Applications of The Perron-Frobenius Theorem

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Motivation

In a finite discrete linear dynamical system

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More generally: When does the power method converge?

$$\mathbf{x}_{n+1} = \frac{A\mathbf{x}_n}{\|A\mathbf{x}_n\|}$$

History

Oskar Perron in 1907 proved the following theorem [Per07] :

Theorem (Perron's Theorem)

Let A be a strictly positive valued $n \times n$ *matrix. Then A has a positive eigenvalue* λ *with* $\lambda > |\mu|$ *for all other eigenvectors* μ *and corresponding right eigenvector* $\mathbf{v}$ *with all positive entries.*

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- His general idea was that if you have a matrix A with A^k strictly positive for some k then the theorem holds.
- In 1948 Kreĭn and Rutman proved a Bannach Space version of the theorem [KR48, KR50].

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- **Caution:** The use of the words **irreducible** and **period** are overloaded and likely don't mean the same thing as other fields.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} A^3 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} A^4 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Weighted Digraphs

A finite **Weighted Directed Graph** (or **digraph**) $\Gamma = (V, E, w)$ is a set of vertices $V = \{1, 2, \ldots, n\}$ with edges $E \subset V^2$ and weight function $w : E \to (0, \infty)$ defined by $w(i, j) = w_{ij}$ for all $(i, j) \in E$.

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We can define adjacency matrix A for digraph Γ in the following way

$$(A)_{ij} = \begin{cases} w_{ij} & \text{if } (i, j) \in E \\ 0 & \text{if } (i, j) \notin E \end{cases}$$

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- $(A)_{ij} > 0$ if $(i, j) \in E$
- $(A^k)_{ij} > 0$ if there is a path of length k from $i \to j$
- The adjacency matrix A for a weighted digraph Γ is primitive iff Γ is strongly connected and aperiodic.

For Nonnegative Irreducible Matrices

Theorem (The Perron-Frobenius theorem)

[Kit98] Let A be an $n \times n$ *real valued nonnegative matrix. If A is irreducible then it has an eigenvalue* $\lambda > 0$ *such that:*

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5. *If $0 \leq B \leq A$ and β is an eigenvalue of B then $|\beta| \leq \lambda$ and equality occurs iff A=B.*

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6. *If A has period p then $\mu = \lambda e^{2\pi ki/p}$ are the p eigenvalues with $|\mu| = \lambda$*

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7. *If A is aperiodic and $\mathbf{v}^T \mathbf{w} = 1$ then*

$$\lim_{k \to \infty} (A/\lambda)^k = \mathbf{w}\mathbf{v}^T$$

The Power Method

Let A be the adjacency matrix of a strongly connected aperiodic digraph Γ. Then A is primitive and:

- $\lambda > 0$ is the Perron-Frobenius eigenvalue with left eigenvector $\mathbf{v}^T > 0$ and right eigenvector $\mathbf{w} > 0$.
- Let $\mathbf{x}_0 \in \mathbb{R}^n_{\geq 0}$ nonzero then

$$\lim_{k\to\infty} (A/\lambda)^k \mathbf{x}_0 = a\mathbf{w}$$

$$\lim_{k\to\infty} \mathbf{x}_0^T (A/\lambda)^k = b\mathbf{v}^T$$

  and $a, b \neq 0$
- Thus both the left and right power methods converge to $\mathbf{v}^T$ and $\mathbf{w}$ respectively.

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*A simple undirected connected graph with at least one odd length closed path has a dominant eigenvector with a corresponding positive valued eigenvector. Once scaled this vector is the **eigencentrality** (or **eigenvector centrality**) ranking for the graph.*

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Note: Larger coordinate entries represent the corresponding vertex in the graph being more central.

PageRank[PBMW99]

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- The adjacency matrix *A*, for PageRank, is related to the weighted adjacency matrix of hyperlinks *H* and the matrix *R* with all ones.

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- The PageRank is the resulting Perron-Frobenius right eigenvector (Appropriately scaled).

Leslie Matrices

A transitional matrix as follows is called a **Leslie Matrix** [Les45]:

$$\begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-2} & s_n \end{bmatrix}$$

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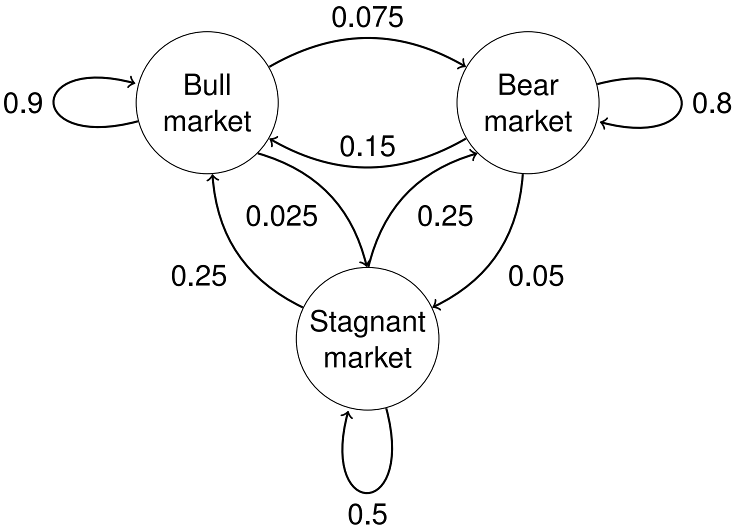
- The Perron-Frobenius eigenvalue λ tells you if the population is stable, growing or going extinct.
- The appropriate eigenvector gives the asymptotic distribution of the cohorts.

Markov Chains Example

$0 < w_{ij} < 1$ and each row (column) adds to 1.



[Jon13]

The Kreĭn-Rutman Theorem

The following is copied from [PD94]:

Theorem (The Kreĭn-Rutman Theorem)

Let X be a Banach Space and $K \subset X$ a convex cone $intK \neq \emptyset$ and $A \in L(X, X)$ a linear bounded operator. If $A(intK) \subset intK$ then there is a nonzero functional $f^* \in K^*$ such that $A^* f^* = \lambda f^*$, $\lambda > 0$.

References: I

Alberto Borobia and Ujué R. Trías.
A geometric proof of the Perron-Frobenius theorem.
Rev. Mat. Univ. Complut. Madrid, 5(1):57–63, 1992.

G. Frobenius.
Uber Matrizen aus positiven Elementen, I and II.
Sitzungberischte der koniglich perussischen Akademie der Wissenchaften zu Berlin, pages 471–479,514–518, 1908,1909.

G. Frobenius.
Uber Matrizen aus nicht negativen Elementen.
Sitzungberischte der koniglich perussischen Akademie der Wissenchaften zu Berlin, pages 456–477, 1912.

Gareth Jones.
File:finance markov chain example state space.svg, 2013.
[Online; accessed 31-Oct-2014].

References: II

Bruce P. Kitchens.
Symbolic dynamics.
Universitext. Springer-Verlag, Berlin, 1998.
One-sided, two-sided and countable state Markov shifts.

M. G. Kreĭn and M. A. Rutman.
Linear operators leaving invariant a cone in a Banach space.
Uspehi Matem. Nauk (N. S.), 3(1(23)):3–95, 1948.

M. G. Kreĭn and M. A. Rutman.
Linear operators leaving invariant a cone in a Banach space.
Amer. Math. Soc. Translation, 1950(26):128, 1950.

P. H. Leslie.
On the use of matrices in certain population mathematics.
Biometrika, 33:183–212, 1945.

References: III

- Lawrence Page, Sergey Brin, Rajeev Motwani, and Terry Winograd.
  The pagerank citation ranking: Bringing order to the web.
  Technical Report 1999-66, Stanford InfoLab, November 1999.
  Previous number = SIDL-WP-1999-0120.
- Vũ Ngọc Phát and Trinh Cong Dieu.
  On the Kreĭn-Rutman theorem and its applications to controllability.
  Proc. Amer. Math. Soc., 120(2):495–500, 1994.
- Oskar Perron.
  Zur Theorie der Matrices.
  Math. Ann., 64(2):248–263, 1907.