})}^n(\mathrm{RInd}_{\mathbb{B}}^{\cup}(k_{\mathbb{B}}(\ell\lambda)), \mathrm{Ind}_{\mathbb{B}}^{\cup}(I_{\mu}^{[1]})),$$

arising from the functoriality of $\mathrm{RInd}_{\mathbb{B}}^{\cup}$.

STEP 3 : Making use of the structure of the modules $I_{\eta}^{[1]}$, recover the desired isomorphisms (3.1.1) from the isomorphisms (3.1.3), and (once again) the dimension equalities (3.1.2).

3.2. Step 1 of the Proof of the Induction Theorems. We largely follow [ABG] for this step, with the exception of the proof of part (ii) of Lemma 3.2 below, which is [ABG, Lem.4.1.1(ii)] in the quantum case. We give a proof in the algebraic groups case in Appendices A and B of this paper. However, the proof [ABG, Lem.4.1.1(ii)] given in [ABG] is not nearly adequate, in our view, even in the quantum case, and we point out in appendices A and B how our proofs there apply to complete it. As mentioned in the introduction, we are grateful to P. Achar and S. Riche for a suggestion to the effect that we look more closely at the [ABG] proof of this result. (In a very preliminary version of this paper, we had assumed the proof in [ABG] was sufficient in the quantum case, and even that it applied to the modular case.) We also thank S. Riche for pointing out an error in our first naive attempt at a correction.

The result Lemma 3.2(ii) below is actually quite strong, in either the algebraic groups or quantum case, and gives, in the regular weight case, a categorification of Rickard’s theorem [R94, Thm. 2.1] on derived equivalences arising from translations. (Essentially, the latter theorem gives Lemma 3.2(i), when the derived equivalences involved in the statement of the theorem are identified in its proof. But the theorem only claims a version of Lemma 3.2(ii) at a character-theoretic level.)

We introduce the lemma by observing, as in the discussion above [ABG, Lem. 4.1.1] that translation functors may be constructed as in the algebraic groups case. This is carried out in [APW, §9], though with the explicit assumption that ℓ be a prime power. This may be removed by appealing to Theorem 3 above. It is easy to check that the resulting constructions have the familiar adjointness and exactness properties of the algebraic groups case [J, II, Lem. 7.6]. Continuing the discussion in [ABG], let $\Xi_{\alpha} : \mathrm{block}(\mathbb{U}) \rightarrow \mathrm{block}(\mathbb{U})$ denote a composition of translation functors first ‘to the wall’ labelled associated to a simple reflection s_{α} and, then back ‘out of the wall’. There are canonical adjunction morphisms $f : id \rightarrow \Xi_{\alpha}$, and $g : \Xi_{\alpha} \rightarrow id$. It is noted in [ABG] that the mapping cone, $C(f)$, of f gives rise to a triangulated functor from $D^b\mathrm{block}(\mathbb{U})$ to itself, denoted θ_{α}^{+} . A similar construction (using $C(g)[-1]$) gives a triangulated functor θ_{α}^{-} . These constructions carry over easily to the algebraic groups case. The statement below is [ABG, Lem. 4.1.1] in both the quantum and algebraic groups cases.

Lemma 3.2. *With the notation discussed above, we have in both the quantum and algebraic groups cases:*

(i) *In $D^b\text{block}(\mathbb{U})$ we have the following canonical isomorphisms $\theta_\alpha^+ \circ \theta_\alpha^- \cong \text{id}$ and $\theta_\alpha^- \circ \theta_\alpha^+ \cong \text{id}$. In particular θ_α^+ and θ_α^- are autoequivalences.*

(ii) *If $\lambda \in W_{\text{aff}} \cdot 0$ and $\lambda^{s_\alpha} > \lambda$ then $\theta_\alpha^+(R\text{Ind}_\mathbb{B}^\mathbb{U} \lambda) \cong R\text{Ind}_\mathbb{B}^\mathbb{U} (\lambda^{s_\alpha})$.*

Remark 3.3. As discussed above, the proof of part (ii) is given in appendices A and B. The proof of part (i) may be obtained from the argument for [R94, Thm. 2.1], or, alternately, from the argument for [ABG, Lem. 4.1.1(i)]. (Both arguments involve similar ingredients.) We mention that [ABG] defines both θ^+ and θ^- as mapping cones. The reader should be aware that the natural definition of θ^- is as a shifted mapping cone, as in the description given here, to obtain property (i).

Lemma 3.4. *For any $\lambda, \mu \in R$ and $n \geq 0$,*

$$(3.4.1) \quad \dim(\text{Ext}_\mathbb{B}^n(k_\mathbb{B}(\ell\lambda), k_\mathbb{B}(\ell\mu))) = \dim(\text{Ext}_{\text{block}(\mathbb{U})}^n(R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\lambda), R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\mu)))$$

Proof. The proof follows [ABG, proof of Lem. 4.2.2]. We include some details for completeness. First, the identity

$$\text{Hom}_\mathbb{B}(k_\mathbb{B}(0), M) \stackrel{\text{adjunction}}{=} \text{Hom}_\mathbb{U}(k_\mathbb{U}(0), R\text{Ind}_\mathbb{B}^\mathbb{U} M) \stackrel{BWB}{=} \text{Hom}_\mathbb{U}(R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(0), R\text{Ind}_\mathbb{B}^\mathbb{U} M)$$

is established, using Borel Weil Bott (BWB) type theory. In fact, Corollary 2.6 is sufficient in the quantum case, and the better known algebraic groups case of that corollary is discussed just above it. To summarize, the identity above holds in both cases, for (at least) any object M in $D_{\text{triv}}(\mathbb{B})$. This proves the lemma in the special case $\lambda = 0$ and arbitrary $\mu \in Y$.

The general case will be reduced to the special case by means of translation functors. For any $\lambda, \mu \in Y$ and $\nu \in Y^+$, we claim (*)

$$R\text{Hom}_{\text{block}(\mathbb{U})}(R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\lambda), R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\mu)) \cong R\text{Hom}_{\text{block}(\mathbb{U})}(R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\lambda + \ell\nu), R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\mu + \ell\nu)).$$

Let $\nu = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r} \in Y \subset W_{\text{aff}}$ be a reduced expression. Then $\ell\nu = 0^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r}}$ and hence $\ell\lambda + \ell\nu = (\ell\lambda)^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r}} > (\ell\lambda)^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{r-1}}} > \cdots > \ell\lambda$. Now the repeated use of the second part of Lemma 3.2 give us $R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\lambda + \ell\nu) \cong \theta_{\alpha_r}^+ \circ \theta_{\alpha_{r-1}}^+ \circ \cdots \circ \theta_{\alpha_1}^+(R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\lambda))$. This together with the first property of Lemma 3.2 gives (*).

For any $\lambda, \mu \in Y$, choose a large $\nu \in Y^+$ such that $\nu - \lambda \in Y^+$. Using $\nu - \lambda$ in place of ν in (*) (i.e. a shift by $\nu - \lambda$), we get

$$R\text{Hom}_{\text{block}(\mathbb{U})}(R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\lambda), R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\mu)) \cong R\text{Hom}_{\text{block}(\mathbb{U})}(R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\nu), R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\mu + \ell\nu - \ell\lambda)) (**).$$

Again, a shift by ν gives

$$R\text{Hom}_{\text{block}(\mathbb{U})}(R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(0), R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\mu - \ell\lambda)) \cong R\text{Hom}_{\text{block}(\mathbb{U})}(R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\nu), R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\mu + \ell\nu - \ell\lambda)) (**).$$

Notice the right hand terms of the two isomorphisms labeled (**) are the same, so that we can view (**) as providing isomorphisms of three expressions, all obtained by applying $R\text{Hom}_{\text{block}(\mathbb{U})}$. Passing, for any fixed i , to $R^i\text{Hom}_{\text{block}(\mathbb{U})}$ expressions, we obtain three vector spaces of the same dimension. One of these vectors spaces $R^i\text{Hom}_{\text{block}(\mathbb{U})}(R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(0), R\text{Ind}_\mathbb{B}^\mathbb{U} k_\mathbb{B}(\ell\mu - \ell\lambda))$ is isomorphic to $R^i\text{Hom}_\mathbb{B}(k_\mathbb{B}(0), k_\mathbb{B}(\ell\mu - \ell\lambda))$, by the special case above with $(\ell\mu - \ell\lambda)$ used in the role of $\ell\mu$. Finally, using the isomorphism $R\text{Hom}_\mathbb{B}(k_\mathbb{B}(0), k_\mathbb{B}(\ell\mu - \ell\lambda)) \cong R\text{Hom}_\mathbb{B}(k_\mathbb{B}(\ell\lambda), k_\mathbb{B}(\ell\mu))$ we complete the proof of the lemma. Q.E.D.

Observe that for any finite-dimensional G -module V , with weights in Y ,

$$\theta_\alpha^+(M \otimes V^{[1]}) = \theta_\alpha^+(M) \otimes V^{[1]}.$$

Thus, using the proof of Step 1 and recalling the generalised tensor identity (see Remark 2.11(ii)), tensoring on the right the first (resp., second) component of each term appearing in the equality given by Lemma 3.4, by any finite-dimensional twisted G -module $M^{[1]}$ (resp., $N^{[1]}$) with weights in Y , preserves

the equality of dimensions:

$$(3.4.2) \quad \begin{aligned} & \dim(\mathrm{Ext}_{\mathbb{B}}^n(M^{[1]}|_{\mathbb{B}} \otimes k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu))) \\ &= \dim(\mathrm{Ext}_{\mathrm{block}(\mathbb{U})}^n(M^{[1]} \otimes \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\mu))), \end{aligned}$$

and

$$\begin{aligned} & \dim(\mathrm{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), N^{[1]}|_{\mathbb{B}} \otimes k_{\mathbb{B}}(\ell\mu))) \\ &= \dim(\mathrm{Ext}_{\mathrm{block}(\mathbb{U})}^n(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\lambda), N^{[1]} \otimes \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\mu))). \end{aligned}$$

3.3. Step 2 of the Proof of the Induction Theorems.

Lemma 3.5. *Let λ be any element of \mathbb{Y} (or \mathbb{X}). Choose $\nu = N\rho$, with ρ as in section 2.4.2, and with $N \in \mathbb{N}$ large enough so that $\ell\tau := \ell(\nu + \lambda)$ is dominant. Let V_{ν} be as in (2.12.1). Then the \mathbb{B} -module $M = V_{\nu}^{[1]} \otimes k_{\mathbb{B}}(\ell\tau)$ satisfies the following three properties:*

- (1) $k_{\mathbb{B}}(\ell\lambda) \subset M$.
- (2) All composition factors of $M/k_{\mathbb{B}}(\ell\lambda)$ have the form $k_{\mathbb{B}}(\ell\eta)$ with $\eta > \lambda$ in the dominance order.
- (3) The map $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M) \rightarrow M$ is \mathfrak{p} -split.

Proof. Once again, the argument uses ideas from [ABG], especially in the analysis of the map in part (3).

Note $M \cong M_0^{[1]}$, where M_0 is the B' -module $V_{\nu} \otimes k_{B'}(\tau)$. The B' -socle of V_{ν} is $k_{B'}(-\nu)$, so $k_{B'}(\lambda) \cong k_{B'}(-\nu) \otimes k_{B'}(\tau)$ is the B' -socle of M_0 . Parts 1) and 2) of the lemma for M follow immediately from corresponding properties of M_0 .

Next, put $F_0 = \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(M_0) \cong V_{\nu} \otimes \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(k_{\mathbb{B}}(\tau))$ and $F = F_0^{[1]}$. The natural \mathbb{B} -map $\varphi_0 : F_0 \rightarrow M_0$ is surjective, as follows from the surjectivity of $\mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(k_{\mathbb{B}}(\tau) \rightarrow k_{\mathbb{B}}(\tau))$.

Consequently, the Frobenius twisted map $\varphi : F \rightarrow M$ is surjective. It is also \mathfrak{p} -split, since both domain and range are completely reducible as \mathfrak{p} -modules. The map φ gives rise (by adjointness) to a map $F \rightarrow \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M)$ which, when composed with the natural map $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M) \rightarrow M$, is the \mathfrak{p} -split surjection φ . Consequently, $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M) \rightarrow M$ is also \mathfrak{p} -split. This proves Property (3) and completes the proof of Lemma 3.5. Q.E.D.

Our arguments now begin to diverge from [ABG].

Lemma 3.6. *Suppose $\mu \in \mathbb{Y}$ and Y is a finite-dimensional \mathbb{B} -module all of whose composition factors are of the form factors $k_{\mathbb{B}}(\ell\lambda)$ with $\lambda \in \mathbb{Y}$. Then for all nonnegative integers n , and any μ ,*

$$(3.6.1) \quad \mathrm{Ext}_{\mathbb{B}}^n(Y, I_{\mu}^{[1]}) \cong \mathrm{Ext}_{\mathbb{U}}^n(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_{\mu}^{[1]})).$$

Proof. First, observe that (3.6.1) is true for n odd, with both sides zero, by Corollary 2.15(1). This greatly simplifies long exact sequence arguments in the remaining n even cases. Now fix n even. Then (3.6.1) is equivalent to the case where Y is one-dimensional. (In fact, for any given Y , (3.6.1) is implied by the corresponding results for each of its composition factors.)

Next, observe in the one-dimensional case, that it is sufficient to check injectivity of the left-to-right map implicit in (3.6.1). This is a consequence of Corollary 2.15(1) and the dimensional equalities (3.4.2). In fact, (3.6.1) will be an isomorphism for any one-dimensional Y and μ for which it is an injection or for any Y and μ (satisfying the hypotheses of the lemma) for which (3.6.1) is an injection on each composition factor of Y .

We are now in a position to treat the one-dimensional case $Y = k_{\mathbb{B}}(\ell\lambda)$, for our fixed even n , by downward induction on the height of λ . Note that (3.6.1) is true (with both sides zero) for λ sufficiently large, by Corollary 2.15(2). We may, hence, assume inductively that (3.6.1) holds for $Y = k_{\mathbb{B}}(\ell\eta)$ with η of larger height than λ , or, more generally for all finite-dimensional Y with composition factors satisfying this height condition.

Let M be the B -module guaranteed by Lemma 3.5. Let N be the cokernel of the B -module inclusion $k_B(\ell\lambda) \hookrightarrow M$. By our height induction, there is an isomorphism (3.6.1) for $Y = N$ and our fixed even n .

Also, the p -split map $\mathrm{RInd}_B^{\cup}(M) \rightarrow M$ from Lemma (3.5) gives an injection

$$\mathrm{Ext}_p^n(M, k_p(\ell\mu)) \hookrightarrow \mathrm{Ext}_p^n(\mathrm{RInd}_B^{\cup}(M), k_p(\ell\mu)),$$

or, equivalently,

$$(3.6.2) \quad \begin{aligned} \mathrm{Ext}_B^n(M, I_\mu^{[1]}) &\hookrightarrow \mathrm{Ext}_B^n(\mathrm{RInd}_B^{\cup}(M), I_\mu^{[1]}) \\ &\cong \mathrm{Ext}_U^n(\mathrm{RInd}_B^{\cup}(M), \mathrm{RInd}_B^{\cup}(I_\mu^{[1]})). \end{aligned}$$

Likewise, the adjunction map $\mathrm{RInd}_B^{\cup}(k_B(\ell\lambda)) \rightarrow k_B(\ell\lambda)$ gives a morphism

$$(3.6.3) \quad \gamma : \mathrm{Ext}_B^n(k_B(\ell\lambda), I_\mu^{[1]}) \rightarrow \mathrm{Ext}_U^n(\mathrm{RInd}_B^{\cup}(k_B(\ell\lambda)), \mathrm{RInd}_B^{\cup}(I_\mu^{[1]})).$$

Let

$$\begin{aligned} A &= \mathrm{Ext}_B^n(N, I_\mu^{[1]}), & A' &= \mathrm{Ext}_B^{n+1}(N, I_\mu^{[1]}) = 0, \\ \tilde{A} &= \mathrm{Ext}_U^n(\mathrm{RInd}_B^{\cup}(N), \mathrm{RInd}_B^{\cup}(I_\mu^{[1]})), & \tilde{A}' &= \mathrm{Ext}_U^{n+1}(\mathrm{RInd}_B^{\cup}(N), \mathrm{RInd}_B^{\cup}(I_\mu^{[1]})) = 0, \\ B &= \mathrm{Ext}_B^n(M, I_\mu^{[1]}), & \tilde{B} &= \mathrm{Ext}_U^n(\mathrm{RInd}_B^{\cup}(M), \mathrm{RInd}_B^{\cup}(I_\mu^{[1]})), \\ C &= \mathrm{Ext}_B^n(k_B(\ell\lambda), I_\mu^{[1]}), & \tilde{C} &= \mathrm{Ext}_U^n(\mathrm{RInd}_B^{\cup}(k_B(\ell\lambda)), \mathrm{RInd}_B^{\cup}(I_\mu^{[1]})) \end{aligned}$$

Then the B -module exact sequence $0 \rightarrow k_B(\ell\lambda) \rightarrow M \rightarrow N \rightarrow 0$ gives rise to a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A' = 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ \tilde{A} & \xrightarrow{\tilde{u}} & \tilde{B} & \xrightarrow{\tilde{v}} & \tilde{C} & \xrightarrow{\tilde{w}} & \tilde{A}' = 0 \end{array}$$

with γ the morphism in (3.6.3), α the isomorphism given by the induction argument so far, and β an injection given by (3.6.2). Both rows are exact. By a standard diagram chase, these conditions force γ to be an injection. As discussed above, this implies γ is an isomorphism, and completes the induction for our fixed n . Since n was an arbitrary even nonnegative integer, and since the odd case has already been handled, the proof of the lemma is complete. Q.E.D.

To be clear: As a consequence of Lemma 3.6, we immediately obtain the isomorphisms (3.1.3). using $Y = k_B(\ell\lambda)$. This completes Step 2.

3.4. Step 3 of the Proof of the Induction Theorems. Recall that by Lemma 3.1, it suffices to establish that there are isomorphisms (3.1.1):

$$\mathrm{Ext}_B^n(k_B(\ell\lambda), k_B(\ell\mu)) \cong \mathrm{Ext}_{\mathrm{block}(U)}^n(\mathrm{RInd}_B^{\cup} k_B(\ell\lambda), \mathrm{RInd}_B^{\cup} k_B(\ell\mu))$$

arising from the application of RInd_B^{\cup} .

It suffices to fix throughout an otherwise arbitrary weight $\lambda \in Y$. For a given $\mu \in Y$, all nonnegative integers n will be treated simultaneously. As a notational convenience, all Ext_B^m -groups with a negative index m are equal to zero by definition.

Starting from §3.3, we have isomorphisms (3.1.3) (arising, as noted, from the functoriality of RInd_B^{\cup}), and wish to pass to analogous isomorphisms with $k_B(\ell\mu)$ in place of the terms $I_\mu^{[1]}$ appearing in (3.1.3).

To begin, note that “twisting” the canonical B -module injection $\mu \hookrightarrow I_\mu$ of $k_B(\mu)$ into its injective hull I_μ leads to a s.e.s.

$$(3.6.4) \quad 0 \longrightarrow k_B(\ell\mu) \longrightarrow I_\mu^{[1]} \longrightarrow \frac{I_\mu^{[1]}}{k_B(\ell\mu)} =: \Sigma_\mu \longrightarrow 0,$$

wherein the module $I_\mu^{[1]}$ has $k_B(\ell\mu)$ as its socle, and the B -composition factors of Σ_μ have the form $k_B(\ell\tau)$, with $ht(\tau) > ht(\mu)$, using the usual height function. Set $ht_\lambda(\Sigma_\mu)$ to be sum of all submodules

M of Σ_μ for which $ht(\eta) \leq ht(\lambda)$ whenever $\ell\eta$ is a composition factor of M . Then $ht_\lambda(\Sigma_\mu)$ is the largest submodule of Σ_μ with this property. In the corresponding s.e.s.

$$(3.6.5) \quad 0 \longrightarrow ht_\lambda(\Sigma_\mu) \longrightarrow \Sigma_\mu \longrightarrow \frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)} \longrightarrow 0,$$

one has $ht_\lambda(\frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}) = 0$.

Since $\text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\zeta), k_{\mathbb{B}}(\ell\eta)) \neq 0$ implies $\zeta \geq \eta$, one has, for any weights $\zeta, \eta \in X$,

$$(3.6.6) \quad \text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\zeta), k_{\mathbb{B}}(\ell\eta)) = 0 \text{ if } ht(\eta) > ht(\zeta).$$

Thus, from (3.6.6), it follows that for any finite-dimensional submodule $N \subset \frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}$,

$$(3.6.7) \quad \text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), N) = 0 \quad \forall n \geq 0$$

Moreover, since $k_{\mathbb{B}}(\ell\lambda)$ is finite dimensional, $\text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), -)$ commutes with direct limits. Since I_μ is a direct limit of its finite dimensional submodules, so is Σ_μ , and also $\frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}$; consequently, the vanishing property (3.6.7) yields the vanishing results

$$(3.6.8) \quad \text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), \frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}) = 0 \quad \forall n \geq 0.$$

The s.e.s. (3.6.5) and the vanishing results (3.6.8) yield the l.e.s.

$$\begin{aligned} \cdots \longrightarrow \text{Ext}_{\mathbb{B}}^{n-1}(k_{\mathbb{B}}(\ell\lambda), ht_\lambda(\Sigma_\mu)) &\longrightarrow \text{Ext}_{\mathbb{B}}^{n-1}(k_{\mathbb{B}}(\ell\lambda), \Sigma_\mu) \longrightarrow \text{Ext}_{\mathbb{B}}^{n-1}(k_{\mathbb{B}}(\ell\lambda), \frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}) = 0 \\ &\longrightarrow \text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), ht_\lambda(\Sigma_\mu)) \longrightarrow \text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), \Sigma_\mu) \longrightarrow \text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), \frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}) = 0 \longrightarrow \cdots \end{aligned}$$

whence isomorphisms

$$(3.6.9) \quad \text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), ht_\lambda(\Sigma_\mu)) \xrightarrow{\cong} \text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), \Sigma_\mu), \quad \forall n \geq 0.$$

Consider now the distinguished triangle obtained from (3.6.5) under $\text{RInd}_{\mathbb{B}}^{\cup}$:

$$(3.6.10) \quad \cdots \longrightarrow \text{RInd}_{\mathbb{B}}^{\cup}(ht_\lambda(\Sigma_\mu)) \longrightarrow \text{RInd}_{\mathbb{B}}^{\cup}(\Sigma_\mu) \longrightarrow \text{RInd}_{\mathbb{B}}^{\cup}\left(\frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}\right) \longrightarrow \cdots$$

Since the functor $\text{RInd}_{\mathbb{B}}^{\cup}$ commutes with taking direct limits, $\text{RInd}_{\mathbb{B}}^{\cup}(\Sigma_\mu/ht_\lambda(\Sigma_\mu)) = \varinjlim \text{RInd}_{\mathbb{B}}^{\cup}(N)$ over finite dimensional submodules $N \subset \Sigma_\mu/ht_\lambda(\Sigma_\mu)$. Thus, if we can replicate the following vanishing result, analogous to (3.6.7):

$$(3.6.11) \quad \text{Ext}_{\text{block}(\cup)}^n(\text{RInd}_{\mathbb{B}}^{\cup}(k_{\mathbb{B}}(\ell\lambda)), \text{RInd}_{\mathbb{B}}^{\cup}(N)) = 0 \quad \forall n \geq 0,$$

then, from the preceding arguments, mutatis mutandis, we will obtain the following isomorphisms:

$$(3.6.12) \quad \text{Ext}_{\mathbb{B}}^n(\text{RInd}_{\mathbb{B}}^{\cup}(k_{\mathbb{B}}(\ell\lambda)), \text{RInd}_{\mathbb{B}}^{\cup}(ht_\lambda(\Sigma_\mu))) \xrightarrow{\cong} \text{Ext}_{\mathbb{B}}^n(\text{RInd}_{\mathbb{B}}^{\cup}(k_{\mathbb{B}}(\ell\lambda)), \text{RInd}_{\mathbb{B}}^{\cup}(\Sigma_\mu)), \quad \forall n \geq 0.$$

Recall that the key step in producing the isomorphisms (3.6.7) was the earlier vanishing result (3.6.6), but now from (3.6.6) and the dimension equality (3.1.2),

$$(3.6.13) \quad \text{Ext}_{\text{block}(\cup)}^n(\text{RInd}_{\mathbb{B}}^{\cup}(k_{\mathbb{B}}(\ell\zeta)), \text{RInd}_{\mathbb{B}}^{\cup}(k_{\mathbb{B}}(\ell\eta))) = 0 \text{ if } ht(\eta) > ht(\zeta).$$

By applying this vanishing result of (3.6.13) to composition factors of N , (3.6.11), and hence (3.6.12), do indeed follow as claimed.

We now carry out a descending induction on $ht(\mu)$. Assume for all weights η for which $ht(\eta) > ht(\mu)$, $\text{RInd}_{\mathbb{B}}^{\cup}$ induces, for all n , isomorphisms

$$(3.6.14) \quad \text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\eta)) \cong \text{Ext}_{\text{block}(\cup)}^n(\text{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\lambda), \text{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\eta)).$$

Then by the definition of $ht_\lambda(\Sigma_\mu)$ and its finite dimensionality, it follows from (3.6.14) that

$$(3.6.15) \quad \text{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), ht_\lambda(\Sigma_\mu)) \cong \text{Ext}_{\text{block}(\cup)}^n(\text{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\lambda), \text{RInd}_{\mathbb{B}}^{\cup}(ht_\lambda(\Sigma_\mu))).$$

From the s.e.s. (3.6.4), upon letting \widehat{V} denote $\mathrm{RInd}_{\mathbb{B}}^{\cup}(V)$, and $b(\mathbb{U})$ denote $\mathit{block}(\mathbb{U})$, we obtain two l.e.s.s tied together:

$$\begin{array}{ccccccccccc} \cdots & \rightarrow & \mathrm{Ext}_{\mathbb{B}}^{n-1}(k_{\mathbb{B}}(\ell\lambda), I_{\mu}^{[1]}) & \rightarrow & \mathrm{Ext}_{\mathbb{B}}^{n-1}(k_{\mathbb{B}}(\ell\lambda), \Sigma_{\mu}) & \rightarrow & \mathrm{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu)) & \rightarrow & \mathrm{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), I_{\mu}^{[1]}) & \rightarrow & \mathrm{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), \Sigma_{\mu}) & \rightarrow & \cdots \\ & & \downarrow & & \\ \cdots & \rightarrow & \mathrm{Ext}_{b(\mathbb{U})}^{n-1}(\widehat{k_{\mathbb{B}}(\ell\lambda)}, \widehat{I_{\mu}^{[1]}}) & \rightarrow & \mathrm{Ext}_{b(\mathbb{U})}^{n-1}(\widehat{k_{\mathbb{B}}(\ell\lambda)}, \widehat{\Sigma_{\mu}}) & \rightarrow & \mathrm{Ext}_{b(\mathbb{U})}^n(\widehat{k_{\mathbb{B}}(\ell\lambda)}, \widehat{k_{\mathbb{B}}(\ell\mu)}) & \rightarrow & \mathrm{Ext}_{b(\mathbb{U})}^n(\widehat{k_{\mathbb{B}}(\ell\lambda)}, \widehat{I_{\mu}^{[1]}}) & \rightarrow & \mathrm{Ext}_{b(\mathbb{U})}^n(\widehat{k_{\mathbb{B}}(\ell\lambda)}, \widehat{\Sigma_{\mu}}) & \rightarrow & \cdots \end{array}$$

In the above diagram, all vertical morphisms arise from the functoriality of $\mathrm{RInd}_{\mathbb{B}}^{\cup}$. The first and fourth vertical maps shown are isomorphisms, as given by Lemma 3.6. By (3.6.9) and (3.6.12), Σ_{μ} in the second and fifth vertical morphisms shown can be replaced with $ht_{\lambda}(\Sigma_{\mu})$, and the resulting morphisms in the second and fifth vertical spots are isomorphisms. By the Five Lemma the third vertical morphism is an isomorphism, i.e., $\mathrm{RInd}_{\mathbb{B}}^{\cup}$ determines

$$\mathrm{Ext}_{\mathbb{B}}^n(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu)) \cong \mathrm{Ext}_{b(\mathbb{U})}^n(\mathrm{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\cup} k_{\mathbb{B}}(\ell\mu)),$$

for each $n \geq 0$, as desired. This completes the proof of Step 3, and, consequently of both induction Theorems 1 and 2.

3.5. Summary, and comparison with the approach in [ABG].

Although a natural approach, [ABG] were unable to use $S := \{k_{\mathbb{B}}(\ell\lambda) \mid \lambda \in R\}$ directly as a set of generators for the triangle category equivalence tool given by Theorem 3.1. This roadblock apparently motivated their attempt to use the set $S' := \{I_{\lambda}^{[1]} \mid \lambda \in R\}$ in place of S . (See [ABG, Remark 4.2.7].) However, the corresponding claim in [ABG, Lem. 4.3.6] that S' (equivalently the set $\{\mathrm{Ind}_{\mathbb{B}}^{\mathbb{B}}(\ell\lambda) \mid \lambda \in R\}$) is inaccurate, since these modules do not actually lie in $D_{\mathit{triv}}(\mathbb{B})$. Nevertheless, in the characteristic 0 setting of [ABG], it is true that $D_{\mathit{triv}}(\mathbb{B})$ is contained in the triangulated category generated by S' , so that a line of argument establishing isomorphisms

$$(3.6.16) \quad \mathrm{Ext}_{\mathbb{B}}^n(I_{\lambda}^{[1]}, I_{\mu}^{[1]}) \cong \mathrm{Ext}_{b(\mathbb{U})}^n(\mathrm{RInd}_{\mathbb{B}}^{\cup}(I_{\lambda}^{[1]}), \mathrm{RInd}_{\mathbb{B}}^{\cup}(I_{\mu}^{[1]})),$$

as pursued in [ABG, Lem. 4.3.6] *would* imply the existence of isomorphisms (3.1.1). Unfortunately, there is no such inclusion of $D_{\mathit{triv}}(\mathbb{B})$ in characteristic $p > 0$. In particular, the first line of the proof of [ABG][Lem. 4.3.6], asserting that the universal enveloping algebra $\mathcal{U}\mathfrak{n}$ (for \mathfrak{n} a nilpotent Lie algebra in a triangular decomposition) has finite global dimension is not true for the correctly analogous characteristic p situation. It is a question of what modules are to be pulled back under the Frobenius morphism. In the characteristic p situation, it is necessary to use modules for the distribution algebra of a positive characteristic unipotent algebraic group, not its unrestricted enveloping algebra, and the finite global dimension property is lost. Overcoming this obstacle, while using much of the apparatus of [ABG], is not trivial, and our proof eventually involves parity properties for \mathfrak{b} -cohomology [AJ, Prop. 2.3]. See above Corollary 2.15 and the proof of Lemma 3.6, which also present our argument in the quantum case.

4. APPENDIX A

The discussion below, in the algebraic groups case, is based on Jantzen's book ([J], pp. 258-259). We follow the notations there. Comments on the quantum case are given in Remark 4.3, to which the reader might look ahead, now. In this appendix and the next we will provide a proof of Lemma 3.2(ii). A closely linked goal is to understand the adjunction map $id \circ \mathrm{RInd}_B^G \longrightarrow T_\mu^\lambda \circ T_\lambda^\mu \circ \mathrm{RInd}_B^G$ in the spirit of the long exact sequences on [J, p. 259]. Put $L = L(\nu_1)$ as on the cited page. The functor $T_\lambda^\mu, T_\mu^\lambda$ are constructed from tensor product functors $L \otimes (-), L^* \otimes (-)$, respectively, using block projections pr_λ, pr_μ :

$$T_\mu^\lambda = pr_\lambda(L^* \otimes -) \circ pr_\mu, \quad T_\lambda^\mu = pr_\mu(L \otimes -) \circ pr_\lambda.$$

These are the definitions given in Jantzen, familiar to many readers. All functors are regarded as functors from the category of rational G -modules to itself. The functor $L \otimes - := L \otimes (-)$ is (left and right) adjoint to $L^* \otimes - := L^* \otimes (-)$, while pr_λ and pr_μ are both self-adjoint (left and right). Consequently, T_λ^μ is (left and right) adjoint to T_μ^λ .

Construction of the adjunction map

So far, this is all standard, but we can go a little further.

(1) Let $X, Y \in G - \mathrm{Mod}$, the category of rational G -modules. Then the identifications $\mathrm{Hom}_G(pr_\lambda X, Y) \cong \mathrm{Hom}_G(X, pr_\lambda Y)$ are quite canonical: Write X, Y , respectively, as direct sums of submodules

$$X = pr_\lambda X \oplus pr'_\lambda X, \quad Y = pr_\lambda Y \oplus pr'_\lambda Y,$$

where $pr'_\lambda X$ has no composition factors in the block associated with λ and is maximal, as a submodule of X , with that property. The submodule $pr'_\lambda Y$ of Y is defined similarly. Obviously, any G -homomorphism $pr_\lambda X \rightarrow Y$ has image in $pr_\lambda Y$. Consequently, it identifies with a map $pr_\lambda X \rightarrow pr_\lambda Y$. Also, any G -homomorphism $X \rightarrow pr_\lambda Y$ sends pr'_λ to 0 and sends $pr_\lambda X$ to $pr_\lambda Y$. Thus,

$$\mathrm{Hom}_G(pr_\lambda X, Y) \cong \mathrm{Hom}_G(pr_\lambda X, pr_\lambda Y) \cong \mathrm{Hom}_G(X, pr_\lambda Y)$$

with each identification very obvious and canonical. We also record

$$\mathrm{Hom}_G(X, Y) \cong \mathrm{Hom}_G(pr_\lambda X, pr_\lambda Y) \oplus \mathrm{Hom}_G(pr'_\lambda X, pr'_\lambda Y)$$

All the the observations in the above paragraph hold if λ is replaced by μ .

(2) In particular, suppose we are given a natural transformation $\eta = \{\eta_{X,Y}\}_{X,Y \in G - \mathrm{Mod}}$ from $\mathrm{Hom}_G(L \otimes Y$ to $\mathrm{Hom}_G(X, L^* \otimes Y)$. Then η gives maps

$$\eta_{pr_\lambda X, pr_\mu Y} : \mathrm{Hom}_G(L \otimes pr_\lambda X, pr_\mu Y) \longrightarrow \mathrm{Hom}_G(pr_\lambda X, L^* \otimes pr_\mu Y).$$

This induces, using **(1)**, a natural transformation we will call $\tilde{\eta}$, again defined on $G - \mathrm{Mod} \times G - \mathrm{Mod}$, with $\tilde{\eta}_{X,Y}$ a map from $\mathrm{Hom}_G(pr_\lambda L \otimes pr_\lambda X, pr_\mu Y)$ to $\mathrm{Hom}_G(X, pr_\lambda(L^* \otimes pr_\mu Y))$. Explicitly,

$$\begin{array}{ccc} \mathrm{Hom}_G(L \otimes pr_\lambda X, pr_\mu Y) & \xrightarrow{\eta_{pr_\lambda X, pr_\mu Y}} & \mathrm{Hom}_G(pr_\lambda X, L^* \otimes pr_\mu Y) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_G(pr_\mu(L \otimes pr_\lambda X) & \xrightarrow{\tilde{\eta}_{X,Y}} & \mathrm{Hom}_G(X, pr_\lambda(L^* \otimes pr_\mu Y)) \\ \downarrow = & & \downarrow = \\ \mathrm{Hom}_G(T_\lambda^\mu X, Y) & & \mathrm{Hom}_G(X, T_\mu^\lambda Y) \end{array}$$

with the vertical isomorphisms between the top two rows given by **(1)**. Note there is a similar diagram with $pr_\mu Y$ replacing Y in the bottom two rows (using $\tilde{\eta}_{X, pr_\mu Y}$).

(3) Using the naturality of η , we can put another row and commutative diagram(s) on top of the top row above:

$$\begin{array}{ccc} \mathrm{Hom}_G(L \otimes pr_\lambda X, Y) & \xrightarrow{\eta_{pr_\lambda X, Y}} & \mathrm{Hom}_G(pr_\lambda X, L^* \otimes Y) \\ \updownarrow & & \updownarrow \\ \mathrm{Hom}_G(L \otimes pr_\lambda X, pr_\mu Y) & \xrightarrow{\eta_{pr_\lambda X, pr_\mu Y}} & \mathrm{Hom}_G(pr_\lambda X, L^* \otimes pr_\mu Y) \end{array}$$

Here the pair of vertical maps pointing upward are indexed by the inclusion $pr_\mu \rightarrow Y$ and yield a commutative diagram. Similarly the pair of downward arrows are indexed by the projection $Y \rightarrow pr_\mu Y$ and give a commutative diagram. The composite of the homomorphisms represented by the upward pointing arrows with the homomorphism represented by the corresponding downward pointing arrows are identities.

We can now prove

Proposition 4.1. *Assume the natural transformation $\eta = \{\eta_{X,Y}\}_{X,Y \in G-Mod}$ gives natural isomorphisms*

$$\eta_{X,Y} : \mathrm{Hom}_G(L \otimes X, Y) \longrightarrow \mathrm{Hom}_G(X, L^* \otimes Y),$$

and let $\tilde{\eta} = \{\tilde{\eta}_{X,Y}\}_{X,Y \in G-Mod}$ be the corresponding natural transformation constructed above. Then $\tilde{\eta}$ gives natural isomorphisms

$$\tilde{\eta}_{X,Y} : \mathrm{Hom}_G(T_\lambda^\mu X, Y) \longrightarrow \mathrm{Hom}_G(X, T_\mu^\lambda Y).$$

Moreover, the corresponding adjunction transformation \tilde{adj} from the identity functor on $G-Mod$ to the functor $T_\mu^\lambda T_\lambda^\mu$ may be constructed from the adjunction map adj similarly associated with η . In fact, for each $X \in G-Mod$, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{adj_X} & L^* \otimes L \otimes X \\ \downarrow = & & \downarrow \\ X & \xrightarrow{\tilde{adj}_X} & T_\mu^\lambda T_\lambda^\mu X \end{array}$$

where the down arrow on the right is the composite projection

$$L^* \otimes L \otimes X \rightarrow L^* \otimes L \otimes pr_\lambda X \rightarrow L^* \otimes pr_\mu(L \otimes pr_\lambda X) \rightarrow pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda X)) = T_\mu^\lambda T_\lambda^\mu X.$$

Proof. We use the (noted) alternate version of the diagram in (2) in which prY replaces Y , and use the diagram in (3) as given. The combination gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_G(L \otimes pr_\lambda X, Y) & \xrightarrow{\eta_{pr_\lambda X, Y}} & \mathrm{Hom}_G(pr_\lambda X, L^* \otimes Y) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_G(L \otimes pr_\lambda X, pr_\mu Y) & \xrightarrow{\eta_{pr_\lambda X, pr_\mu Y}} & \mathrm{Hom}_G(pr_\lambda X, L^* \otimes pr_\mu Y) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_G(pr_\mu(L \otimes pr_\lambda X), pr_\mu Y) & \xrightarrow{\tilde{\eta}_{X, pr_\mu Y}} & \mathrm{Hom}_G(X, pr_\lambda(L^* \otimes pr_\mu Y)) \\ \downarrow = & & \downarrow = \\ \mathrm{Hom}_G(T_\lambda^\mu X, Y) & & \mathrm{Hom}_G(X, T_\mu^\lambda Y) \end{array}$$

Now take $Y = L \otimes pr_\lambda X$. Thus $pr_\mu Y = T_\lambda^\mu X$, and $\tilde{\eta}_{X, pr_\mu Y}(1_{T_\lambda^\mu X}) = \tilde{\eta}_{X, T_\lambda^\mu X}(1_{T_\lambda^\mu X}) = \tilde{adj}_X$. Chasing the element $1_{T_\lambda^\mu X}$ up to the second row gives an element which is the projection $Y \rightarrow pr_\mu Y$ in $\mathrm{Hom}_G(L \otimes pr_\lambda X, pr_\mu Y) = \mathrm{Hom}_B(Y, pr_\mu Y)$. This element is also the image of $1_{L \otimes pr_\lambda X} = 1_Y$ under the downward

vertical map on the left. Observe $\eta_{pr_\lambda X, Y}(1_Y) = adj_{pr_\lambda X}$. Following the right hand vertical maps in the case $X = pr_\lambda X$ gives a commutative diagram

$$\begin{array}{ccc} pr_\lambda X & \xrightarrow{adj_{pr_\lambda X}} & L^* \otimes L \otimes pr_\lambda X \\ \downarrow = & & \downarrow \\ pr_\lambda X & \xrightarrow{\widetilde{adj}_{pr_\lambda X}} & T_\mu^\lambda T_\lambda^\mu pr_\lambda X \end{array}$$

with the right hand map the composite of projections

$$L^* \otimes L \otimes pr_\lambda X \rightarrow L^* \otimes pr_\mu(L \otimes pr_\lambda X) \rightarrow pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda X)) = T_\mu^\lambda T_\lambda^\mu X.$$

Now return to the case of a general X and apply functoriality¹¹ of the adjunction maps adj , \widetilde{adj} to obtain a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{adj_X} & L^* \otimes L \otimes X \\ \downarrow & & \downarrow \\ pr_\lambda X & \xrightarrow{adj_{pr_\lambda X}} & L^* \otimes L \otimes pr_\lambda X \\ \downarrow = & & \downarrow \\ pr_\lambda X & \xrightarrow{\widetilde{adj}_{pr_\lambda X}} & T_\mu^\lambda T_\lambda^\mu pr_\lambda X \\ \uparrow & & \uparrow = \\ X & \xrightarrow{\widetilde{adj}_X} & T_\mu^\lambda T_\lambda^\mu X \end{array}$$

In this diagram, the middle rectangle is identical to the diagram just discussed. All the unlabeled vertical maps are evident projections. In particular, the whole commutative diagram could be extended on the left, preserving commutativity, by a long downward equality map from the upper left X to the lower left X . Also, the lower right equality arrow can be reversed, still preserving commutativity. With these changes, the perimeter rectangle becomes the commutative diagram required in the proposition. This completes its proof. Q.E.D.

We remark that the adjunction obtained from the usual natural isomorphism

$$\mathrm{Hom}_G(L \otimes X, Y) \cong \mathrm{Hom}_G(X, L^* \otimes X)$$

is quite explicit: For $x \in X$, $adj_X(x) = 1_L \otimes x$, if $L^* \otimes L$ is identified with $\mathrm{Hom}_k(L, L)$. Even if we do not use that identification, we can just write

$$adj_X(x) = \sum_{\epsilon \in I} \epsilon^* \otimes \epsilon \otimes x,$$

¹¹It is a general property of adjunction maps $Id \rightarrow EF$ where E is a right adjoint to a functor F , that any map $\phi : X \rightarrow X'$ in the underlying category gives a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & EF(X') \\ \uparrow \phi & & \uparrow EF(\phi) \\ X & \longrightarrow & EF(X) \end{array}$$

where both horizontal maps are adjunctions. We include a brief proof: $F(\phi)$ is the value at $1_{F(X)}$ of the evident map $\mathrm{Hom}(F(X), F(X)) \rightarrow \mathrm{Hom}(F(X), F(X'))$ and also the value at $1_{F(X')}$ of the evident map $\mathrm{Hom}(F(X'), F(X')) \rightarrow \mathrm{Hom}(F(X), F(X'))$. Applying the (natural) adjointness isomorphism $\mathrm{Hom}(F(-), F(-)) \cong \mathrm{Hom}(-, EF(-))$ to $F(\phi)$ yields a map $X \rightarrow EF(X')$ which, correspondingly, factors in two different ways, giving the desired commutative diagram.

We remark that there is a dual commutative diagram for the ‘‘counital adjunction’’ $FE \rightarrow Id$. The formulation and proof may be given using dual categories and the adjunction case.

where ϵ ranges over any basis I of L , and ϵ^* denotes the corresponding dual basis element. The sum on the right is independent of the basis I chosen.

As a corollary to the proposition, we have

Corollary 4.2. *Let $X, Y \in G\text{-mod}$, and identify $L^* \otimes L \otimes (X \otimes Y) \cong (L^* \otimes L \otimes X) \otimes Y$. Then $\widetilde{adj}_{X \otimes Y}(-) = (adj_X(-)) \otimes Y$. If all weights of Y lie in the root lattice, then $\widetilde{adj}_{X \otimes Y^{[1]}}(-) = \widetilde{adj}_X(-) \otimes Y^{[1]}$, identifying $T_\mu^\lambda T_\lambda^\mu(X \otimes Y^{[1]})$ with $(T_\mu^\lambda T_\lambda^\mu X) \otimes Y^{[1]}$.*

Proof. The first equality is immediate from the formula for $adj_X(x)$, $x \in X$ above, applied to $X \otimes Y$ and $adj_{X \otimes Y}$.

We can argue with adj to handle \widetilde{adj} : First, observe the rearrangements

$$pr_\lambda(X \otimes Y^{[1]}) = pr_\lambda X \otimes Y^{[1]}, \text{ and}$$

$$pr_\mu(L \otimes pr_\lambda(X \otimes Y^{[1]})) = pr_\mu(L \otimes pr_\lambda X \otimes Y^{[1]}) = pr_\mu(L \otimes pr_\lambda X) \otimes Y^{[1]}. \text{ Also,}$$

$$pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda(X \otimes Y^{[1]}))) = pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda X)) \otimes Y^{[1]}.$$

Here we have heavily used the fact that the operator $- \otimes Y^{[1]}$ commutes with our “block” projections. (Recall the latter are formulated in terms of the affine Weyl group, which contains translations by p -multiples of the root lattice.) We have regarded $pr_\lambda X$ as a submodule of X , and have taken a similar viewpoint with all the projections in these equalities. (Similar equalities hold for complementary projections, Thus, $pr'_\lambda(X \otimes Y^{[1]}) = pr'_\lambda X \otimes Y^{[1]}$, etc.)

Recall that we have described \widetilde{adj}_X in Proposition 4.1 as the composition of adj_X followed by a sequence of projections

$$L^* \otimes L \otimes X \rightarrow L^* \otimes L \otimes pr_\lambda X \rightarrow L^* \otimes pr_\mu(L \otimes pr_\lambda X) \rightarrow pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda X)).$$

Tensoring $adj_X(-)$ on the right with $Y^{[1]}$ gives $\widetilde{adj}_{X \otimes Y^{[1]}}(-)$ as shown above (even with $Y^{[1]}$ any G -module). Next, tensor the sequence of projections displayed above with $Y^{[1]}$, obtaining

$$\begin{aligned} L^* \otimes L \otimes X \otimes Y^{[1]} &\rightarrow L^* \otimes L \otimes pr_\lambda(X \otimes Y^{[1]}) \rightarrow L^* \otimes pr_\mu(L \otimes pr_\lambda(X \otimes Y^{[1]})) \\ &\rightarrow pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda X)) \otimes Y^{[1]}. \end{aligned}$$

Using the rearrangements discussed above, we get

$$\begin{aligned} L^* \otimes L \otimes X \otimes Y^{[1]} &\rightarrow L^* \otimes L \otimes pr_\lambda X \otimes Y^{[1]} \rightarrow L^* \otimes pr_\mu(L \otimes pr_\lambda X) \otimes Y^{[1]} \\ &\rightarrow pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda(X \otimes Y^{[1]}))). \end{aligned}$$

Here we have identified tensor products isomorphic through the associative law. The lower display above is easily recognized as the sequence in Proposition 4.1 whose composition with $adj_{X \otimes Y^{[1]}}$ gives $\widetilde{adj}_{X \otimes Y^{[1]}}$. Combining this with the equality $adj_{X \otimes Y^{[1]}}(-) = adj_X(-) \otimes Y^{[1]}$ noted above gives the identification $\widetilde{adj}_{X \otimes Y^{[1]}}(-) = \widetilde{adj}_X(-) \otimes Y^{[1]}$, completing the proof of the corollary. Q.E.D.

Remark 4.3. The quantum case

The lack of cocommutativity requires some care in treating the quantum case, and it becomes important to distinguish right from left. For example, consider the “usual” natural isomorphism

$$\text{Hom}_G(L \otimes X, Y) \cong \text{Hom}_G(X, L^* \otimes X)$$

in the algebraic groups case. It may be given in more detail as a composite

$$(4.3.1) \quad \text{Hom}_G(L \otimes X, Y) \cong \text{Hom}_G(X, \text{Hom}_k(L, Y)) \cong \text{Hom}_G(X, L^* \otimes Y).$$

The right hand isomorphism depends on the isomorphism of G -modules

$$\text{Hom}_k(L, Y) \cong L^* \otimes Y.$$

The usual way to identify a simple tensor element $f(-) \otimes y$ on the right ($f \in L^*, y \in Y$) with a function on the left is to let it send $v \in L$ to $f(v)y$. Let $g \in G$, and suppose for a moment that G is not a group, but a Hopf algebra with antipode S , and that L, Y are left G -modules. Using Sweedler (implicit sum) notation, with g mapping to $g_1 \otimes g_2$ under comultiplication, the action of g on the right hand element gives $f(Sg_1(-) \otimes g_2y)$, but, on the left, it gives a function sending v to $f(Sg_2(v))g_1y$. The latter is not formally the function corresponding to the right hand element without cocommutativity.

This can be fixed by either using the right action of the Hopf algebra G or by keeping the left action and changing the tensor product $L^* \otimes Y$ to $Y \otimes L^*$. We prefer the latter approach, since left actions are often implicitly used—e.g., in [J]. In keeping with the spirit of previous sections of this paper, define an “opposite” tensor product \otimes^{op} by

$$X \otimes^{op} Y := Y \otimes X,$$

A similar analysis can be carried out on the left hand isomorphism of the display (4.3.1). We find that the standard correspondence gives an isomorphism of left G -modules

$$\mathrm{Hom}_k(L \otimes^{op} X, Y) \cong \mathrm{Hom}_k(X, \mathrm{Hom}_k(L, Y)).$$

Here the left action of the Hopf algebra G on the various modules $\mathrm{Hom}_k(-, -)$ is given by “conjugation.” That is, if $g \in G$ and f is a linear function from one left G -module to another, the action of g on f gives a linear function $g_1f(Sg_2(-))$. When the antipode is surjective (as it is for all the Hopf algebras we consider), the space of “fixed points” of this action of G (all f for which each $g \in G$ acts through the counit) results precisely in the space of G -homomorphisms. (A general statement and proof of this fact may be found in [APW, 2.9].) In particular, we have a general version of (4.3.1) which holds for any such Hopf algebra:

$$(4.3.2) \quad \mathrm{Hom}_G(L \otimes^{op} X, Y) \cong \mathrm{Hom}_G(X, \mathrm{Hom}_k(L, Y)) \cong \mathrm{Hom}_G(X, L^* \otimes^{op} Y).$$

Finally, notice that \otimes^{op} is just as associative an operation as \otimes , which is strictly associative, if standard identifications are made in iterated tensor products of k -spaces.

Thus, **the results and arguments of this section hold in the quantum case.** The reader can even read or reread the statements and arguments in both the algebraic groups case and quantum case simultaneously, after replacing \otimes with \otimes^{op} , and using the same simultaneous notations U, B, k, \dots , as in previous sections, in place of G, B, k, \dots

5. APPENDIX B

We now return to Jantzen [J], pp. 258-259. Proposition 7.11 there implies, if T_λ^μ is “to a wall,” then

$$T_\lambda^\mu \mathrm{RInd}_B^G(w \cdot \lambda) \cong \mathrm{RInd}_B^G(w \cdot \mu).$$

Recall Jantzen denotes one dimensional weight modules by the weights alone.

The argument for the above isomorphism is helpful: $T_\lambda^\mu \mathrm{RInd}_B^G(w \cdot \lambda) = pr_\mu(L \otimes pr_\lambda \mathrm{RInd}_B^G(w \cdot \lambda)) = pr_\mu(L \otimes \mathrm{RInd}_B^G(w \cdot \lambda)) = pr_\mu \mathrm{RInd}_B^G(L \otimes w \cdot \lambda)$. At this point a B composition series of L is examined and it is found that there is only one composition factor, call it λ_l appearing with multiplicity one, such that $pr_\mu \mathrm{RInd}_B^G(\lambda_l \otimes w \cdot \lambda) \neq 0$. It is determined that $\lambda_l \otimes w \cdot \lambda$ is $w \cdot \mu$, completing the proof.

Next, let us come “out of the wall” with T_μ^λ . We assume μ is on a true “wall” with stabilizer $\{1, s\}$ for a simple reflection s . We want to know what happens to $T_\mu^\lambda \mathrm{RInd}_B^G(w \cdot \mu)$. Again, write

$$\begin{aligned} T_\mu^\lambda \mathrm{RInd}_B^G(w \cdot \mu) &= pr_\lambda(L^* \otimes pr_\mu \mathrm{RInd}_B^G(w \cdot \mu)) \\ &= pr_\lambda(L^* \otimes \mathrm{RInd}_B^G(w \cdot \mu)) \\ &= pr_\lambda \mathrm{RInd}_B^G(L^* \otimes w \cdot \mu). \end{aligned}$$

This time L^* has two composition factors exactly, $\gamma = \nu$ and $\gamma = \nu'$, each appearing with multiplicity 1, such that $pr_\lambda \mathrm{RInd}_B^G(\gamma \otimes w \cdot \mu) \leq 0$. The two weights ν, ν' (not Jantzen’s notation) satisfy $\{\nu + w \cdot \mu, \nu' + w \cdot \mu\} = \{w \cdot \lambda, ws \cdot \lambda\}$.

Jantzen treats the case $ws \cdot \lambda < w \cdot \lambda$ (with the roles of λ, μ reversed) in [J, Prop. 7.12]. For our purposes, to be compatible with Lemma 3.2(ii), we will consider the case $ws \cdot \lambda > w \cdot \lambda$, which requires different arguments (in the same setting).

5.1. The main issue. In this case there is an exact sequence of B -modules

$$0 \longrightarrow M \longrightarrow L^* \otimes sw \cdot \mu \longrightarrow M' \longrightarrow 0$$

in which the weight $w \cdot \lambda$ appears in M and $ws \cdot \lambda$ appears in M' . These appearances are each with multiplicity 1, and no other weight τ with $pr_\lambda \text{RInd}_B^G(\tau) \neq 0$ appears in either M or M' . Apply $pr_\lambda \text{RInd}_B^G$ to the above short exact sequence.

The result is a distinguished triangle

$$(*) \quad \cdots \longrightarrow \text{RInd}_B^G(w \cdot \lambda) \longrightarrow T_\mu^\lambda \text{RInd}_B^G(w \cdot \mu) \longrightarrow \text{RInd}_B^G(ws \cdot \lambda) \longrightarrow \cdots$$

As previously noted, the middle term is isomorphic to

$$T_\mu^\lambda T_\lambda^\mu \text{RInd}_B^G(w \cdot \lambda).$$

This leads to the question as to whether or not the resulting map

$$\text{RInd}_B^G(w \cdot \lambda) \longrightarrow T_\mu^\lambda T_\lambda^\mu \text{RInd}(w \cdot \lambda)$$

is the adjunction map. **We claim** that it is, indeed, the adjunction map, at least up to a nonzero scalar multiple. (Thus, there is a distinguished triangle $(*)$ in which the left hand map is adjunction.)

The proof of this claim will essentially occupy the rest of this appendix! We will use the context and notation of the algebraic groups case, and treat the quantum case at the end in Remark 5.15. Part (ii) of Lemma 3.2 then follows, since $\theta_\alpha^+(\text{RInd}_B^U \lambda)$, from its mapping cone definition, fits into a triangle with the same two objects and left hand map as $(*)$, but replacing the object $\text{RInd}_B^U(\lambda^{s_\alpha})$ as the third object. Standard triangulated category axioms then give a map $\theta_\alpha^+(\text{RInd}_B^U \lambda) \longrightarrow \text{RInd}_B^U(\lambda^{s_\alpha})$, part of a commutative diagram with identify maps on the two left hand objects in $(*)$ and their translations under [1]. The “five lemma” then gives the desired isomorphism $\theta_\alpha^+(\text{RInd}_B^U \lambda) \cong \text{RInd}_B^U(\lambda^{s_\alpha})$, completing the proof of part (ii) of Lemma 3.2. Since the a proof of part (i) has already been given, this will complete the proof of the lemma.

triangles which is the identity on the two left hand objects. of the algeb

For the moment, we prove the claim in the case where both $w \cdot \lambda$ and $ws \cdot \lambda$ are dominant: Note that $\text{RInd}_B^G(w \cdot \lambda) \cong \text{Ind}_B^G(w \cdot \lambda)$ and $\text{RInd}_B^G(ws \cdot \lambda) \cong \text{Ind}_B^G(ws \cdot \lambda)$ by Kempf’s theorem. Also, of course, the functor RInd_B^G is right adjoint to restriction. In particular

$$\text{Hom}_{D^b(G)}(\text{RInd}_B^G(w \cdot \lambda), \text{RInd}_B^G(ws \cdot \lambda)) = \text{Hom}_B(\text{Ind}_B^G(w \cdot \lambda), ws \cdot \lambda)$$

so that any map from $\text{RInd}(w \cdot \lambda)$ to the middle term of $(*)$ factors through the left hand map. However, $\text{Hom}_B(\text{RInd}_B^G(w \cdot \lambda), w \cdot \lambda) \cong k$. The claim follows. In our argument we have used the fact that RInd_B^G is right adjoint to restriction.

5.2. We will now try to exploit the validity of the dominant case, by using it to build well-behaved resolutions in the general $w \cdot \lambda < ws \cdot \lambda$ case, to which we now return.

The B -modules we will use to resolve $w \cdot \lambda$ will be sums of those of the form $w \cdot \lambda \otimes p\tau \otimes V^{[1]}$, where τ is in the root lattice and $V^{[1]}$ is a Frobenius twisted G -module (restricted to B) with V having all weights in the root lattice.

Lemma 5.3. *The trivial module $k = k(0)$ has a positive resolution $k \xrightarrow{\sim} K^\bullet$, where each K^n is a direct sum of B -modules $p\tau \otimes V^{[1]}$, with τ and $V^{[1]}$ as above. Moreover, we may assume all τ are dominant and that $q \cdot \lambda + p\tau$, $w \cdot \mu + p\tau$, are $ws \cdot \lambda + p\tau$ are also dominant.*

In fact, we can assume $\nu + p\tau$ is dominant for all ν in any fixed finite list of weights.

Proof. Notice that k and all $p\tau \otimes V^{[1]}$ are Frobenius-twisted B -modules. Each (Frobenius-)twisted injective B -module hull $I_\mu^{[1]}$ for a weight μ in the root lattice is a direct union of modules $p(\mu + \sigma) \otimes V^{[1]}$; see(e:Itwist). Thus, any finite dimensional B -module $N^{[1]}$, with N having all weights in the root lattice, can be embedded in a direct sum of these, with the weights $\tau = \mu + \sigma$, as large as we like. The cokernel of the embedding will also be a finite dimensional twisted B -module of the same form as $N^{[1]}$ above. Hence the process can continue. Starting with $k = k(0)$ in the initial role of $N^{[1]}$, we obtained the desired resolution. Q.E.D.

We now describe some of the main issues we face at this point. Let τ be any weight in the root lattice such that $w \cdot \lambda + p\tau$, $w \cdot \mu + p\tau$, and $ws \cdot \lambda + p\tau$ are dominant, as well as $p\tau$. Form the composite of the adjunction map

$$\mathrm{Ind}_B^G(w \cdot \lambda + p\tau) \longrightarrow T_\mu^\lambda T_\lambda^\mu \mathrm{Ind}_B^G(w \cdot \mu + p\tau)$$

and the usual isomorphism

$$T_\mu^\lambda T_\lambda^\mu \mathrm{Ind}_B^G(w \cdot \lambda + p\tau) \xrightarrow{\sim} T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu + p\tau)$$

Note $w \cdot \lambda + p\tau = w' \cdot \lambda$ and $w \cdot \mu + p\tau = w' \cdot \mu$ for w' , the composite of w followed by translation by $p\tau$. We will discuss the ‘‘usual’’ isomorphism later in some details, but its exact nature may be regarded as unknown at the moment, together with any details regarding the adjunction map. We do, however, note that the latter map is nonzero. The composite then gives a nonzero map

$$(5.3.1) \quad \mathrm{Ind}_B^G(w \cdot \lambda + p\tau) \longrightarrow T_\lambda^\mu \mathrm{Ind}_B^G(w \cdot \mu + p\tau)$$

‘‘Another’’ map with the same domain and target objects is obtained, as in 5.1, by applying $pr_\lambda \mathrm{Ind}_B^G(-)$ to the sequence $0 \longrightarrow M \longrightarrow L^* \otimes w \cdot \mu$ there, but with $w' \cdot \mu = w \cdot \mu + p\tau$ playing the role of $w \cdot \mu$. We will call the resulting map the ‘‘Jantzen map’’ $\mathrm{Jan}_Y^{w \cdot \lambda}$ for Y the B -module $p\tau = k(p\tau)$. (We will shortly generalize this notation.) To discuss $\mathrm{Jan}_Y^{w \cdot \lambda}$ and (5.3.1) in a parallel way, denote the latter as $\mathrm{Adj}_Y^{w \cdot \lambda}$ (for the same Y). Then the discussion at the end of 5.1 gives, using w' in place of w there:

Proposition 5.4. *The maps $\mathrm{Adj}_Y^{w \cdot \lambda}$ and $\mathrm{Jan}_Y^{w \cdot \lambda}$ differ by at most a nonzero scalar multiple, for $Y = p\tau$, when $p\tau$, $w \cdot \lambda + p\tau$, $w \cdot \mu + p\tau$, and $ws \cdot p\tau$ are dominant, and τ with the root lattice.*

Now let $\mathcal{Y}^{w \cdot \lambda}$ denote the full subcategory of B -modules $p\tau \otimes V^{[1]}$ with $p\tau$ as in the proposition and $V^{[1]}$ a finite dimensional Frobenius twisted G -module with tall weights of V in the root lattice. Also write $V^{[1]}$ for its restriction $Y^{[1]}|_B$ depending on context.

We will usually abbreviate $\mathcal{Y} := \mathcal{Y}^{w \cdot \lambda}$.

Our next goal is to extend the maps $\mathrm{Adj}_Y^{w \cdot \lambda}$, $\mathrm{Jan}_Y^{w \cdot \lambda}$ to all $y \in \mathcal{Y}$ and regard them as natural transformations $\mathrm{Adj}^{w \cdot \lambda} = \{\mathrm{Adj}_Y^{w \cdot \lambda}\}_{Y \in \mathcal{Y}}$, $\mathrm{Jan}^{w \cdot \lambda} = \{\mathrm{Jan}_Y^{w \cdot \lambda}\}_{Y \in \mathcal{Y}}$

$$\mathrm{Adj}^{w \cdot \lambda} : \mathrm{Ind}_B^G(w \cdot \lambda \otimes -) \longrightarrow T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes -) \quad \text{and}$$

$$\mathrm{Jan}^{w \cdot \lambda} : \mathrm{Ind}_B^G(w \cdot \lambda \otimes -) \longrightarrow T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes -)$$

These functors and natural transformations will then automatically extend to $\mathrm{add} \mathcal{Y}$, the additive full subcategory of $B\text{-mod}$ consisting of all finite direct sums of objects in \mathcal{Y} . Notice that all the K^n from the previous lemma belong to $\mathrm{add} \mathcal{Y}$, so that the (to be demonstrated) naturality will result in two maps of complexes

$$\mathrm{Ind}_B^G(w \cdot \lambda \otimes K^\bullet) \longrightarrow T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet)$$

resulting in maps

$$\mathrm{RInd}_B^G(w \cdot \lambda) \longrightarrow T_\mu^\lambda \mathrm{RInd}_B^G(w \cdot \mu)$$

to which we want to compare. We will return to this point after achieving the goal above.

We treat first the Jantzen maps.

The Jantzen maps $Jan_Y^{w \cdot \lambda}(y \in \mathcal{Y})$ and their naturality

Recall the short exact sequence

$$0 \longrightarrow M \longrightarrow L^* \otimes w \cdot \mu \longrightarrow M' \longrightarrow 0$$

in 5.1. Let $Y = p\tau \otimes V^{[1]} \in \mathcal{Y}$. Tensor on the right with Y and apply $\mathrm{RInd}_B^G(-)$ to get a distinguished triangle

$$\cdots \longrightarrow \mathrm{RInd}_B^G(M \otimes Y) \longrightarrow \mathrm{RInd}_B^G(L^* \otimes w \cdot \mu \otimes Y) \longrightarrow \mathrm{RInd}_B^G(M' \otimes Y) \longrightarrow \cdots$$

The middle term naturally identifies with $L^* \otimes \mathrm{RInd}_B^G(w \cdot \mu \otimes \gamma)$ through the “generalized tensor identity” (discussed in this paper in Remark 2.11(ii)). Note $L^* \otimes \mathrm{RInd}_B^G(w \cdot \mu \otimes \gamma) \cong L^* \otimes \mathrm{Ind}_B^G(w \cdot \mu \otimes \gamma)$, by the construction of \mathcal{Y} . As discussed earlier in this appendix, M has one weight $\nu \in W_{\mathrm{aff}} \cdot \lambda$, namely $\nu = w \cdot \lambda$, appearing with multiplicity 1. Also note that $\nu + \eta$ is in the same (dot action) affine Weyl group orbit as ν for any weight ν and weight η of \mathcal{Y} . Consequently, $pr_\lambda \mathrm{RInd}_B^G(M \otimes \gamma) \cong \mathrm{RInd}_B^G(w \cdot \lambda \otimes Y) \cong \mathrm{Ind}_B^G(w \cdot \lambda \otimes Y)$. A specific construction of an isomorphism may be given from any full flag of B -submodules of M with one dimensional sections. If such a flag is fixed, we obtain an isomorphism natural in Y of $Y \in \mathcal{Y}$. Similar remarks apply for M' and isomorphism $pr_\lambda \mathrm{RInd}_B^G(M' \otimes Y) \cong \mathrm{Ind}_B^G(ws \cdot \lambda \otimes Y)$.

As a consequence of the discussion above, we have exact sequences, natural in $Y \in \mathcal{Y}$

$$0 \longrightarrow \mathrm{Ind}_B^G(w \cdot \lambda \otimes Y) \longrightarrow T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes Y) \longrightarrow \mathrm{Ind}_B^G(ws \cdot \lambda \otimes Y) \longrightarrow 0$$

We define the map on the left (ignoring the obvious zero map) to be $Jan_Y^{w \cdot \lambda}$.

We summarize some of its main properties (in addition to the above exact sequence).

Proposition 5.5. (i) *The maps $Jan_Y^{w \cdot \lambda}$, $y \in \mathcal{Y}$, collectively define a natural transformation of functors.*

$$\mathrm{Ind}_B^G(w \cdot \lambda \otimes -) \longrightarrow T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes -)$$

on the category \mathcal{Y} (whose morphisms are B -maps).

(ii) *For any fixed $Y = p\tau \otimes V^{[1]}$, there is a commutative diagram with “obvious” vertical isomorphisms, natural in V*

$$\begin{array}{ccc} \mathrm{Ind}_B^G(w \cdot \lambda \otimes \tau \otimes V^{[1]}) & \xrightarrow{Jan_Y^{w \cdot \lambda}} & T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes p\tau \otimes V^{[1]}) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} & \xrightarrow{Jan_{p\tau}^{w \cdot \lambda} \otimes V^{[1]}} & (T_\mu^\lambda(w \cdot \mu \otimes p\tau)) \otimes V^{[1]} \end{array}$$

Proof. Part(i) has been proved already. For part (ii), it is enough to check the commutativity after identifying the right hand terms with $pr_\lambda \mathrm{Ind}_B^G(L^* \otimes w \cdot \mu \otimes p\tau \otimes V^{[1]})$ and $pr_\lambda \mathrm{Ind}_B^G(L^* \otimes w \cdot \mu \otimes p\tau) \otimes V^{[1]}$, respectively. The top row in this revised diagram may be obtained by applying $pr_\lambda \mathrm{RInd}_B^G(-)$ to the inclusion $M \otimes \gamma \longrightarrow L^* \otimes w \cdot \mu \otimes \gamma$, by construction. Similarly the bottom row may be obtained by applying $pr_\lambda \mathrm{RInd}_B^G(-)$ to the inclusion $M \otimes p\tau \leq L^* \otimes w \cdot \mu$, then tensoring on the right with $V^{[1]}$. Now, naturality of the generalized tensor identity gives commutativity of the closed rectangle in the diagram below.

$$\begin{array}{ccccc}
\mathrm{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \xrightarrow{\cong} & pr_\lambda \mathrm{RInd}_B^G(M \otimes Y) & \longrightarrow & pr_\lambda \mathrm{RInd}_B^G(L^* \otimes w \cdot \mu \otimes p\tau \otimes V^{[1]}) \\
& & \downarrow \cong & & \downarrow \cong \\
& & pr_\lambda(\mathrm{RInd}_B^G(M \otimes p\tau) \otimes V^{[1]}) & \longrightarrow & pr_\lambda(\mathrm{RInd}_B^G(L^* \otimes w \cdot \mu \otimes p\tau) \otimes V^{[1]}) \\
& & \downarrow \cong & & \downarrow \cong \\
\mathrm{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} & \xrightarrow{\cong} & pr_\lambda \mathrm{RInd}_B^G(M \otimes p\tau) \otimes V^{[1]} & & pr_\lambda \mathrm{RInd}_B^G(L^* \otimes w \cdot \mu \otimes p\tau) \otimes V^{[1]}
\end{array}$$

The identity and its naturality may also be used to complete the open rectangle on the left to a commutative rectangle¹², using the “obvious” tensor identity isomorphism for a vertical map. Finally, all the “ RInd_B^G ” symbols in the diagram may be replaced with “ Ind_B^G ,” and the bottom row completed to make a lower-right commutative rectangle.

The bottom row then agrees with that of the revised diagram. That is, its composition gives the composition of $Jan_{p\tau}^{w \cdot \lambda} \otimes V^{[1]}$ and the identification $(T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes p\tau) \otimes V^{[1]}) \cong pr_\lambda \mathrm{Ind}_B^G(L^* \otimes w \cdot \mu \otimes p\tau) \otimes V^{[1]}$. We have now shown that both top and bottom rows of the now completed and commutative outer rectangle agree with those of the revised version of the diagram in (ii). The left hand columns also agree, and the “obvious” isomorphism on the right in the outer rectangle define, through composition, an obvious isomorphism in the “revised” diagram, making the latter commutative. In the original diagram in ii), the composition is

$$T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes p\tau \otimes V^{[1]}) \cong T_\mu^\lambda (\mathrm{Ind}_B^G(w \cdot \mu \otimes p\tau) \otimes V^{[1]}) \cong (T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes p\tau)) \otimes V^{[1]}$$

which may be taken as the definition of the right hand column “obvious” isomorphism in the original diagram in ii). The latter diagram then becomes commutative, and the proposition is proved. Q.E.D.

This completes our treatment of $Jan^{w \cdot \lambda}$. Before turning to $Adj^{w \cdot \lambda}$, we discuss some isomorphisms $T_\lambda^\mu \mathrm{Ind}_B^G(w \cdot \lambda \otimes Y) \xrightarrow{\sim} \mathrm{Ind}_B^G(w \cdot \mu \otimes Y)$ and $T_\mu^\lambda T_\lambda^\mu \mathrm{Ind}_B^G(w \cdot \lambda \otimes Y) \xrightarrow{\sim} T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes Y)$, $Y \in \mathcal{Y}$, which enter into the definition and discussion of $Adj^{w \cdot \lambda}$. We will call the first isomorphism above $Iso_\gamma^{w \cdot \lambda}$, and the second, $T Iso_Y^{w \cdot \lambda} (= T_\mu^\lambda \circ Iso_Y^{w \cdot \lambda})$.

The isomorphism $Iso_Y^{w \cdot \lambda}$ is obtained in a similar spirit to our construction above of the isomorphism $pr_\lambda \mathrm{RInd}_B^G(M \otimes \gamma) \cong \mathrm{Ind}_B^G(w \cdot \lambda \otimes Y)$, except we apply $pr_\mu \mathrm{RInd}_B^G$ to $L \otimes -$. The module L has only one weight, which we called λ_ℓ at the beginning of this appendix (in the discussion of the “to a wall” isomorphism), with the property that $\lambda_\ell + w \cdot \lambda$ belongs to $W_{aff} \cdot \mu$. The same is true if any weight of Y is added to $w \cdot \lambda$. We have $w \cdot \mu = \lambda_\ell + w \cdot \lambda$, and so $w \cdot \mu \otimes Y \cong \lambda_\ell \otimes w \cdot \lambda \otimes Y$. The weight λ_ℓ appears with multiplicity one in L , so

$$pr_\mu \mathrm{RInd}_B^G(L \otimes w \cdot \lambda \otimes Y) \cong \mathrm{RInd}_B^G(\lambda_\ell \otimes w \cdot \lambda \otimes Y) \cong \mathrm{RInd}_B^G(w \cdot \mu \otimes Y) \cong \mathrm{Ind}_B^G(w \cdot \mu \otimes Y)$$

The first isomorphism can be constructed by using any B -flag of L with λ_ℓ as a section and applying $pr_\mu \mathrm{RInd}_B^G$ to the various sub and factor modules associated to the flag terms. If we fix the flag and procedure, the first isomorphism becomes natural in $Y \in \mathcal{Y}$. The other isomorphisms obviously are natural in Y , as are the isomorphisms

$$T_\lambda^\mu \mathrm{Ind}_B^G(w \cdot \lambda \otimes Y) = pr_\mu(L \otimes \mathrm{Ind}_B^G(w \cdot \lambda \otimes Y)) \cong pr_mu(\mathrm{Ind}_B^G(L \otimes w \cdot \lambda \otimes Y))$$

$$\text{and } pr_mu(\mathrm{Ind}_B^G(L \otimes w \cdot \lambda \otimes Y)) \cong pr_\mu(\mathrm{RInd}_B^G(L \otimes w \cdot \lambda \otimes Y)).$$

¹²It is carried out by using a B -module flag of $M \otimes p\tau$ and applying naturality to the various inclusion and factor maps involved.

This latter isomorphism arises from the vanishing of $(R^n \text{Ind}_B^G)(L \otimes w \cdot \lambda \otimes Y) = 0$ for $n > 0$ (a consequence of our construction of \mathcal{Y} and the generalized tensor identity).

The composition of all these isomorphisms (in an evident order) is defined to be

$$\text{Iso}_Y^{w \cdot \lambda} : T_\lambda^\mu \text{Ind}_B^G(w \cdot \lambda \otimes Y) \xrightarrow{\sim} \text{Ind}_B^G(w \cdot \mu \otimes Y)$$

The construction shows it is natural in $Y \in \mathcal{Y}$ as is $T\text{Iso}_Y^{w \cdot \lambda} := T_\mu^\lambda \circ \text{Iso}_Y^{w \cdot \lambda}$. This gives part (i) of the following proposition.

Proposition 5.6. (i) *The maps $\text{Iso}_Y^{w \cdot \lambda}$ and $T\text{Iso}_Y^{w \cdot \lambda}$ ($Y \in \mathcal{Y}$) collectively define natural isomorphisms of functors on the category \mathcal{Y}*

$$T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes -) \longrightarrow \text{Ind}_B^G(w \cdot \mu \otimes -) \quad \text{and} \quad T_\mu^\lambda T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes -) \longrightarrow T_\mu^\lambda \text{Ind}_B^G(w \cdot \mu \otimes -)$$

(ii) *For any fixed $Y = p\tau \otimes V^{[1]} \in \mathcal{Y}$, these are commutative diagrams, with “obvious” vertical isomorphisms, natural in V .*

$$\begin{array}{ccc} T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \xrightarrow{\text{Iso}_Y^{w \cdot \lambda}} & \text{Ind}_B^G(w \cdot \mu \otimes p\tau \otimes V^{[1]}) \\ \downarrow \cong & & \downarrow \cong \\ T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} & \xrightarrow{\text{Iso}_Y^{w \cdot \lambda} \otimes V^{[1]}} & \text{Ind}_B^G(w \cdot \mu \otimes p\tau) \otimes V^{[1]} \end{array}$$

and

$$\begin{array}{ccc} T_\mu^\lambda T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \xrightarrow{T\text{Iso}_Y^{w \cdot \lambda}} & T_\mu^\lambda \text{Ind}(w \cdot \mu \otimes p\tau \otimes V^{[1]}) \\ \downarrow \cong & & \downarrow \cong \\ T_\mu^\lambda T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} & \xrightarrow{T\text{Iso}_Y^{w \cdot \lambda} \otimes V^{[1]}} & T_\mu^\lambda \text{Ind}(w \cdot \mu \otimes p\tau) \otimes V^{[1]} \end{array}$$

Proof. Part (i) already has been proved. Next, note that a commutative lower diagram in (ii) can be obtained by first applying T_μ^λ to a commutative upper diagram, then using the natural isomorphism $T_\mu^\lambda(- \otimes V^{[1]}) \cong T_\mu^\lambda(- \otimes V^{[1]})$ on the lower row of the upper diagram. the reader may convince him/her self that the entire procedure preserves the “obvious” property of the vertical maps!

Thus, it is suffice to treat the upper diagram in (ii). The first thing to do here is to note the “obvious” isomorphism $T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) \cong T_\lambda^\mu (\text{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]}) \cong T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes p\tau) \otimes V^{[1]}$. This gives the first column in the upper diagram. The isomorphism may be regarded as the (by now “obvious”) process of “pulling out” $V^{[1]}$, from inductions of tensor products, block projection or translation functors, or some combination of these operators. The row of isomorphism above requires two steps to fully “pullout” $V^{[1]}$. If we continue with the several steps required to define $\text{Iso}_Y^{w \cdot \lambda}$, we see at every step along the way there is an opportunity to “pull out” $V^{[1]}$. This gives a series of possibly commutative diagrams, written below in top to bottom order.

$$\begin{array}{ccc}
& T_\lambda^\mu \operatorname{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \xrightarrow{\cong} T_\lambda^\mu \operatorname{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} \\
(1) & \parallel & \parallel \\
& pr_\mu(L \otimes \operatorname{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]})) & \xrightarrow{\cong} pr_\mu(L \otimes \operatorname{Ind}_B^G(w \cdot \lambda \otimes p\tau)) \otimes V^{[1]} \\
(2) & \downarrow \cong & \downarrow \cong \\
& pr_\mu(\operatorname{Ind}_B^G(L \otimes w \cdot \lambda \otimes p\tau \otimes V^{[1]})) & \xrightarrow{\cong} pr_\mu(\operatorname{Ind}_B^G(L \otimes w \cdot \lambda \otimes p\tau)) \otimes V^{[1]} \\
(3) & \downarrow \cong & \downarrow \cong \\
& pr_\mu(\operatorname{RInd}_B^G(L \otimes w \cdot \lambda \otimes p\tau \otimes V^{[1]})) & \xrightarrow{\cong} pr_\mu(\operatorname{RInd}_B^G(L \otimes w \cdot \lambda \otimes p\tau)) \otimes V^{[1]} \\
(4) & \downarrow \cong & \downarrow \cong \\
& \operatorname{RInd}_B^G(\lambda_\ell \otimes w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \xrightarrow{\cong} \operatorname{RInd}_B^G(\lambda_\ell \otimes w \cdot \lambda \otimes p\tau) \otimes V^{[1]} \\
(5) & \downarrow \cong & \downarrow \cong \\
& \operatorname{RInd}_B^G(w \cdot \mu \otimes p\tau \otimes V^{[1]}) & \xrightarrow{\cong} \operatorname{RInd}_B^G(w \cdot \mu \otimes p\tau) \otimes V^{[1]} \\
(6) & \downarrow \cong & \downarrow \cong \\
& \operatorname{RInd}_B^G(w \cdot \mu \otimes p\tau \otimes V^{[1]}) & \xrightarrow{\cong} \operatorname{Ind}_B^G(w \cdot \mu \otimes p\tau) \otimes V^{[1]}
\end{array}$$

Diagram (1) commutes as a matter of notation, identifying the functor $T_\lambda^\mu(-)$ with $pr_\lambda L(-)$, when applied to the “block” associated to $W_{aff} \cdot \lambda$ (the top row isomorphism has already been given in T_λ^μ notation.) For diagram (2), note that the isomorphism in its top row may formally be applied to the same row with pr_μ removed. Next, remove pr_μ from the bottom row of (2) also. If we can get commutativity in the resulting rectangle

$$\begin{array}{ccc}
L \otimes \operatorname{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \xrightarrow{\cong} & L \otimes \operatorname{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} \\
\downarrow \cong & & \downarrow \cong \\
\operatorname{Ind}_B^G(L \otimes w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \xrightarrow{\cong} & \operatorname{Ind}_B^G(L \otimes w \cdot \lambda \otimes p\tau) \otimes V^{[1]}
\end{array}$$

We get it for (2) by applying pr_μ to the whole diagram, then pulling out $V^{[1]}$ on the right.

To get commutativity of the rectangle itself note that all four of its corners are induced modules, by the tensor identity, isomorphism to the lower left hand corner. Using the formalism in [J, I.3.4], every induced module $\operatorname{Ind}_B^G M$ (where M here just denotes some B -module) is equipped with a B -module map $\epsilon_M : \operatorname{Ind}_B^G M \rightarrow M$. If N is G -module the tensor identity isomorphism $\operatorname{Ind}_B^G(M \otimes N) \cong (\operatorname{Ind}_B^G M) \otimes N$ composed with $\epsilon_M \otimes N$ gives $\epsilon_{M \otimes N}$. (This can be extracted from the discussion in [J, I.3.6].) This implies that the usual universal property of induction (see [J, I. Prop. 3.46]) applies directly to $(\operatorname{Ind}_B^G M) \otimes N$ using $\epsilon_M \otimes N$ in the role of a “counit” adjunction (terminology of Wikipedia). Note that the target of $\epsilon_M \otimes N$ is $M \otimes N$. We will just call $\epsilon_M \otimes N$ the evaluation map associated with $(\operatorname{Ind}_B^G M) \otimes N$ and $M \otimes N$ the associated evaluation target. Returning to the rectangle above, all four of its corners, all obtained from the induced module in the lower left corner by various applications of the tensor identity, have the same target (up to associativity isomorphisms). Consequently, all maps in the rectangle may be viewed as “induced” from the identity map on their (common) target. (This certainly true in the case of an individual application of the tensor identity, from which it follows in the case of the tensor identity applied within a tensor product of several factors. All individual maps in the rectangle arise this way, and the property of being “induced” from the identity map on their (common) target. (This is certainly

true in the case of an individual application of the tensor identity, from which it follows in the case of the tensor identity applied within a tensor product of several factors. All individual maps in the rectangle arise this way, and the property of being “induced” from the identity map on a common target carries over to composition.) It follows now that the rectangle above is (thoroughly) commutative, as in (2).

Commutativity of (3) is easily seen to hold, since the derived functor RInd_B^G on both sides is applied to objects acyclic for Ind_B^G (i.e., their “higher derived functors vanish”). The meaning of the vertical maps in (4) was discussed in the construction of $\mathrm{Iso}_Y^{w \cdot \lambda}$. The horizontal maps in the bottom row as obtained from the generalized tensor identity. The top row map is obtained similarly, after pulling $V^{[1]}$ out of the block projection. Both column constructions may be viewed, before applying pr_μ , as arising from maps $L \rightarrow L' \leftarrow \lambda_\ell$ where L' is a quotient of L , tensoring with $w \cdot \lambda \otimes p\tau$ or $w \cdot \lambda \otimes p$ and applying $\mathrm{RInd}_B^G(-)$ or $\mathrm{RInd}_B^G(-) \otimes V^{[1]}$. Since the generalized tensor identity may be regarded as a natural transformation of functors. We obtain a commutative diagram rising from maps $L \rightarrow L' \leftarrow \lambda_\ell$ where L' is a quotient of L , tensoring with $w \cdot \lambda \otimes p\tau$ or $w \cdot \lambda \otimes p$ and applying $\mathrm{RInd}_B^G(-)$ or $\mathrm{RInd}_B^G(-) \otimes V^{[1]}$. Since the generalized tensor identity may be regarded as a natural transformation of functors. We obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{RInd}_B^G(L \otimes w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \cong & \mathrm{RInd}_B^G(L \otimes w \cdot \lambda \otimes p\tau) \otimes V^{[1]} \\
\downarrow & & \downarrow \\
\mathrm{RInd}_B^G(L' \otimes w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \cong & \mathrm{RInd}_B^G(L' \otimes w \cdot \lambda \otimes p\tau) \otimes V^{[1]} \\
\uparrow & & \uparrow \\
\mathrm{RInd}_B^G(\lambda_\ell \otimes w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \cong & \mathrm{RInd}_B^G(\lambda_\ell \otimes w \cdot \lambda \otimes p\tau) \otimes V^{[1]}
\end{array}$$

Now apply pr_μ and pullout $V^{[1]}$ on the right. All column isomorphism become the column isomorphisms in (4), equating the objects in the bottom row of the latter with the same objects with pr_μ applied. The top row of (4) has been discussed and agrees with the top row of the diagram above, after the modification.

The commutativity of diagram (5) is easy, since λ_ℓ is equal to $w \cdot \mu$ as a weight. It is interesting to note that the construction of $\mathrm{Iso}_Y^{w \cdot \lambda}$ must fix an isomorphism between the 1-dimensional section λ_ℓ of L , and the abstract 1-dimensional weight space $w \cdot \mu$.

Observation: In this sense $\mathrm{Iso}_Y^{w \cdot \lambda}$ can be modified by a nonzero scalar multiplication, and remain a version obtained by the “same” construction (still a natural transformation defined on the category \mathcal{Y}). Such a modification carries over to $T\mathrm{Iso}_Y^{w \cdot \lambda}$.

The pull-out operation in (6) is the generalized tensor identity in both rows, except that $\mathrm{RInd}_B^G(-)$ may be identified with $\mathrm{Ind}_B^G(-)$ on the bottom row. The columns just reflect this identification and the diagram is clearly commutative.

Note that the bottom row in (6) is precisely the right hand column in the upper diagram in (ii). The right hand column of the iterated rectangles (1), (2), \dots , (6) is by construction, $\mathrm{Iso}_{pr_\tau}^{w \cdot \lambda} \otimes V^{[1]}$. Thus, the outer perimeter of (1), (2), \dots , (6) gives a commutative version of the upper diagram in (ii), after turning the perimeter diagram on its side (left hand side put on top). This completes the proof of the proposition. Q.E.D.

The maps $\mathrm{Adj}_Y^{w \cdot \lambda}(Y \in \mathcal{Y})$ and their naturality

The map $Adj_Y^{w \cdot \lambda} : \text{Ind}_B^G(w \cdot \lambda \otimes Y) \longrightarrow T_\mu^\lambda(w \cdot \mu \otimes Y)$ is defined as the composition of adjunction $adj_X : X \longrightarrow T_\mu^\lambda T_\lambda^\mu X$, with $X = \text{Ind}_B^G(w \cdot \lambda \otimes Y)$, $Y \in \mathcal{Y}$, and the previously discussed isomorphism

$$T\text{Iso}_Y^{w \cdot \lambda} : T_\mu^\lambda T_\lambda^\mu \text{Ind}_B^G(w \cdot \lambda \otimes Y) \xrightarrow{\sim} T_\mu^\lambda \text{Ind}_B^G(w \mu \cdot \lambda \otimes Y)$$

The adjunction map \widetilde{adj}_X , $X \in G\text{-mod}$, is defined as the image of $1 \in \text{Hom}_G(T_\lambda^\mu X, T_\lambda^\mu X)$ under a natural isomorphism $\text{Hom}_G(T_\lambda^\mu -, -) \xrightarrow{\sim} \text{Hom}_G(-, T_\mu^\lambda -)$, with $X, T_\lambda^\mu X$ as the variables. (Thus $\text{Hom}_G(T_\lambda^\mu X, T_\lambda^\mu X) \cong \text{Hom}_G(X, T_\mu^\lambda T_\lambda^\mu X)$). In Appendix A, we have given a thorough discussion of \widetilde{adj}_X , constructing it from a similar adjunction map adj_X associated to the adjoint functors $L \otimes -$ and $L^* \otimes -$. We will quote from Appendix A to prove the proposition below.

Proposition 5.7. (i) *The maps \widetilde{adj}_X ($X \in G\text{-mod}$) collectively give the adjunction natural transformation from the identity functor to $T_\mu^\lambda T_\lambda^\mu$.*

(ii) *For any V in $G\text{-mod}$ with all weights in the root lattice, there is a commutative diagram.*

$$\begin{array}{ccc} X \otimes V^{[1]} & \xrightarrow{\widetilde{adj}_{X \otimes V^{[1]}}} & T_\mu^\lambda T_\lambda^\mu (X \otimes V^{[1]}) \\ \downarrow = & & \downarrow \cong \\ X \otimes V^{[1]} & \xrightarrow{\widetilde{adj}_{X \otimes V^{[1]}}} & (T_\mu^\lambda T_\lambda^\mu X) \otimes V^{[1]} \end{array}$$

The right hand column is morphism becomes equality, if both right hand objects are viewed as a submodules of $L^ \otimes L \otimes X \otimes V^{[1]}$.*

Proof. Part (i) has already been discussed. Note the obvious fact that adjunctions are natural transformations. (A proof is written down in footnote 11 of this paper, noted in the proof of Proposition 4.1.)

Part (ii) follows from Corollary 4.2.

Q.E.D.

We can now give parallel properties of $Adj^{w \cdot \lambda}$, meant especially to mirror Proposition 5.5 for $Jan^{w \cdot \lambda}$.

Proposition 5.8. (i) *The maps $Adj_Y^{w \cdot \lambda}, Y \in \mathcal{Y}$, collectively define a natural transformation of functors*

$$\text{Ind}_B^G(w \cdot \lambda \otimes -) \longrightarrow T_\mu^\lambda \text{Ind}_B^G(w \cdot \mu \otimes -)$$

on the category \mathcal{Y}

(ii) *For any fixed $Y = p\tau \otimes V^{[1]}$ in \mathcal{Y} , there is a commutative diagram with “obvious” vertical isomorphisms, natural in V :*

$$\begin{array}{ccc} \text{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \xrightarrow{Adj_Y^{w \cdot \lambda}} & T_\mu^\lambda (w \cdot \mu \otimes p\tau \otimes V^{[1]}) \\ \downarrow \cong & & \downarrow \cong \\ \text{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} & \xrightarrow{Adj_{p\tau \otimes V^{[1]}}^{w \cdot \lambda}} & T_\mu^\lambda (w \cdot \mu \otimes p\tau) \otimes V^{[1]} \end{array}$$

Proof. Part (i) follows from the definition $Adj_Y^{w \cdot \lambda} = T\text{Iso}_Y^{w \cdot \lambda} \circ \widetilde{adj}_X$ with $X = \text{Ind}_B^G(w \cdot \lambda \otimes Y)$.

For part (ii), note that the left hand column of the lower diagram in Proposition 3(ii) may be written as a composition

$$T_\mu^\lambda T_\lambda^\mu \text{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) \cong T_\mu^\lambda T_\lambda^\mu (\text{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]}) \cong T_\mu^\lambda T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes p\tau) \otimes V^{[1]}.$$

The second isomorphism is the right hand column of Proposition 5.7(ii). To deal with the first isomorphism, we need the following Lemma.

Lemma 5.9. *For $Y = p\tau \otimes V^{[1]} \in \mathcal{Y}$, there is a commutative diagram*

$$\begin{array}{ccc} \text{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \xrightarrow{\widetilde{\text{adj}}_{\text{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]})}} & T_\mu^\lambda T_\lambda^\mu \text{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) \\ \downarrow \cong & & \downarrow \cong \\ \text{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} & \xrightarrow{\widetilde{\text{adj}}_{\text{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]}}} & T_\mu^\lambda T_\lambda^\mu (\text{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]}) \end{array}$$

Proof. This is just naturality of $\widetilde{\text{adj}}_X$ with respect to $X \in G\text{-mod}$, applied to the G -module isomorphism comprising the left column. Q.E.D.

We now return to the proof of Proposition 5.8. Put the diagram of Lemma 5.9 on top of that of Proposition 5.7 (ii), taking $X = \text{Ind}_B^G(w \cdot \lambda \otimes Y)$. In this case, the top horizontal edge of the diagram in Proposition 5.7(ii) agrees with the bottom edge of the lemma, so concatenation makes sense. Moreover the right hand column of the concatenated diagram agrees with the left hand column of the lower diagram in Proposition 5.6(ii). (The latter column was discussed above as composition of isomorphisms.) This allows a further concatenation, a giving commutative diagram

$$\begin{array}{ccccc} \text{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \longrightarrow & T_\mu^\lambda T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \longrightarrow & T_\mu^\lambda \text{Ind}(w \cdot \mu \otimes p\tau \otimes V^{[1]}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \text{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} & \longrightarrow & T_\mu^\lambda T_\lambda^\mu (\text{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]}) & & \downarrow \cong \\ \downarrow = & & \downarrow \cong & & \downarrow \\ \text{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} & \longrightarrow & T_\mu^\lambda T_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} & \longrightarrow & T_\mu^\lambda \text{Ind}(w \cdot \mu \otimes p\tau \otimes V^{[1]}). \end{array}$$

The top row maps are $\widetilde{\text{adj}}_{\text{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]})}$, on the left and $T\text{Iso}_Y^{w \cdot \lambda}$, with $Y = p\tau \otimes V^{[1]}$, on the right. In the same order, the bottom row maps are $\widetilde{\text{adj}}_{\text{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]}}$ and $T\text{Iso}_{p\tau}^{w \cdot \lambda} \otimes V^{[1]}$. The composition of the two top row maps is $\text{Adj}_Y^{w \cdot \lambda}$, and the composition of the bottom row maps is $\text{Adj}_{p\tau}^{w \cdot \lambda} \otimes V^{[1]}$. The commutativity of the outer rectangle now gives the desired commutativity of the diagram in (ii) Q.E.D.

We are just about ready for a vast improvement to Proposition 5.4. First we need an easy but key observation.

Lemma 5.10. *For any $Y = p\tau \otimes V^{[1]}$ in \mathcal{Y} , the left column “obvious” isomorphisms in the diagrams of Proposition 5.5(ii) and Proposition 5.8(ii) are equal as are the right column.*

Proof. On the left, both isomorphisms just pull out $V^{[1]}$ using the tensor identity. A similar isomorphism is used on the right (in both cases) except it is also necessary to commute $T_\mu^\lambda(-)$ and $(-)\otimes V^{[1]}$. Q.E.D.

We can now prove a main theorem.

Theorem 5.11. *There is a nonzero scalar $c \in k$ such that $\text{Adj}_Y^{w \cdot \lambda} = c\text{Jan}_Y^{w \cdot \lambda}$ for all $y \in \mathcal{Y}$.*

Proof. Proposition 5.4 gives a nonzero scalar that works in the especial case $Y = p\tau$. The constant it gives is possibly dependent on Y and call it $c(p\tau)$.

Propositions 5.5(ii) and 5.8(iii), together with Lemma 5.10, show we claim, an equality

$$\text{adj}_Y^{w \cdot \lambda} = c(p\tau)\text{Jan}_{p\tau}^{w \cdot \lambda}.$$

whenever $Y = p\tau \otimes V^{[1]} \in \mathcal{Y}$. To prove this equality, note $Adj^{w,\lambda} = c(p\tau)Jan_{p\tau}^{w,\lambda}$. Tensor on the right with $V^{[1]}$ to get

$$Adj_{p\tau}^{w,\lambda} \otimes V^{[1]} = c(p\tau)Jan_{p\tau}^{w,\lambda} \otimes V^{[1]}$$

Precompose each side with the downward left column isomorphism common to the diagrams in Propositions 5.5(ii) and 5.8(ii), and postcompose with the upward right column isomorphism. This gives the claimed equality (reading it off from the two commutative diagrams and the previous equality.)

It remains to prove $c(p\tau) = c(p\tau')$ whenever $p\tau, p\tau'$ are 1-dimensional objects in \mathcal{Y} . Note that $p(\tau + \tau')$ with necessarily, also belong to \mathcal{Y} . We will show $c(p\tau) = c(p(\tau + \tau'))$. This is enough, since the equality $c(\tau') = c(p(\tau' + \tau))$ will follow by re-choosing notations.

Let $V = \text{Ind}_B^G(\tau')$. The $p\tau \otimes V^{[1]}$ and $p\tau \otimes p\tau' = p(\tau + \tau')$ both belong to Y . There a B - module map (“evaluation”) from $V = \text{Ind}_B^G(\tau')$ onto τ' , and we twist it by the Frobenius to get a surjective map $V^{[1]} \rightarrow p\tau'$. Tensor on the left with $p\tau$ to get a surjective map $p\tau \otimes V^{[1]} \rightarrow p(\tau + \tau')$. We will call this map ϕ . It is a map in the category \mathcal{Y} .

We have commutative diagram by naturality of $Jan^{w,\lambda}$ with respect to \mathcal{Y}

$$\begin{array}{ccc} \text{Ind}_B^G(w \cdot \lambda \otimes p(\tau + \tau')) & \xrightarrow{Jan_{p(\tau+\tau')}^{w,\lambda}} & T_\mu^\lambda \text{Ind}(w \cdot \mu \otimes p(\tau + \tau')) \\ \text{Ind}_B^G(w \cdot \lambda \otimes \phi) \uparrow & & \uparrow T_\mu^\lambda \text{Ind}(w \cdot \mu \otimes \phi) \\ \text{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes V^{[1]}) & \xrightarrow{Jan_{p\tau \otimes V^{[1]}}^{w,\lambda}} & T_\mu^\lambda \text{Ind}(w \cdot \mu \otimes p\tau \otimes V^{[1]}) \end{array}$$

The left column map is nonzero, since its composition with the evaluation map $\text{Ind}_B^G(w \cdot \lambda \otimes p(\tau + \tau')) \rightarrow w \cdot \lambda \otimes p(\tau + \tau')$ is nonzero. The top row map is injective, an instance of the left part of the short exact sequence displayed above Proposition 5.5. Hence the composition

$$Jan_{p(\tau+\tau')}^{w,\lambda} \circ \text{Ind}_B^G(w \cdot \lambda \otimes \phi)$$

is not zero.

However, there is a similar diagram, identical to the above, but with “Jan” replaced by “Adj”. We have

$$Adj_{p(\tau+\tau')}^{w,\lambda} \circ \text{Ind}_B^G(w \cdot \lambda \otimes \phi) = c(p(\tau + \tau'))Jan_{p(\tau+\tau')}^{w,\lambda} \circ \text{Ind}_B^G(w \cdot \lambda \otimes \phi).$$

On the other hand, commutativity of the “Adj” diagram equates the left expression with $T_\mu^\lambda \text{Ind}_B^G(w \cdot \mu \otimes \phi) \circ Adj_{p\tau \otimes V^{[1]}}^{w,\lambda} = T_\mu^\lambda \text{Ind}_B^G(w \cdot \mu \otimes \phi) \circ c(p\tau)Jan_{p\tau \otimes V^{[1]}}^{w,\lambda}$. Now bring out the scalar $c(\phi)$ and apply commutativity in the “Jan” diagram. The right expression becomes $c(p\tau)Jan_{p(\tau+\tau')}^{w,\lambda} \text{Ind}_B^G(w \cdot \lambda \otimes \phi)$. We have shown

$$c(p(\tau + \tau'))Jan_{p(\tau+\tau')}^{w,\lambda} \circ \text{Ind}_B^G(w \cdot \lambda \otimes \phi) = c(p\tau)Jan_{p(\tau+\tau')}^{w,\lambda} \circ \text{Ind}_B^G(w \cdot \lambda \otimes \phi)$$

But we have shown that the map appearing to the right of both $c(p(\tau + \tau'))$ and $c(p\tau)$ above is not zero. So, the only way the equality can occur is to have $c(p(\tau + \tau')) = c(p\tau)$.

This completes the proof of the theorem.

Q.E.D.

Though not entirely necessary, it simplifies notation if we modify $TIso_Y^{w,\lambda}$ by a scalar by changing its construction, as per the observation in the proof of Proposition 5.6. We do this so that the newly constructed $TIso_Y^{w,\lambda}$ is the old one multiplied by c above (uniformly in $Y \in \mathcal{Y}$). This allows us to have actual equalities.

$Adj_Y^{w,\lambda} = Jan_Y^{w,\lambda}$ for all $Y \in \mathcal{Y}$. This is also a good time to enlarge the domain of the natural transformation $Adj^{w,\lambda}$ and $Jan^{w,\lambda}$ from \mathcal{Y} to $add\mathcal{Y}$ (which has objects direct sums of objects of \mathcal{Y} . Similar

domain enlargements can be made for $Iso^{w \cdot \lambda}$ and $TIso^{w \cdot \lambda}$. The domain $add \mathcal{Y}$ can also be extended to complexes of objects from $add \mathcal{Y}$, such as the complex K^\bullet discussed as Lemma 5.3.

In the next proposition, we use this formalism to illuminate the triangle (*) above the claim in 5.1, rewritten below

$$(*) \quad \cdots \longrightarrow \mathrm{RInd}_B^G(w \cdot \lambda) \longrightarrow T_\mu^\lambda \mathrm{RInd}_B^G(w \cdot \lambda) \longrightarrow \mathrm{RInd}_B^G(ws \cdot \lambda) \longrightarrow \cdots$$

Proposition 5.12. *With a suitable choice of the complex K^\bullet in Lemma 5.3, there is an exact sequence of complexes*

$$0 \longrightarrow \mathrm{Ind}(w \cdot \lambda \otimes K^\bullet) \longrightarrow T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet) \longrightarrow \mathrm{Ind}_B^G(ws \cdot \lambda \otimes K^\bullet) \longrightarrow 0$$

which represent (*) at the level of complexes (in the sense that its sequence of these objects and two maps - ignoring the 0's - identifies, after passing to the bounded derived category, with the displayed portion above of (*)).

The left hand complex map is $\mathrm{Jan}_{K^\bullet}^{w \cdot \lambda}$, the extension of $\mathrm{Jan}^{w \cdot \lambda}$ to complexes of $add \mathcal{Y}$ objects, in the particular case of the complex K^\bullet .

Proof. Choose K^\bullet so that each K^n is a direct sum of terms $Y = p\tau \otimes V^{[1]}$ in \mathcal{Y} with also $\nu + p\tau$ dominant for each weight ν of $L^* \otimes w \cdot \mu$. The construction of (*) is from the exact sequence at the top of 5.1

$$0 \longrightarrow M \longrightarrow L^* \otimes w \cdot \mu \longrightarrow M' \longrightarrow 0$$

by applying $pr_\lambda \mathrm{RInd}_B^G(-)$. With our choice of K^\bullet , $\mathrm{RInd}_B^G(-)$ applied to each term is the same as $\mathrm{Ind}_B^G(- \otimes K^\bullet)$. Also, pr_λ applied to $\mathrm{Ind}_B^G(\nu \otimes K^\bullet)$ for $\nu \neq w \cdot \lambda$ in M or $\nu \neq ws \cdot \lambda$ in M' , is the zero complex.

Thus, $pr_\lambda \mathrm{Ind}_B^G(- \otimes K^\bullet)$, applied to the displayed sequence, gives an exact sequence

$$0 \longrightarrow \mathrm{Ind}_B^G(w \cdot \lambda \otimes K^\bullet) \longrightarrow pr_\lambda \mathrm{Ind}(L^* \otimes w \cdot \mu \otimes K^\bullet) \longrightarrow \mathrm{Ind}_B^G(ws \cdot \lambda \otimes K^\bullet) \longrightarrow 0$$

The middle term identifies with $pr_\lambda(L^* \otimes \mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet))$ via the tensor identity. Such an identification must also be made in the construction of (*), though with $\mathrm{RInd}_B^G(w \cdot \mu)$ replacing $\mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet)$. Note also $pr_\lambda(L^* \otimes \mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet)) = pr_\lambda(L^* \otimes pr_\mu \mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet)) = T_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet)$, and a similar equality holds for $\mathrm{RInd}_B^G(w \cdot \mu)$.

Following each step above gives the identifications claimed in the proposition. An alternate argument could be made by replacing K^\bullet in the short exact sequence displayed above with a complex of injective B -modules (a resolution of $k = k(0)$ also). This gives a semisplit short exact sequence of complexes. Its three term sequence then automatically becomes a three term sequence in a distinguished triangle, upon passing to the derived category. In more detail, let $K^\bullet \longrightarrow I^\bullet$ be an isomorphism of complexes, with I^\bullet a B -module injective resolution of $k = k(0)$. There are resulting commutative diagram of map of G -module complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & pr_\tau \mathrm{Ind}(M \otimes I^\bullet) & \longrightarrow & pr_\lambda \mathrm{Ind}_B^G(L^* \otimes w \cdot \mu \otimes I^\bullet) & \longrightarrow & pr_\tau \mathrm{Ind}(M \otimes I^\bullet) \longrightarrow 0 \\
& & \uparrow & & \searrow & & \uparrow \\
& & & & pr_\lambda(L^* \otimes \mathrm{Ind}_B^G(w \cdot \mu \otimes I^\bullet)) & & \\
& & & & \uparrow & & \\
0 & \longrightarrow & pr_\tau \mathrm{Ind}(M \otimes K^\bullet) & \longrightarrow & pr_\lambda \mathrm{Ind}_B^G(L^* \otimes w \cdot \mu \otimes K^\bullet) & \longrightarrow & pr_\tau \mathrm{Ind}(M \otimes K^\bullet) \longrightarrow 0 \\
& & & & \searrow & & \uparrow \\
& & & & pr_\lambda(L^* \otimes \mathrm{Ind}_B^G(w \cdot \mu \otimes I^\bullet)) & &
\end{array}$$

The skew maps are isomorphism of complexes, and the vertical maps are all quasi-isomorphisms. The top row is an exact sequence of complexes of injective objects, is therefore semi-split, and therefore becomes part of a distinguished triangle (ignoring the zeros and zero maps) at the derived category label (D^+ or D^b here). There are a few other commutative squares of quasi isomorphism need to give a complete picture of the identification claimed in the proposition, but we leave then to the reader (who should have the idea by now). On the left, for example, diagrams must be added handling the identifications $\text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet) \cong \text{pr}_\lambda \text{Ind}_B^G(M \otimes K^\bullet)$. (This will require two rectangles, associated with the location of $w \cdot \lambda$ as a section of M .)

The analogous and simpler identification $\text{Ind}_B^G(w \cdot \lambda \otimes Y) \cong \text{pr}_\lambda \text{Ind}_B^G(M \otimes Y)$ is part of the Jantzen map $\text{Jan}_Y^{w \cdot \lambda}$, discussed above Proposition 5.5, partly using “ RInd_B^G ” notation. Sticking to the “ Ind_B^G ” notation, the map $\text{Jan}_Y^{w \cdot \lambda}$ is the composite of $\text{Ind}_B^G(w \cdot \lambda \otimes Y) \cong \text{pr}_\lambda(L^* \otimes \text{Ind}_B^G(w \cdot \mu \otimes Y))$, with $\text{pr}_\lambda \text{Ind}_B^G(M \otimes Y) \rightarrow \text{pr}_\lambda \text{Ind}_B^G(L^* \otimes w \cdot \mu \otimes Y) \cong \text{pr}_\lambda(L^* \otimes \text{Ind}_B^G(w \cdot \mu \otimes Y))$, the latter equal to $\text{pr}_\lambda(L^* \otimes \text{pr}_\mu \text{Ind}_B^G(w \cdot \mu \otimes Y) \cong \text{T}_\mu^\lambda \text{Ind}(w \cdot \mu \otimes Y)$.

Passing from \mathcal{Y} to $\text{add } \mathcal{Y}$ and then to complexes of $\text{add } \mathcal{Y}$ objects, and, following the pathway above, we find that $\text{Jan}_K^{w \cdot \lambda}$ is the composition of the identification $\text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet) \cong \text{pr}_\lambda \text{Ind}_B^G(M \otimes K^\bullet)$, the bottom left map of the above diagram followed by the adjacent skew map, and finally the identification

$$\text{pr}_\lambda(L^* \otimes \text{Ind}_B^G(w \cdot \mu \otimes K^\bullet) = \text{pr}_\lambda(L^* \otimes \text{pr}_\mu(\text{Ind}_B^G(w \cdot \mu \otimes K^\bullet) = \text{T}_\mu^\lambda \text{Ind}_B^G(w \cdot \mu \otimes K^\bullet).$$

This is, altogether, precisely the map

$$\text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet) \longrightarrow \text{T}_\mu^\lambda \text{Ind}_B^G(w \cdot \mu \otimes K^\bullet)$$

which our construction, in completed form, gives for the left hand amp in the exact sequences displayed in the proposition. So that map is $\text{Jan}_K^{w \cdot \lambda}$, and our proof of the proposition is complete. Q.E.D.

The general case of the claim of 5.1

Recall that we noted in 5.1 that the middle term of the distinguished triangle (*) was isomorphic to $\text{T}_\mu^\lambda \text{T}_\lambda^\mu \text{RInd}_B^G(w \cdot \lambda)$. Noting further that this resulted in a map $\text{RInd}_B^G(w \cdot \lambda) \longrightarrow \text{T}_\mu^\lambda \text{T}_\lambda^\mu \text{RInd}_B^G(w \cdot \mu)$, we claimed that this map was adjunction, at least up to scalar multiple.

To some extent, this requires first interpreting what isomorphism of $\text{T}_\mu^\lambda \text{T}_\lambda^\mu \text{RInd}_B^G(w \cdot \lambda)$ with the middle term $\text{T}_\mu^\lambda \text{RInd}_B^G(w \cdot \mu)$ of (*) was intended. The top of subsection 5.1 indicates that an isomorphism $\text{T}_\lambda^\mu \text{RInd}_B^G(w \cdot \lambda) \cong \text{RInd}_B^G(w \cdot \mu)$ be used. We will follow that framework. Represent $\text{RInd}_B^G(w \cdot \lambda)$ by $\text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet)$ and $\text{RInd}_B^G(w \cdot \mu)$ by $\text{Ind}_B^G(w \cdot \mu \otimes K^\bullet)$, as in Proposition 5.12 and its proof. Then an isomorphism $\text{T}_\lambda^\mu \text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet) \cong \text{Ind}_B^G(w \cdot \mu \otimes K^\bullet)$ is s given by $\text{Iso}_K^{w \cdot \lambda}$. Composing with T_μ^λ on both sides gives the isomorphism of complexes

$$\text{T Iso}_K^{w \cdot \lambda} : \text{T}_\mu^\lambda \text{T}_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes K^\bullet) \cong \text{T}_\mu^\lambda \text{Ind}(w \cdot \mu \otimes K^\bullet).$$

Note that the two sides represent $\text{T}_\mu^\lambda \text{T}_\lambda^\mu \text{RInd}_B^G(w \cdot \lambda)$ and $\text{T}_\mu^\lambda \text{RInd}_B^G(w \cdot \mu)$, respectively.

Finally, we need a way to represent the adjunction map from $\text{RInd}_B^G(w \cdot \lambda)$, represented as $\text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet)$, to $\text{T}_\mu^\lambda \text{T}_\lambda^\mu \text{RInd}_B^G(w \cdot \lambda)$, which we have represented as $\text{T}_\mu^\lambda \text{T}_\lambda^\mu \text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet)$. For this, we use the map of complexes

$$\widetilde{\text{adj}}_{\text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet)} : \text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet) \longrightarrow \text{T}_\mu^\lambda \text{T}_\lambda^\mu \text{Ind}(w \cdot \lambda \otimes K^\bullet).$$

This map just applies adjunction to the G -modules in each degree of the complex $\text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet)$. This results in a map of complexes, by naturality of adjunction. In particular, passing to the derived category level (D^+ or D^b), the map of complexes $\widetilde{\text{adj}}_{\text{Ind}_B^G(w \cdot \lambda \otimes K^\bullet)}$ induces “adjunction” as a map

$$\mathrm{RInd}_B^G(w \cdot \lambda) \longrightarrow T_\mu^\lambda T_\lambda^\mu \mathrm{RInd}_B^G(w \cdot \lambda)$$

The result below is a corollary to Theorem 5.11 and Proposition 5.12.

Corollary 5.13. *With the isomorphism $T_\mu^\lambda \mathrm{RInd}_B^G(w \cdot \mu) \xrightarrow{\sim} T_\mu^\lambda T_\lambda^\mu \mathrm{RInd}_B^G(w \cdot \lambda)$ taken as inverse to the isomorphism induced by $T\mathrm{Iso}_{K^\bullet}^{w \cdot \lambda}$ discussed above, the claim of subsection 5.1 is correct. More precisely, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{RInd}_B^G(w \cdot \lambda) & \longrightarrow & T_\mu^\lambda T_\lambda^\mu \mathrm{RInd}_B^G(w \cdot \lambda) \\ \downarrow = & & \downarrow \cong \\ \mathrm{RInd}_B^G(w \cdot \lambda) & \longrightarrow & T_\mu^\lambda \mathrm{RInd}_B^G(w \cdot \mu) \end{array}$$

with the top row adjunction and the bottom row from (*). The right column isomorphism is as above.

Proof. We have $\mathrm{Adj}_Y^{w \cdot \lambda} = T\mathrm{Iso}_Y^{w \cdot \lambda} \circ \widetilde{\mathrm{adj}}_{\mathrm{Ind}_B^G(w \cdot \lambda \otimes Y)}$ for all $Y \in \mathcal{Y}$, by definition. Passing to $\mathrm{adj} \mathcal{Y}$ and complexes such as K^\bullet , we have the similar identity

$$\mathrm{Adj}_{K^\bullet}^{w \cdot \lambda} = T\mathrm{Iso}_{K^\bullet}^{w \cdot \lambda} \circ \widetilde{\mathrm{adj}}_{\mathrm{Ind}(w \cdot \lambda \otimes K^\bullet)}$$

By Theorem 5.11 we also have

$$\mathrm{Adj}_{K^\bullet}^{w \cdot \lambda} = c \mathrm{Jan}_{K^\bullet}^{w \cdot \lambda}$$

where c is a non zero scalar. If we adjust $T\mathrm{Iso}^{w \cdot \lambda}$ as per the (bold-faced), observation in the proof of Proposition 5.6, we may assume $c = 1$. Assume that this adjustment is in force. Then we have a commutative diagram at complexes

$$\begin{array}{ccc} \mathrm{Ind}_B^G(w \cdot \lambda \otimes K^\bullet) & \xrightarrow{\widetilde{\mathrm{adj}}_{\mathrm{Ind}_B^G(w \cdot \lambda \otimes K^\bullet)}} & T_\mu^\lambda T_\lambda^\mu \mathrm{Ind}(w \cdot \lambda \otimes K^\bullet) \\ \downarrow = & & \downarrow T\mathrm{Iso}_{K^\bullet}^{w \cdot \lambda} \\ \mathrm{Ind}_B^G(w \cdot \lambda \otimes K^\bullet) & \xrightarrow{\mathrm{Jan}_{K^\bullet}^{w \cdot \lambda}} & T_\mu^\lambda \mathrm{Ind}(w \cdot \mu \otimes K^\bullet) \end{array}$$

By Proposition 5.6 the bottom arrow represents the map $\mathrm{RInd}_B^G(w \cdot \lambda) \longrightarrow T_\mu^\lambda \mathrm{RInd}_B^G(w \cdot \mu)$ in (*). We discussed, above the statement of the corollary, the fact that the top row becomes the adjunction map $\mathrm{RInd}_B^G(w \cdot \lambda) \longrightarrow T_\mu^\lambda T_\lambda^\mu \mathrm{RInd}_B^G(w \cdot \lambda)$ upon passing to the derived category. The right column map is, as discussed in the statement of the corollary the right column derived derived category, isomorphism in the corollary's diagram. Altogether, the commutativity of the diagram of complexes above gives the commutativity of the diagram in the corollary. This completes its proof. Q.E.D.

Remark 5.14. Without the adjustment observed in the proof of Proposition 5.6, we only get commutativity up to a scalar, as allowed in the claim.

Remark 5.15. The quantum case The same changes of \otimes to \otimes^{op} observed in Appendix A need to be made in this appendix, in the quantum case. In addition it is necessary to replace the references to [J] in the proof of Proposition 5.6 with references to Remarks 2.11(d),(e). Remark 2.11(f) helps explain the differences in the formalism of these remarks (which also could be used in the algebraic groups case) with that of [J]. Recall also that Remark 2.11(d) provides both right and left generalized tensor identities in the quantum case, heavily used in the arguments above (for example, in the proofs of Propositions 5.5 and 5.6. With these changes and observations, all of the proofs and results in this appendix carry over to the quantum case.

In particular, the claim of subsection 5.1 holds in both the algebraic groups and quantum cases. As argued below the claim, this completes the proof of Lemma 3.2.

6. Appendix C

The purpose of this appendix is to supplement, and, indeed, to “fix,” the statement and proof of [ABG, Lem. 9.10.5], as a service to the reader. This is all in characteristic 0, and not part of the induction theorem (except in the way of application), but it is important to [ABG] as a whole and to the discussion in [PS2, ft.13] concerning Koszulity in the quantum case. The proof given in [ABG] of the lemma, corrected for misprints and issues with the induction theorem proof, still seemed inaccurate to us, but we found it could be fixed using an algebraic result from [PS2]. The latter result is nontrivial, but relatively elementary, not using the Lusztig quantum conjecture. This seems desirable, so that [ABG] could have the latter conjecture, in the $\ell > h$ case, as a corollary.

Our notation in this appendix largely follows [ABG], with two major changes: The formula for the “dot” notation \bullet is replaced by that for the standard “dot” action \cdot in [J] and subsection 2.4.2 above. Thus, the new formula reads, for w in the Weyl group or affine Weyl group, and $\lambda \in X$,

$$w \bullet \lambda = w(\lambda + \rho) - \rho.$$

Also, we will use Borel subalgebras \mathbf{B} whose associated roots are negative, rather than positive. With these two changes, [ABG, Lem. 3.5.1], which we will use below, is correct as stated. (It actually was not, before, even for $w = 1$.) The statement of the quantum induction theorem [ABG, Thm. 3.5.5], which we will also use, is unchanged. Finally, the change from positive to negative Borels on the quantum side is deliberately not repeated on the Langlands dual side, when choosing Borel objects there (associated to Grassmanian varieties).

At the point the result [ABG, Lem. 9.10.5] in question is introduced in [ABG] the authors have established an equivalence of derived categories [ABG, (9.10.1)]

$$\gamma : D^b \mathbf{block}(\mathbf{U}) \rightarrow D^b \mathcal{Perv}(\mathbf{Gr}),$$

and it is desired to show the functor γ induces an equivalence from $\mathbf{block}(\mathbf{U})$ to $\mathcal{Perv}(\mathbf{Gr})$. To this end, categories $D_{\leq \lambda}^b \mathbf{block}(\mathbf{U})$ and $D_{\leq \lambda}^p \mathcal{Perv}$ are introduced “for each $\lambda \in Y^{++}$.” This appears to be a misprint, repeated several times on [ABG, p. 668], and the definition of $D_{\leq \lambda}^b \mathbf{block}(\mathbf{U})$ is incorrect with any choice of λ . Instead, these categories should be introduced for each $\lambda \in Y$, with $\mu \leq \lambda$ interpreted to mean $\tilde{\mu} \uparrow \tilde{\lambda}$, where $\tilde{\mu} \in W \bullet \ell \mu$ is, in our notation here, the (unique) dominant weight, and $\tilde{\lambda}$ is defined similarly.

The order \uparrow above is that discussed in [J, II,6.1-6.11]; note that p there is allowed to be any positive integer. The order \uparrow should replace the order \preceq in [ABG, (3.4.5)]. The following equivalence (in the case $\ell \geq h$) follows from the more general theorem [PS2, Thm. 9.6], which also has a formulation for $\ell < h$

$$y \bullet 0 \uparrow w \bullet 0 \text{ iff } y \leq' w,$$

whenever $y \bullet 0, w \bullet 0$ are dominant and $y, w \in W_{\text{aff}}$. The order \leq' is the Bruhat–Chevalley order with respect to the dominant standard chamber fundamental reflections. This equivalence seems essential to correct the lemma.

We shall use \leq for the Bruhat–Chevalley order with respect to the antidominant standard chamber. Thus, for $y, w \in W_{\text{aff}}$, $y \leq w$ iff $w_0 y w_0 \leq' w_0 w w_0$, with w_0 the long word in W . When y, w are in W , $y \leq w$ means the same as $y \leq' w$. When $\nu, \mu \in Y$, $\nu y \leq \mu w$ iff $(-\nu)y \leq' (-\mu)w$. (The classical root system has an automorphism $y \mapsto -w_0(y)$, preserving positive roots.) Notice there is an implicit change from \leq' to \leq in the proof of [ABG, Cor. 8.3.2]. (This occurs in the assertion that “ λw^{-1} is minimal in the right coset $\lambda W \leq W_{\text{aff}}^n$.” The hypothesis of Cor. 8.3.2(ii) gives minimality of $w\lambda$ with respect to \leq' . Passing to inverses gives minimality of $(-\lambda)w^{-1}$ with respect to \leq' . Now it is necessary, it seems, to use \leq to get minimality of λw^{-1} .)

The definition of $D_{\leq \lambda}^b \mathcal{Perv}$ is correct as given in [ABG, p. 668] provided it is allowed that $\lambda \in \Upsilon$. Similarly, λ should be taken in Υ in the statement of the lemma, which we provide below, with this change. Note the direction of Υ is reverse to the equivalence in part (i).

Lemma 6.1. ([ABG, 9.10.5]) *For any $\lambda \in \Upsilon$, we have*

(i) *the functor Υ induces an equivalence*

$$D_{\leq \lambda}^b \mathcal{Perv} \xrightarrow{\sim} D_{\leq \lambda}^b \mathbf{block}(\mathbf{U}).$$

Moreover,

(ii) *the induced functor*

$$D_{\leq \lambda}^b \mathcal{Perv} / D_{< \lambda}^b \mathcal{Perv} \longrightarrow D_{\leq \lambda}^b \mathbf{block}(\mathbf{U}) / D_{< \lambda}^b \mathbf{block}(\mathbf{U})$$

sends the class of IC_λ to the class of L_λ .

Proof. We follow [ABG], taking into account the changes above, and also the misprints noted in [ABG, p. 675]. There are also some inaccuracies in [ABG, Cor. 8.2.4, Cor. 8.3.2] which we address as they arise.

We know for any $\lambda \in \Upsilon$, the functor Υ sends, by construction, the object $R \operatorname{Ind}_{\mathbb{B}}^{\mathbf{U}}(\ell\lambda)$ to \overline{W}_λ . Fix $\lambda \in \Upsilon$, and let $w \in W$ be the element with $w\lambda \bullet 0 = w \bullet \ell\lambda$ dominant. Then, by [ABG, Lem. 3.5.1]—see our Remark 2.11(a) and subsections 2.5, 2.6 for additional details—we have that $R^{\ell(w)} \operatorname{Ind}_{\mathbb{B}}^{\mathbf{U}}(\ell\lambda)$ has L_λ as a composition factor with multiplicity one, and all other composition factors L_μ of $R^{\ell(w)} \operatorname{Ind}_{\mathbb{B}}^{\mathbf{U}}(\ell\lambda)$, or any composition factor L_μ of $R^j \operatorname{Ind}_{\mathbb{B}}^{\mathbf{U}}(\ell\lambda)$ with $j \neq \ell(w)$, satisfy $y\mu \bullet 0 \uparrow w\lambda \bullet 0$, with $y \in W_{\text{aff}}$, $y\mu \bullet 0$ dominant, and $y\mu \bullet 0 \neq w\lambda \bullet 0$. For any such μ , we have $y\mu < w\lambda$, as noted above, and $\mu y^{-1} < \lambda w^{-1}$. As observed in the proof of [ABG, Cor. 8.3.2], this implies $\operatorname{supp} \overline{W}_\mu = \overline{\operatorname{Gr}}_\mu \subseteq \overline{\operatorname{Gr}}_\lambda = \operatorname{supp} \overline{W}_\lambda$, and the inclusion is proper. Thus, Υ takes $D_{\leq \lambda}^b \mathbf{block}(\mathbf{U})$ into $D_{\leq \lambda}^b \mathcal{Perv}$. By induction (on, say, the height of the dominant weight $w \bullet \ell\lambda$), we may assume Υ induces an equivalence of triangulated categories, when $\leq \lambda$ is replaced by $< \lambda$. (Here $\mu < \lambda$ is taken to mean $y\mu \bullet 0 \uparrow w\lambda \bullet 0$ as above, with $y \in W$ and $y\mu \bullet 0 \uparrow w\lambda \bullet 0$, $y\mu \bullet 0 \neq w\lambda \bullet 0$. Equivalently, $\overline{\operatorname{Gr}}_\mu$ is properly contained in $\overline{\operatorname{Gr}}_\lambda$, as we have seen.) By [ABG, §9.1, p. 655] $\mathcal{Perv}(\operatorname{Gr})$ is generated by simple objects IC_ν , $\nu \in \Upsilon$, each with support contained in $\overline{\operatorname{Gr}}_\nu$.

We take this opportunity to mention there are errors of sign in [ABG, Cor. 8.2.4, Cor. 8.3.2], where $C_{yw}[-\dim \mathcal{B}_{yw}]$ should be replaced by $C_y[\dim \mathcal{B}_{yw}]$ and $C_\lambda[-\dim \operatorname{Gr}_\lambda - \ell(w)]$ should be replaced by $C_\lambda[\dim \operatorname{Gr} - \ell(w)]$.¹³

With these changes, the conclusion of [ABG, Cor. 8.3.2(ii)] shows \overline{W}_λ and $IC_\lambda[-\ell(w)]$ have the same restriction to $\operatorname{Gr}_\lambda$ (from $\overline{\operatorname{Gr}}_\lambda$). This shows, together with the fact that Υ induces an equivalence $D_{< \lambda}^b \mathbf{block}(\mathbf{U}) \rightarrow D_{< \lambda}^b \mathcal{Perv}$, which we obtained above by induction, that the strict image under Υ of $D_{\leq \lambda}^b \mathbf{block}(\mathbf{U})$ is $D_{\leq \lambda}^b \mathcal{Perv}$. Since we already know Υ provides an equivalence $D^b \mathbf{block}(\mathbf{U}) \rightarrow D^b \mathcal{Perv}$, it follows now that it induces one between $D_{\leq \lambda}^b \mathbf{block}(\mathbf{U})$ and $D_{\leq \lambda}^b \mathcal{Perv}$. This proves (i).

Moreover, we also get (ii), since we have shown that $\Upsilon(R \operatorname{Ind}_{\mathbb{B}}^{\mathbf{U}}(\ell\lambda)) = \overline{W}_\lambda$ is $IC_\lambda[-\ell(w)]$ in the quotient category $D_{\leq \lambda}^b \mathcal{Perv} / D_{< \lambda}^b \mathcal{Perv}$. Our remarks on the composition factors of the cohomology groups of the preimage $R \operatorname{Ind}_{\mathbb{B}}^{\mathbf{U}}(\ell\lambda)$ of \overline{W}_λ show that the image in the quotient category $D_{\leq \lambda}^b \mathbf{block}(\mathbf{U}) / D_{< \lambda}^b \mathbf{block}(\mathbf{U})$ is $L_\lambda[-\ell(w)]$. This completes the proof of (ii) and the lemma. Q.E.D.

7. ACKNOWLEDGEMENTS

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¹³This latter change needs to be made both in the statement and proof of [ABG, Cor. 8.3.2]. The error is in ignoring the shift in degree that can accompany direct images with proper maps.

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