Applications of The Perron-Frobenius Theorem

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Motivation

In a finite discrete linear dynamical system

\[ x_{n+1} = Ax_n \]

What are sufficient conditions for \( x_{n+1} \) to converge?
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In a finite discrete linear dynamical system

\[ x_{n+1} = A x_n \]

What are sufficient conditions for \( x_{n+1} \) to converge?

More generally: When does the power method converge?

\[ x_{n+1} = \frac{A x_n}{\| A x_n \|} \]
Oskar Perron in 1907 proved the following theorem [Per07]:

**Theorem (Perron’s Theorem)**

Let $A$ be a strictly positive valued $n \times n$ matrix. Then $A$ has a positive eigenvalue $\lambda$ with $\lambda > |\mu|$ for all other eigenvectors $\mu$ and corresponding right eigenvector $v$ with all positive entries.
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- In 1948 Kreĭn and Rutman proved a Banach Space version of the theorem [KR48, KR50].
Conditions for $A^k > 0$

- If $A \geq 0$ has $A^k > 0$ (coordinatewise) for some $k$ then $A$ is primitive.
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- $A \geq 0$ is \textit{primitive} then it is both \textit{irreducible} and \textit{aperiodic}.

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A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad
A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad
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- If $A \geq 0$ has $A^k > 0$ (coordinatewise) for some $k$ then $A$ is primitive.
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- $A$ is irreducible if for each $i, j$ indices there is a $k$ such that $(A^k)_{ij} > 0$.

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- this quantity $p(i) = \gcd\{k \mid (A^k)_{ii} > 0\}$ is called the **period of index** $i$. If $A$ is irreducible each index has the same period $p$ so we would call $p$ the **period of $A$**.

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Caution: The use of the words irreducible and period are overloaded and likely don’t mean the same thing as other fields.

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A finite **Weighted Directed Graph** (or digraph) $\Gamma = (V, E, w)$ is a set of vertices $V = \{1, 2, \ldots, n\}$ with edges $E \subset V^2$ and weight function $w : E \rightarrow (0, \infty)$ defined by $w(i, j) = w_{ij}$ for all $(i, j) \in E$. 

We can define adjacency matrix $A$ for digraph $\Gamma$ in the following way: $A_{ij} = \begin{cases} w_{ij} & \text{if } (i, j) \in E \\ 0 & \text{if } (i, j) \notin E \end{cases}$
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Digraphs and Primitivity

- Γ is **strongly connected** if for any two $i, j \in V$ there is a path $i \rightarrow j$ and another path $j \rightarrow i$.
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- $\Gamma$ is **aperiodic** if there is no integer $k > 1$ that divides the length of every closed path $i \rightarrow i$.

- The adjacency matrix $A$ for a weighted digraph $\Gamma$ is primitive iff $\Gamma$ is strongly connected and aperiodic.
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- $(A)_{ij} > 0$ if $(i, j) \in E$
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- \( (A)_{ij} > 0 \) if \( (i, j) \in E \)
- \( (A^k)_{ij} > 0 \) if there is a path of length \( k \) from \( i \rightarrow j \)
Digraphs and Primitivity

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For Nonnegative Irreducible Matrices

Theorem (The Perron-Frobenius theorem)

[Kit98] Let $A$ be an $n \times n$ real valued nonnegative matrix. If $A$ is irreducible then it has an eigenvalue $\lambda > 0$ such that:

1. $\lambda$ is a simple root of the characteristic polynomial.
2. $\lambda$ has strictly positive left eigenvector $v^T$ and right eigenvector $w$.
3. The eigenvectors of $\lambda$ are unique up to a scalar.
4. Any eigenvalue $\mu$ of $A$ has $|\mu| \leq \lambda$.
5. If $0 \leq B \leq A$ and $\beta$ is an eigenvalue of $B$ then $|\beta| \leq \lambda$ and equality occurs iff $A = B$.
6. If $A$ has period $p$ then $\mu = \lambda e^{2\pi ki/p}$ are the $p$ eigenvalues with $|\mu| = \lambda$.
7. If $A$ is aperiodic and $v^T w = 1$ then $\lim_{k \to \infty} (A/\lambda)^k = wv^T$. 
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7. If $A$ is aperiodic and $v^T w = 1$ then

$$\lim_{k \to \infty} (A/\lambda)^k = wv^T$$
The Power Method

Let $A$ be the adjacency matrix of a strongly connected aperiodic digraph $\Gamma$. Then $A$ is primitive and:

$\lambda > 0$ is the Perron-Frobenius eigenvalue with left eigenvector $v^T > 0$ and right eigenvector $w > 0$.

Let $x_0 \in \mathbb{R}^n_{\geq 0}$ nonzero then

$$\lim_{k \to \infty} (A/\lambda)^k x_0 = aw$$

$$\lim_{k \to \infty} x_0^T (A/\lambda)^k = bv^T$$

and $a, b \neq 0$

Thus both the left and right power methods converge to $v^T$ and $w$ respectively.
Eigenvector centrality

Lemma

A connected simple undirected graph \( \Gamma \) with at least one odd length closed path has an aperiodic irreducible nonnegative valued adjacency matrix \( A \).
**Eigenvector centrality**

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**Theorem (Eigencentrality)**

A simple undirected connected graph with at least one odd length closed path has a dominant eigenvector with a corresponding positive valued eigenvector. Once scaled this vector is the **eigencentrality** (or **eigenvector centrality**) ranking for the graph.

Note: Larger coordinate entries represent the corresponding vertex in the graph being more central.
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PageRank [PBMW99]

Let vertices be webpages.
PageRank\cite{PBMW99}

Let vertices be webpages. PageRank graph has two types of directed edges:

1. Hyperlinks weighted (with probability $\alpha$)
2. Random jump to any other vertex (with probability $1-\alpha$)

Where $0<\alpha<1$ is called the damping

The adjacency matrix $A$, for PageRank, is related to the weighted adjacency matrix of hyperlinks $H$ and the matrix $R$ with all ones.

$A = \alpha H + (1-\alpha)R$

Since every vertex is connected to every other, the graph is strongly connected and if there are 3 webpages aperiodic.

The PageRank is the resulting Perron-Frobenius right eigenvector ( Appropriately scaled).
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- The PageRank is the resulting Perron-Frobenius right eigenvector ( Appropriately scaled).
Leslie Matrices

A transitional matrix as follows is called a **Leslie Matrix** [Les45]:

\[
\begin{bmatrix}
  f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\
  s_1 & 0 & 0 & \cdots & 0 & 0 \\
  0 & s_2 & 0 & \cdots & 0 & 0 \\
  0 & 0 & s_3 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & s_{n-2} & s_n \\
\end{bmatrix}
\]

- The $s_i < 1$ are survival rates and $f_i \geq 0$ are fecundity rates.
- $s_i > 0$ for all $i$ and $f_n > 0$ guarantee the underlying state graph is strongly connected and aperiodic.
- The Perron-Frobenius eigenvalue $\lambda$ tells you if the population is stable, growing or going extinct.
- The appropriate eigenvector gives the asymptotic distribution of the cohorts.
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Markov Chains Example

$0 < w_{ij} < 1$ and each row (column) adds to 1.

Bull market

Stagnant market

Bear market

$0.25$

$0.075$

$0.15$

$0.025$

$0.25$

$0.05$

$0.9$

$0.8$

$0.8$

$0.5$

Jonathan [Jon13]
The Kreĭn-Rutman Theorem

The following is copied from [PD94]:

**Theorem (The Kreĭn-Rutman Theorem)**

*Let* $X$ *be a Banach Space and* $K \subset X$ *a convex cone* $\text{int}K \neq \emptyset$ *and* $A \in \mathcal{L}(X, X)$ *a linear bounded operator. If* $A(\text{int}K) \subset \text{int}K$ *then there is a nonzero functional* $f^* \in K^*$ *such that* $A^*f^* = \lambda f^*$, *where* $\lambda > 0$. 

Alberto Borobia and Ujué R. Trías.
A geometric proof of the Perron-Frobenius theorem.

G. Frobenius.
Uber Matrizen aus positiven Elementen, I and II.

G. Frobenius.
Uber Matrizen aus nicht negativen Elementen.

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*Uspehi Matem. Nauk (N. S.)*, 3(1(23)):3–95, 1948.

M. G. Kreĭn and M. A. Rutman.  
Linear operators leaving invariant a cone in a Banach space.  

P. H. Leslie.  
On the use of matrices in certain population mathematics.  
Lawrence Page, Sergey Brin, Rajeev Motwani, and Terry Winograd.

Vũ Ngọc Phát and Trinh Cong Dieu.

Oskar Perron.