A Mathematical Introduction to the Rubik’s Cube

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Outline

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Final Notes
6 faces, 12 edges, 8 corners
54 stickers
Scrambling faces is equivalent to rotating the whole cube.
We can consider the faces fixed.
It’s a good idea to fix your orientation (at least to start).
Think of operations on blocks rather than stickers.
There are 8 corner blocks with 3 orientations each.
There are 12 edge blocks with 2 orientations each.
Number of positions by assembling and disassembling cube

\[8!(3^8)12!(2^{12}) = 519,024,039,293,878,272,000\]
The last corner and edge orientations are determined by the previous blocks.
You can’t change the orientation of one corner or edge. The corners and edges are linked in this parity.

So the number of configurations is

$$8!(3^7)12!(2^{11})/2 = 43,252,003,274,489,856,000$$

Dividing by the 24 rotations gives

$$1,802,166,803,103,744,000$$
Some Notation

- **Up, Down, Right, Left, Front, Back**
- Rotation about face center keeping cube orientation fixed.
- The Forward Direction is clockwise from outside cube.
- Half-turns $U^2$ (or $U^2$)
- Inverse $R'$ (or $R^{-1}$)
Socks and Shoes

- \((fg)′ = g′f′\)
- Taking the inverse of an composite operation where each part is invertible is the same thing as reversing the order and inverting each part
- Note \((f′)′ = f\)
- \((RFBU2R′)′ = RU2B′F′R′\)
- So if you know exactly what operations have been done to a cube. You can solve it by Socks and Shoes.
A Choice of Optimization

- **Minimize Time**: Easily defined. Can learn from youtube. Dominant algorithm CFOP requires 17-78 algorithms.

- **Minimize Operations**: How do you count? This is actually quite hard.

- **Minimize Memory Footprint**: Less algorithms and structure of each algorithm to make easier to remember. A viable long term solution strategy. Can also be applied to similar puzzles. Quite slow in time and operations.
The Beginners Method

Basically build the solved cube a layer at a time.

1. Solve the **Cross. Intuitive**
2. Solve the **First 2 Layers - F2L Intuitive**
3. Solve the **Last Layer - LL** There’s some work here.
The Cross

- Pick a side and ignore everything except the faces and the edges from the chosen side in solved position.
- Put these edges in the correct position and orientation.
- There are only 7920 possible positions.
- This strategy is useful: break, do something, fix-it
- Solved state is a cross (ignore gray):
First 2 Layer - F2L

- Identify the corners that would go into the cross. Ignore all others
- **Goal:** Pair these corners with their 2nd layer edges. These are called **F2L pairs**. Then put into their correct slot.
- **Note:** You want this method to preserve the cross.
- This strategy is useful: break, do something, fix-it
- There are only 4 F2L pairs.
- These connect the F2L pair below U, F'. Then U', F slot the pair.
Last Layer

So you’ve solved the cube to something that looks like this:

- Want algorithms to preserve first 2 layers.
- There are 2 types of algorithms people use: **Permutations** and **Orientations**.
- **Permutations** change the position.
- **Orientations** rotate corners and flip edges.
- There are also orientation preserving permutations.
Last Layer - Speed Strategy

1. Orient Last Layer \textbf{OLL}
2. Permute Last Layer \textbf{PLL}

- Use only orientation preserving permutations

- The dominant speed solving method is call \textbf{CFOP}:
  - \textbf{C}ross, \textbf{F}2\textbf{L}, \textbf{O}LL, \textbf{P}LL
- The minimum number of algorithms for OLL + PLL is 17.
Last Layer - Alternative Strategy

1. Permute Corners
2. Permute Edges
3. Orient Corners
4. Orient Edges

- This can be accomplished with 4 algorithms that all have the same structure.
- Note here the permutations are allowed to change the orientation (e.g., in CFOP).
A Little more Abstraction

Think of the algorithms on the cube as functions on the cube. We call them permutations. The following properties are met:

- **Associative** $f(gh) = (fg)h$
- **Identity** There is an $e$ with $ef = fe = f$ for all $f$
- **Inverse** Every $f$ has an inverse $f'$ with $ff' = f'f = e$

We adopt left to right function notation. Let $x$ be the red corner on the left then $x\mathbf{RUF}$ is the red corner on the right.
Not Commutative

The commutative property says $fg = gf$ for all $f$ and $g$
Fixing the First 2 Layers

The key is to find elements that move as few pieces as possible. On a Rubik’s cube that happens to be a 3-cycle:

So \( F' R' F L F' R F L' \) is a 3-cycle on corner blocks.
Commutators

A **Commutator** for 2 permutations $f, g$ is

$$[f, g] = f'g'fg$$

- $[f, g]' = [g, f]$ (by socks and shoes)
- $[f, g]$ is the identity if $f$ and $g$ commute that is $fg = gf$
- Call the support of a permutation $f$ all elements moved by $f$.

$$\text{supp}\{f\} = \{x \in \text{cube} \mid xf \neq x\}$$

- **Theorem (The Commutator Theorem)**

*If $f, g$ are permutations with $\text{supp}\{f\} \cap \text{supp}\{g\} = \{x\}$ then $[f, g]$ is a 3-cycle.*
Sketch of Proof of the Commutator Theorem

Let $\text{supp}\{f\} \cap \text{supp}\{g\} = \{x\}$.

- Note since $xf \neq x$ and $xg \neq x$ that $g$ and $f$ have to move distinct elements to and from $x$.

$$a \xrightarrow{f} x \xrightarrow{f} b \quad \alpha \xrightarrow{g} x \xrightarrow{g} \beta$$

- $f'$ and $g'$ are just the diagrams in reverse.
- Note $a, b \notin \text{supp}\{g\}$ and $\alpha, \beta \notin \text{supp}\{f\}$
- It is straightforward to compute
  - $b[f, g] = x$,
  - $x[f, g] = \beta$,
  - $\beta[f, g] = x$ and
  - $y[f, g] = y$ for any other $y$.
- So $[f, g]$ is the 3-cycle $(b, x, \beta)$
Commutator Example

\[ [M, S] = M' S' M S \]
Commutator Example II - “The Sexy Move”

$$[R,F] = R'F'R$$
Commutator Example III - “The Sledgehammer”

\[
[R, F'] = R'FRF'
\]
Conjugation to the Rescue

We define the conjugation of $f$ by $g$ to be

$$f^g = g'fg$$

- **Note:** Conjugation does the break-it, do something, fix-it strategy.
- You can think $f^g$ has the same structure as $f$ just moved to different places by $g$
- $(f^g)' = (f')^g = g'f'g$ by Socks-and-Shoes
- You can use conjugation to find a move $f^g$ that who only moves one element in common with a simple rotation such as $R$. 
An Example 3-cylce

\[ [R^F, L'] = (F'R'F)L(F'RFL) \]

\[ R^F = F'R'F \]

\[ [R^F, L'] \]
Orientation changing moves

**Strategy:** Come up with two 3-cycle commutators that move the same 3 things but change the orientation differently. For example:

\[[L,R^U] \rightarrow [U',D^F]\]
Final Notes:

- Commutators are almost enough to solve the entire cube. You may need a quarter turn on the last layer.
- The commutator theorem and conjugation applies to all sorts of twisty puzzles.
- Adding orientation of faces makes a cube slightly more challenging.
- Shape modifications of a cube can be solved with the same algorithms but may need face orientation moves also.
- Try conjugation with slicing to get edge 3-cycles. e.g. $[M, R^U]$
- Conjugation with cube rotation can move your 3-cycle shapes all over the place.