A Mathematical Introduction to the Rubik's Cube

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Sept 29th, 2015

Outline

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- Some Notation
- Socks and Shoes

The Beginners Method
- The Cross
- First 2 Layer - F2L
- Last Layer

Group Theory
- Commutators
- Conjugation

Final Notes

- 6 faces, 12 edges, 8 corners
- 54 stickers
- Scrambling faces is equivalent to rotating the whole cube.
- We can consider the faces fixed.
- It's a good idea to fix your orientation (at least to start).
- Think of operations on blocks rather than stickers.
  - There are 8 **corner blocks** with 3 orientations each.
  - There are 12 **edge blocks** with 2 orientations each.
- Number of positions by assembling and dissembling cube

$$8!(3^8)12!(2^{12}) = 519,024,039,293,878,272,000$$

- The last corner and edge orientations are determined by the previous blocks.

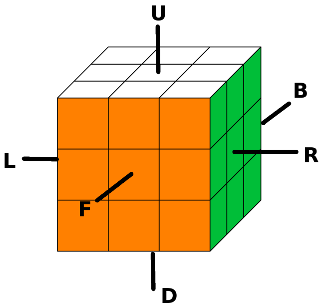
- You can't change the orientation of one corner or edge. The corners and edges are linked in this parity.
- So the number of configurations is

$$8!(3^7)12!(2^{11})/2 = 43,252,003,274,489,856,000$$

- Dividing by the 24 rotations gives

$$1,802,166,803,103,744,000$$

Some Notation



- **U**p,**D**own,**R**ight,**L**eft,**F**ront,**B**ack
- Rotation about face center keeping cube orientation fixed.
- The Forward Direction is clockwise from outside cube.
- Half-turns $\mathbf{U}2$ (or $\mathbf{U}^2$)
- Inverse $\mathbf{R}'$ (or $\mathbf{R}^{-1}$)

Socks and Shoes

- $(fg)' = g'f'$
- Taking the inverse of an composite operation where each part is invertible is the the same thing as reversing the order and inverting each part
- Note $(f')' = f$
- $(\mathbf{RFBU}2\mathbf{R}')' = \mathbf{RU}2\mathbf{B}'\mathbf{F}'\mathbf{R}'$
- So if you know exactly what operations have been done to a cube. You can solve it by **Socks and Shoes**.

A Choice of Optimization

- **Minimize Time:** Easily defined. Can learn from youtube. Dominant algorithm CFOP requires 17-78 algorithms.
- **Minimize Operations:** How do you count? This is actually quite hard.
- **Minimize Memory Footprint:** Less algorithms and structure of each algorithm to make easier to remember. A viable long term solution strategy. Can also be applied to similar puzzles. Quite slow in time and operations.

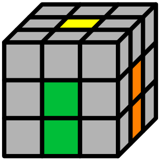
The Beginners Method

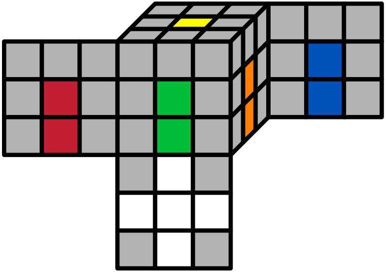
Basically build the the solved cube a layer at a time.

1. Solve the **Cross**. *Intuitive*
2. Solve the **First 2 Layers - F2L** *Intuitive*
3. Solve the **Last Layer - LL** There's some work here.

The Cross

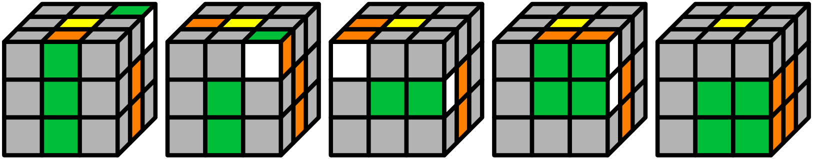
- Pick a side and ignore everything except the faces and the edges from the chosen side in solved position.
- Put these edges in the correct position and orientation.
- There are only 7920 possible positions.
- This strategy is useful: break, do something, fix-it
- Solved state is a cross (ignore gray):





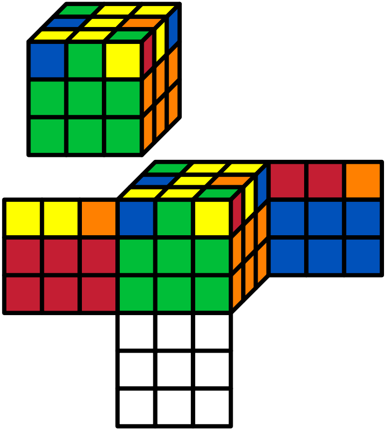
First 2 Layer - F2L

- Identify the corners that would go into the cross. Ignore all others
- **Goal:** Pair these corners with their 2nd layer edges. These are called **F2L pairs**. Then put into their correct slot.
- **Note:** You want this method to preserve the cross.
- This strategy is useful: break, do something, fix-it
- There are only 4 F2L pairs.
- These connect the F2L pair below $\mathbf{U}, \mathbf{F}'$. Then $\mathbf{U}', \mathbf{F}$ slot the pair.



Last Layer

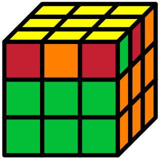
So you've solved the cube to something that looks like this:



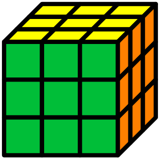
- Want algorithms to preserve first 2 layers.
- There are 2 types of algorithms people use: **Permutations** and **Orientations**.
- **Permutations** change the position.
- **Orientations** rotate corners and flip edges.
- There are also orientation preserving permutations.

Last Layer - Speed Strategy

1. Orient Last Layer **OLL**



2. Permute Last Layer **PLL**

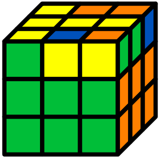


Use only orientation preserving permutations

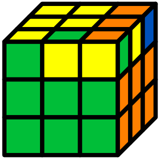
- The dominant speed solving method is call **CFOP**: **C**ross, **F**2L, **O**LL, **P**LL
- The minimum number of algorithms for OLL + PLL is 17.

Last Layer - Alternative Strategy

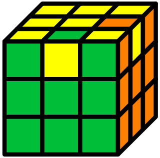
1. Permute Corners



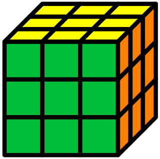
2. Permute Edges



3. Orient Corners



4. Orient Edges



- This can be accomplished with 4 algorithms that all have the same structure.
- Note here the permutations are allowed to change the

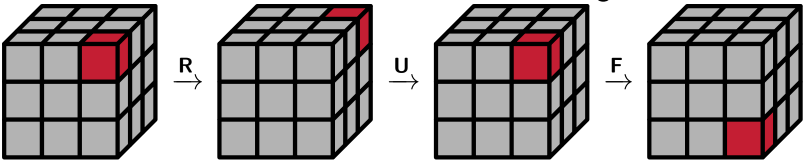
A Little more Abstraction

Think of the algorithms on the cube as functions on the cube. We call them **permutations**.

The following properties are met:

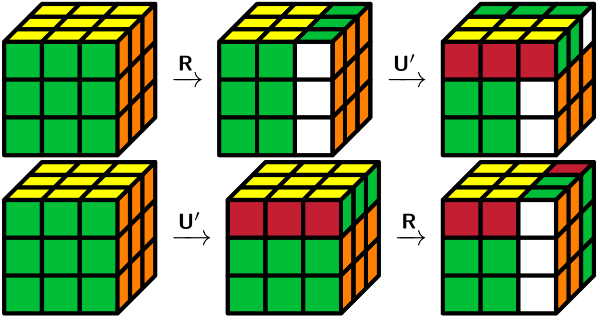
- **Associative** $f(gh) = (fg)h$
- **Identity** There is an e with $ef = fe = f$ for all f
- **Inverse** Every f has an inverse f' with $ff' = f'f = e$

We adopt left to right function notation. Let x be the red corner on the left then x**RUF** is the red corner on the right



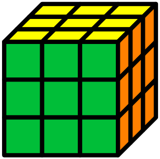
Not Commutative

The commutative property says $fg = gf$ for all f and g

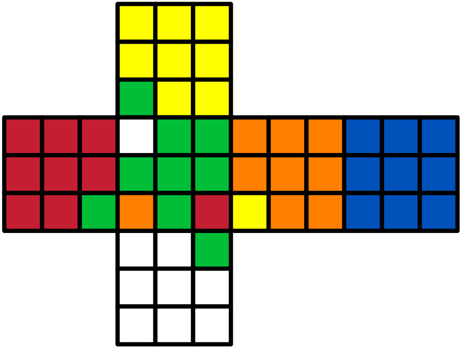


Fixing the First 2 Layers

The key is to find elements that move as few pieces as possible. On a Rubik's cube that happens to be a 3-cycle:



$\xrightarrow{\mathbf{F'R'FLF'RFL'}}$



So $\mathbf{F'R'FLF'RFL'}$ is a 3-cycle on corner blocks.

Commutators

A **Commutator** for 2 permutations f, g is

$$[f, g] = f'g'fg$$

- $[f, g]' = [g, f]$ (by socks and shoes)
- $[f, g]$ is the identity if f and g commute that is $fg = gf$
- Call the support of a permutation f all elements moved by f.

$$\text{supp}\{f\} = \{x \in \text{cube} \mid xf \neq x\}$$

- Theorem (The Commutator Theorem)

If f, g are permutations with $supp\{f\} \cap supp\{g\} = \{x\}$ then $[f, g]$ is a 3-cycle.

Sketch of Proof of the Commutator Theorem

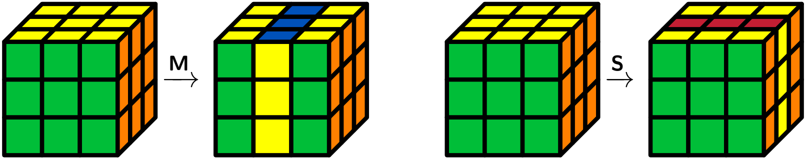
Let $\mathrm{supp}\{f\} \cap \mathrm{supp}\{g\} = \{x\}$.

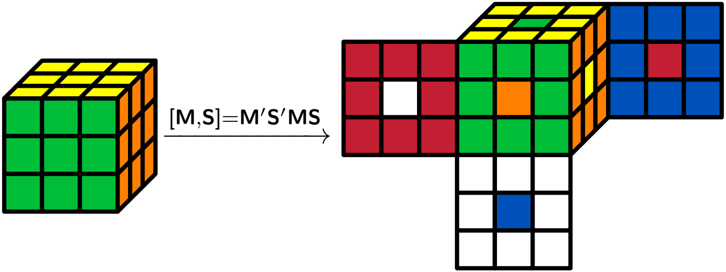
- Note since $xf \neq x$ and $xg \neq x$ that g and f have to move distinct elements to and from x.

$$a \xrightarrow{f} x \xrightarrow{f} b \qquad \alpha \xrightarrow{g} x \xrightarrow{g} \beta$$

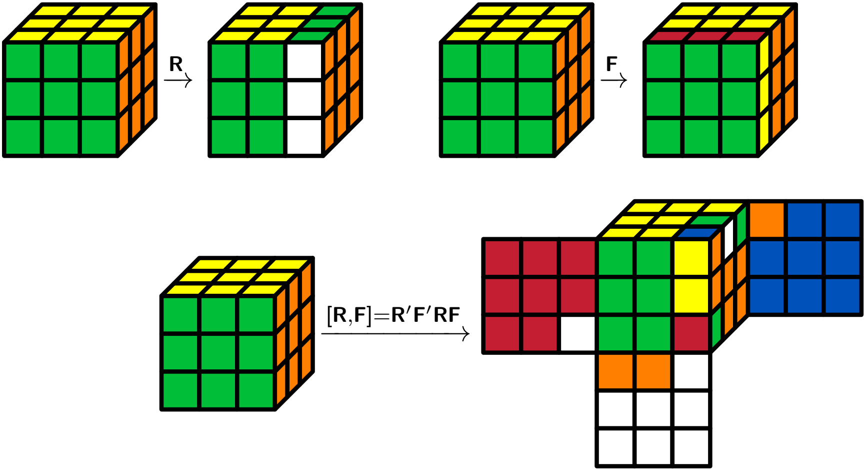
- f' and g' are just the diagrams in reverse.
- Note $a, b \notin \mathrm{supp}\{g\}$ and $\alpha, \beta \notin \mathrm{supp}\{f\}$
- It is straightforward to compute
  - $b[f, g] = x$,
  - $x[f, g] = \beta$,
  - $\beta[f, g] = x$ and
  - $y[f, g] = y$ for any other y.
- So $[f, g]$ is the 3-cycle (b, x, β)

Commutator Example

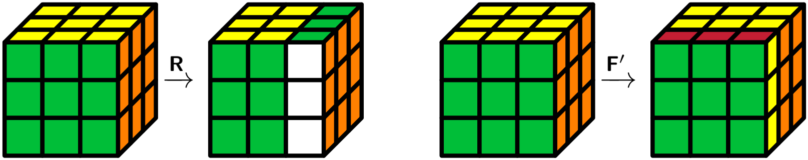


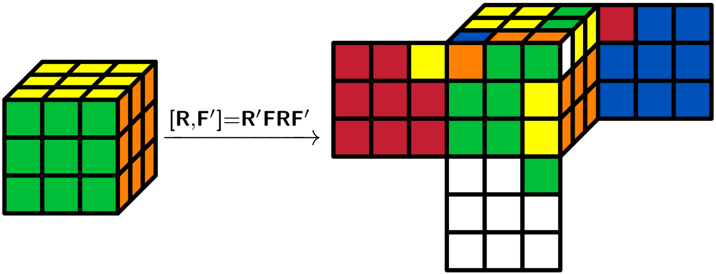


Commutator Example II - "The Sexy Move"



Commutator Example III - "The Sledgehammer"





Conjugation to the Rescue

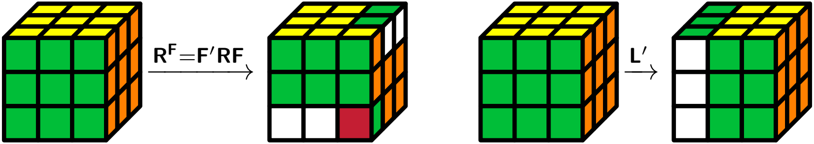
We define the conjugation of f by g to be

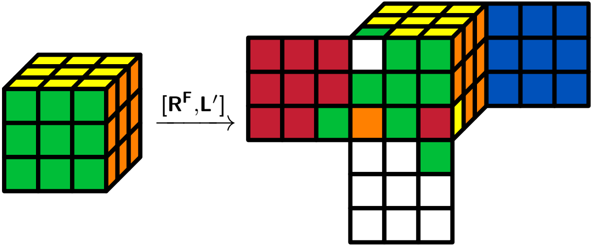
$$f^g = g' f g$$

- **Note:** Conjugation does the break-it, do something, fix-it strategy.
- You can think f^g has the same structure as f just moved to different places by g
- $(f^g)' = (f')^g = g' f' g$ by Socks-and-Shoes
- You can use conjugation to find a move f^g that who only moves one element in common with a simple rotation such as **R**.

An Example 3-cylce

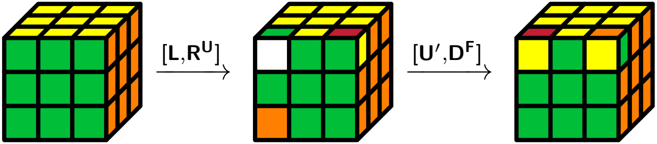
$$[\mathbf{R}^{\mathbf{F}}, \mathbf{L}'] = (\mathbf{F}'\mathbf{R}'\mathbf{F})\mathbf{L}(\mathbf{F}'\mathbf{R}\mathbf{F})\mathbf{L}'$$





Orientation changing moves

Strategy: Come up with two 3-cycle commutators that move the same 3 things but change the orientation differently. For example:



Final Notes:

- Commutators are **almost** enough to solve the entire cube. You may need a quarter turn on the last layer.
- The commutator theorem and conjugation applies to all sorts of twisty puzzles.
- Adding orientation of faces makes a cube slightly more challenging.
- Shape modifications of a cube can be solved with the same algorithms but may need face orientation moves also.
- Try conjugation with slicing to get edge 3-cycles. e.g. $[\mathbf{M}, \mathbf{R}^{\mathbf{U}}]$
- Conjugation with cube rotation can move your 3-cycle shapes all over the place.