Solution to Review Problems for Midterm III

Midterm III: Friday, November 19 in class
Topics: 3.8-3.11, 4.1,4.3

1. Find the derivative of the following functions and simplify your answers.

(a) $x(\ln(4x))^3+\ln(5\cos^3(x))$ (b) $\ln\frac{e^{3x}}{(1+e^{3x})^5}$ (c) $\log_4((\frac{x+3}{x-3})^{\ln 4})$ (d) $(x^2+1)^{2x}$

(e) $(\sin(x))^{\ln(x)}$ (f) $\frac{\cos(x)^3(2x+1)^3\sin^{-1}(x)}{\sqrt{2x+1}e^{3x}}$ (g) $\tan^{-1}(e^{3x})$ (i) $\sec^{-1}(x^2)$

(j) $\csc^{-1}(x^2)\cot^{-1}(2x)+x\cos^{-1}(2x)$ (k) $\frac{(x^2-1)^2(2x+1)^3x^5}{(x^2+1)^3(x+1)^4\sin^5(2x)}$

Solution: (a) $f(x)=x(\ln(4x))^3+\ln(5\cos^3(x))=x(\ln(4x))^3+\ln(5)+\ln(\cos^3(x))=x(\ln(4x))^3+\ln(5)+3\ln(\cos(x))$. $f'(x)=(\ln(4x))^3+3\,(\ln(4x))^2-3\,\frac{\sin(x)}{\cos(x)}$.

(b) $f(x)=\ln\frac{e^{3x}}{(1+e^{3x})^5}=\ln(e^{3x})-5\ln(1+e^{3x})=3x-5\ln(1+e^{3x})$. $f'(x)=3-\frac{15e^{3x}}{1+e^{3x}}$.

(c) $f(x)=\log_4((\frac{x+3}{x-3})^{\ln 4})=\ln 4\log_4((\frac{x+3}{x-3}))=\ln 4\frac{\ln((\frac{x+3}{x-3}))}{\ln 4}=\ln((\frac{x+3}{x-3}))=\ln(x+3)-\ln(x-3)$. Then $f'(x)=\frac{1}{x+3}-\frac{1}{x-3}$.

(d) Let $y=(x^2+1)^{2x}$. Then $\ln(y)=\ln(x^2+1)^{2x}=2x\ln(x^2+1)$ and $\frac{y'}{y}=(2x)'\ln(x^2+1)+2x(\ln(x^2+1))'=2\ln(x^2+1)+2x\cdot\frac{2x}{x^2+1}==2\ln(x^2+1)+\frac{4x^2}{x^2+1}$. This implies that $y'(x)=y\cdot(2\ln(x^2+1)+\frac{4x^2}{x^2+1})=(x^2+1)^{2x}(2\ln(x^2+1)+\frac{4x^2}{x^2+1})$.

(e) Let $y=(\sin(x))^{\ln(x)}$, Then $\ln(y)=\ln((\sin(x))^{\ln(x)})=\ln(x)\ln((\sin(x))$ and $\frac{y'}{y}=(\ln(x))'\ln((\sin(x))+\ln(x)(\ln((\sin(x)))'=\frac{1}{x}\ln((\sin(x))+\ln(x)\frac{\cos(x)}{\sin(x)}$. This implies that

$y'=y(\frac{1}{x}\ln((\sin(x))+\ln(x)\frac{\cos(x)}{\sin(x)})=(\sin(x))^{\ln(x)}(\frac{1}{x}\ln((\sin(x))+\ln(x)\cot(x))$.

(f) Let $y=\frac{\cos(x)^3(2x+1)^3\sin^{-1}(x)}{\sqrt{2x+1}e^{3x}}$. Then

$\ln(y)=\ln(\frac{\cos(x)^3(2x+1)^3\sin^{-1}(x)}{\sqrt{2x+1}e^{3x}})$

$=3\ln(\cos(x))+3\ln(2x+1)+\ln(\sin^{-1}(x))-\ln((2x+1)^{\frac{1}{2}})-\ln(e^{3x})$

$=3\ln(\cos(x))+3\ln(2x+1)+\ln(\sin^{-1}(x))-\frac{1}{2}\ln(2x+1)-3x)$. This implies

that $\frac{y'}{y}=3\frac{-\sin(x)}{\cos(x)}+3\cdot\frac{2}{2x+1}+\frac{\frac{1}{\sqrt{1-x^2}}}{(\sin^{-1}(x)}-\frac{2}{2(2x+1)}-3$

$=-3\tan(x)+\frac{6}{2x+1}+\frac{1}{\sqrt{1-x^2}(\sin^{-1}(x)}-\frac{1}{(2x+1)}-3$ and

$y'=(\frac{\cos(x)^3(2x+1)^3\sin^{-1}(x)}{\sqrt{2x+1}e^{3x}})(-3\tan(x)+\frac{6}{2x+1}+\frac{1}{\sqrt{1-x^2}(\sin^{-1}(x)}-\frac{1}{(2x+1)}-3)$.

(g) $f(x)=\tan^{-1}(e^{3x})$, $f'(x)=\frac{(e^{3x})'}{1+(e^{3x})^2}=\frac{3e^{3x}}{1+e^{6x}}$.

(i) $f(x)=\sec^{-1}(x^2)$ $f'(x)=(\sec^{-1}(x^2))'=\frac{(x^2)'}{|x^2|\sqrt{x^4-1}}=\frac{2x}{x^2\sqrt{x^4-1}}==\frac{2}{x\sqrt{x^4-1}}$.

(j) $f(x)=\csc^{-1}(x^2)\cot^{-1}(2x)+x\cos^{-1}(2x)$ $f'(x)=(\csc^{-1}(x^2))'\cot^{-1}(2x)+\csc^{-1}(x^2)(\cot^{-1}(2x))'+\cos^{-1}(2x)+x(\cos^{-1}(2x))'$

$= -\frac{(x^2)'}{|x^2|\sqrt{x^4-1}}\cot^{-1}(2x) + \csc^{-1}(x^2)\cdot(-\frac{(2x)'}{1+x^4}) + \cos^{-1}(2x) + x\cdot\frac{-1}{\sqrt{1-x^2}} = -\frac{2}{x\sqrt{x^4-1}}\cot^{-1}(2x) - \csc^{-1}(x^2)(\frac{2}{1+x^4}) + \cos^{-1}(2x) - x\cdot\frac{1}{\sqrt{1-x^2}}$

(k) $y = \frac{(x^2-1)^2(2x+1)^3x^5}{(x^2+1)^3(x+1)^4\sin^5(2x)}$

$\ln(y) = 2\ln(x^2-1) + 3\ln(2x+1) + 5\ln(x) - 3\ln(x^2+1) - 4\ln(x+1) - 5\ln(\sin(2x))$.
This implies that $\frac{y'}{y} = 2\frac{2x}{x^2-1} + 3\frac{2}{2x+1} + 5\frac{1}{x} - 4\frac{1}{x+1} - 5\frac{2\cos(2x)}{\sin(2x)}$
$= \frac{4x}{x^2-1} + \frac{6}{2x+1} + \frac{5}{x} - \frac{4}{x+1} - 10\cot(2x)$.
Thus $y'(x) = (\frac{(x^2-1)^2(2x+1)^3x^5}{(x^2+1)^3(x+1)^4\sin^5(2x)})(\frac{4x}{x^2-1} + \frac{6}{2x+1} + \frac{5}{x} - \frac{4}{x+1} - 10\cot(2x))$.

2. (a) $\sin^{-1}(\frac{1}{2}) = -\pi/6$ (b) $\cos^{-1}(-\frac{\sqrt{3}}{2}) = 5\pi/6$ (c) $\sin^{-1}(-1) = -\pi/2$ (d) $\cos^{-1}(-\frac{1}{\sqrt{2}}) = 3\pi/4$ (e) $\sec^{-1}(-2) = 2\pi/3$ (b) $\csc^{-1}(-\frac{2}{\sqrt{3}}) = -pi/3$ (f) $\cot^{-1}(-\sqrt{3}) = 5\pi/6$ (g) $\tan^{-1}(-1) = -\pi/4$

(h) $\tan(\sec^{-1}(\frac{5}{3}))$

Let $\theta = \sec^{-1}(\frac{5}{3})$. Then $\sec(\theta) = \sec(\sec^{-1}(\frac{5}{3})) = \frac{5}{3} = \frac{hyp}{adj}$. From $adj^2 + opp^2 = hyp^2$, we have $3^2 + opp^2 = 5^2$ and $opp = 4$. So $\tan(\sec^{-1}(\frac{5}{3})) = \tan(\theta) = \frac{opp}{adj} = \frac{4}{3}$.

(i) $\cos(\tan^{-1}(-\frac{2}{3}))$

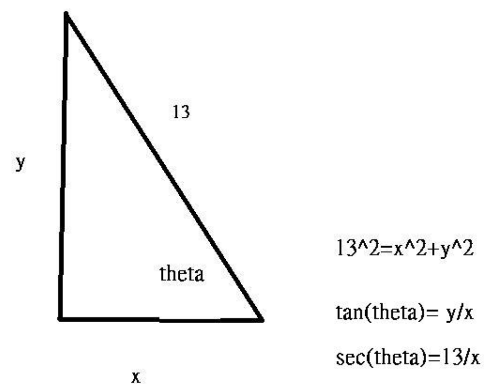
Let $\theta = \tan^{-1}(-\frac{2}{3})$. Then $\tan(\theta) = \tan(\tan^{-1}(-\frac{2}{3})) = \frac{-2}{3} = \frac{opp}{adj}$. We have $opp = -2$ and $adj = 3$ From $adj^2 + opp^2 = hyp^2$, we have $3^2 + (-2)^2 = hyp^2$ and $hyp = \sqrt{13}$. So $\cos(\tan^{-1}(-\frac{2}{3})) = \cos(\theta) = \frac{adj}{hyp} = \frac{3}{\sqrt{13}}$.

(j) $\sec(\csc^{-1}(-\frac{5}{3}))$

Let $\theta = \csc^{-1}(-\frac{5}{3})$. Then $\csc(\theta) = \csc(\csc^{-1}(-\frac{5}{3})) = -\frac{5}{3} = \frac{5}{-3} = \frac{hyp}{opp}$. We have $opp = -3$ and $hyp = 5$ From $adj^2 + opp^2 = hyp^2$, we have $adj^2 + (-3)^2 = 5^2$ and $adj = 4$. So $\sec(\csc^{-1}(-\frac{5}{3})) = \sec(\theta) = \frac{hyp}{adj} = \frac{5}{4}$.

(k) $\cot(\sin^{-1}(-\frac{2}{3}))$

Let $\theta = \sin^{-1}(-\frac{2}{3})$. Then $\sin(\theta) = \sin(\sin^{-1}(-\frac{2}{3})) = -\frac{2}{3} = \frac{-2}{3} = \frac{opp}{hyp}$. We have $opp = -2$ and $hyp = 3$ From $adj^2 + opp^2 = hyp^2$, we have $adj^2 + (-2)^2 = 3^2$ and $adj = \sqrt{5}$. So $\cot(\sin^{-1}(-\frac{2}{3})) = \cot\theta) = \frac{adj}{opp} = \frac{\sqrt{5}}{-2}$.

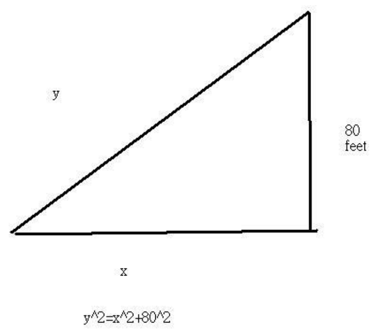


3. A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 5 ft from the house, the base is moving away at the rate of 24 ft/sec.
(a) What is the rate of change of the height of the top of the ladder?
(b) At what rate is the angle between the ladder and the ground changing then?

Solution: Solution: (a) Let x be the distance between the base of the ladder to the house and y be the distance between the ladder and the ground. We have $x^2 + y^2 = 13^2 = 169$. This implies that $2xx'(t) + 2yy'(t) = 0$. We are given $x = 5$ and $x' = 24$. From $2xx'(t) + 2yy'(t) = 0$, we get $xx' + yy' = 0$, $yy' = -xx'$ and $y'(t) = -\frac{xx'(t)}{y}$. From $x^2 + y^2 = 169$ and $x = 5$, we get $y^2 = 169 - 5^2 = 169 - 25 = 144$ and $y = 12$. Thus $y'(t) = -\frac{xx'(t)}{y} = -\frac{5\cdot 24}{12} = -10$ and the rate of change of the height of the top of the ladder is -10ft/sec.
(b) Let θ be the angle between the ladder and the ground. We have $\tan(theta) = \frac{y}{x}$. This implies that $\sec^2(\theta)\theta'(t) = \frac{y'x-yx'}{x^2}$, $\frac{13^2}{x^2}\theta'(t) = \frac{y'x-yx'}{x^2}$ and $\theta'(t) = \frac{y'x-yx'}{169}$. Using $x = 5$, $x' = 24$, $y = 12$ and $y' = -10$, we get $\theta'(t) = \frac{y'x-yx'}{169} = \frac{(-10)\cdot 5-12\cdot 24}{169} = \frac{-50-288}{169} = \frac{-338}{169} = -2$. Thus the rate where the angle between the ladder and the ground changing is $-2\ radian/sec$.

4. A child flies a kite at a height of 80ft, the wind carrying the kite horizontally away from the child at a rate of 34 ft/sec. How fast must the child let out the string when the kite is 170 ft away from the child?

Solution: The height of the kite is 80. The horizontal distance the



wind blows the kite is x. The amount of the string let out to blow the kite x feet is y. We have $y^2 = x^2 + 80^2$. This implies that $2yy'(t) = 2xx'(t)$ and $y' = \frac{xx'(t)}{y}$. We are given $x = 170$ and $x'(t) = 34$. From $y^2 = x^2 + 6400$, we have $y^2 = 170^2 + 6400 = 28900 + 6400 = 35300$ and $y = \sqrt{35300}$. Thus $y' = \frac{170\cdot 34}{\sqrt{35300}} = \frac{170\cdot 34}{\sqrt{35300}} = \frac{5780}{\sqrt{35300}}$.

5. A spherical balloon is inflating with helium at a rate of 180 π $\frac{ft^3}{min}$. (a) How fast is the balloon's radius increasing at the instant the radius is 3 ft?
(b) How fast is the surface area increasing?

Solution: (a) The volume of a sphere with radius r is $V(r) = 4\pi r^3 3$. From this we know that $\frac{dV(r)}{dt} = 4\pi \cdot 3 \cdot r^2 3 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$. We are given $\frac{dV(r)}{dt} = 180\pi \ \frac{ft^3}{min}$ and $r(t) = 3$. This gives $180\pi \ \frac{ft^3}{min} = 4\pi \cdot 3^2 ft^2 \cdot \frac{dr}{dt}$ and $\frac{dr}{dt} = 5\frac{ft}{min}$.
(b) The surface area of a sphere is $A = 4\pi r^2$. Thus $\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi \cdot 3ft \cdot 5\frac{ft}{min} = 120\pi \frac{ft^2}{min}$.

6. (a) Find the linearization of $(27+x)^{\frac{1}{3}}$ at $x=0$.
(b) Use the linearization in part (a) to estimate $(28)^{\frac{1}{3}}$. Solution: (a)

$f(x) = (27+x)^{\frac{1}{3}}$ $f'(x) = \frac{1}{3}(27+x)^{-\frac{2}{3}}$. The linearization of f at $x=0$ is $L(x) = f(0)+f'(0)(x-0) = (27)^{\frac{1}{3}}+\frac{1}{3}(27)^{-\frac{2}{3}}x = 3+\frac{1}{3}\frac{1}{((27)^{\frac{1}{3}})^2}x = 3+\frac{1}{3}\frac{1}{9}x = 3+\frac{x}{27}$.
(b) We can use $L(1) = 3+\frac{1}{27} = \frac{82}{27}$ to approximate $f(1) = (28)^{\frac{1}{3}}$.

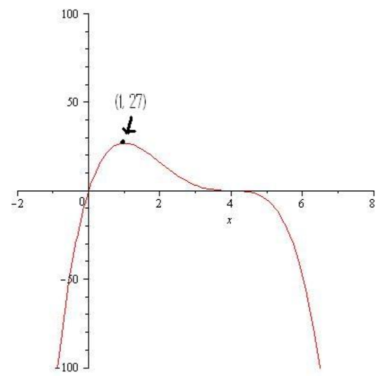
7. (a) Find the linearization of $\sqrt{9+x}$ at $x=0$.
(b) Use the linearization in part (a) to estimate $\sqrt{9.1}$. Solution: (a)

$f(x) = \sqrt{9+x}$ $f'(x) = \frac{1}{2}(9+x)^{-\frac{1}{2}}$. The linearization of f at $x=0$ is $L(x) = f(0)+f'(0)(x-0) = \sqrt{9}+\frac{1}{2}(9)^{-\frac{1}{2}}x = \sqrt{9}+\frac{1}{2}\frac{1}{3}x = 3+\frac{1}{6}x$.
(b) We can use $L(0.1) = 3+\frac{0.1}{6}x = 3+0.0166\cdots = 3.00166$ to approximate $f(0.1) = \sqrt{9.1}$.

8. Find the critical points of the f and identify the intervals on which f is increasing and decreasing. Also find the function's local and absolute extreme values.
(a) $f(x) = x(4-x)^3$ (b) $f(x) = x^2+\frac{2}{x}$ (c) $f(x) = x-3x^{\frac{1}{3}}$ (d) $f(x) = (x^2-2)e^{2x}$.
Solution: (a) First, note that the domain of $f(x) = x(4-x)^3$ is $(-\infty,\infty)$. $f'(x) = (x)'(4-x)^3+x((4-x)^3)' = (4-x)^3+x\cdot 3(4-x)^2(4-x)' = (4-x)^3 - 3x(4-x)^2 = (4-x)^2(4-x-3x) = (4-x)^2(4-4x) = 4(4-x)^2(1-x)$. f' exists everywhere. So the critical point is determined by solving $f'(x)=0 \Leftrightarrow (4-x)^2(4-4x)=0$. So $x=1$ or $x=4$. Thus the critical points are $x=1$ or $x=4$.
We try to find out where f' is positive, and where it is negative by factoring $f'(x) = 4(4-x)^2(1-x)$. Note that the critical points 1 and 4 divide the domain into $(-\infty,1)\cup(1,4)\cup(4,\infty)$. Take $-2\in(-\infty,1)$, $2\in(1,4)$ and $5\in(4,\infty)$. Evaluate $f'(-2) = 4(4+2)^2(1+2) > 0$, $f'(2) = 4(4-2)^2(1-2) = +\cdot - < 0$ and $f'(5) = 4(4-5)^2(1-5) = +\cdot - < 0$. From which we see that $f'(x) > 0$ for $x\in(-\infty,1)$ and $f'(x) < 0$ for $x\in(1,4)\cup(4,\infty)$. Therefore the function f is increasing on $(-\infty,1)$, decreasing on $(1,4)\cup(4,\infty)$. Note that $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} x(4-x)^3 = -\infty\cdot\infty = -\infty$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x(4-x)^3 = \infty\cdot-\infty = -\infty$. So f has a absolute maximum at $x=1$ with $f(1) = 1\cdot(4-1)^3 = 27$.

x	$(-\infty,1)$	(1,4)	$(4,\infty)$
$f'(x)$	$f'(-2)>0$ +	$f'(2)<0$ -	$f'(5)<0$ -
$f(x)$	increasing	decreasing	decreasing

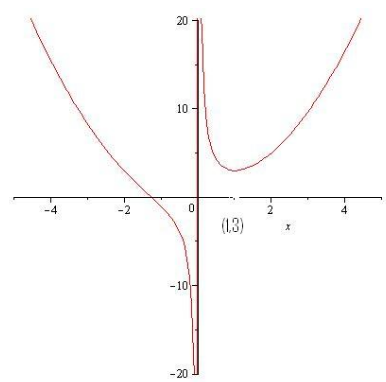
(b) First, note that the domain of $f(x) = x^2+\frac{2}{x}$ is $(-\infty,0)\cup(0,\infty)$. $f'(x) = 2x-\frac{2}{x^2} = \frac{2x^3-2}{x^2} = 2\frac{x^3-1}{x^2} = \frac{(x-1)(x^2+x+1)}{x^2}$. f' exists everywhere in the domain of f. So the critical point is determined by solving $f'(x)=0 \Leftrightarrow$



$\frac{(x-1)(x^2+x+1)}{x^2} = 0$. So $x = 1$. Note that $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} > 0$. Thus the critical points are $x = 1$.
Note that 0 and 1 divide the domain $(-\infty, 0) \cup (0, \infty)$ into $(\infty, 0) \cup (0, 1) \cup (1, \infty)$. Take $-1 \in (-\infty, 0)$, $0.5 \in (0, 1)$ and $2 \in (1, \infty)$. Evaluate $f'(-1) = \frac{(-)(+)}{+} < 0$, $f'(0.5) = \frac{(-)(+)}{+} < 0$ and $f'(2) = \frac{(+)(+)}{+} > 0$ We know that $f'(x) < 0$ for $x \in (-\infty, 0)$, $f'(x) < 0$ for $x \in (0, 1)$ $f'(x) > 0$ for $x \in (1, \infty)$. Therefore the function f is decreasing on $(-\infty, 0) \cup (0, 1)$, increasing on $(1, \infty)$.

x	$(-\infty, 0)$	(0,1)	$(1, \infty)$
$f'(x)$	$f'(-1) < 0$ -	$f'(0.5) < 0$ -	$f'(2) > 0$ +
$f(x)$	decreasing	decreasing	decreasing

Note that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^2 + \frac{2}{x} = \infty$ and $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^2 + \frac{2}{x} = \infty$, $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} x^2 + \frac{2}{x} = \frac{2}{0^-} = -\infty$, $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x^2 + \frac{2}{x} = \frac{2}{0^+} = \infty$ So f has a local minimum at $x = 1$ with $f(1) = 1 + \frac{2}{1} = 3$.

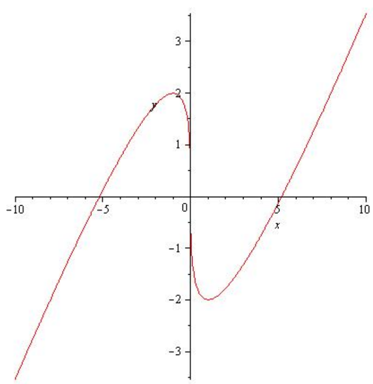


(c) First, note that the domain of $f(x) = x - 3x^{\frac{1}{3}}$ is $(-\infty, \infty)$. $f'(x) = 1 - x^{-\frac{2}{3}} = 1 - \frac{1}{\sqrt[3]{x^2}} = \frac{\sqrt[3]{x^2}-1}{\sqrt[3]{x^2}}$. f' doesn't exist when $x = 0$. Next we solve

$f'(x) = 0 \Leftrightarrow \frac{\sqrt[3]{x^2}-1}{\sqrt[3]{x^2}}$. So $\sqrt[3]{x^2} = 1$, $x^2 = 1$ and $x = 1$ or $x = -1$. Thus the critical points are $x = -1$, $x = 1$ and $x = 0$ ($f'(0)$ doesn't exist and 0 is in the domain).
Note that 0 and ± 1 divide the real line into $(-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$. Take $-2 \in (-\infty, -1)$, $-0.5 \in (-1, 0)$, $0.5 \in (0, 1)$, and $2 \in (1, \infty)$. From $f'(x) = \frac{\sqrt[3]{x^2}-1}{\sqrt[3]{x^2}}$ $f'(-2) = \frac{+}{+} > 0$, $f'(-0.5) = \frac{-}{+} < 0$, and $f'(0.5) = \frac{-}{+} < 0$ $f'(2) = \frac{+}{+} > 0$ We know that $f'(x) > 0$ for $x \in (-\infty, -1) \cup (1, \infty)$, $f'(x) < 0$ for $x \in (-1, 0) \cup (0, 1)$ Therefore the function f is increasing on $(-\infty, -1) \cup (1, \infty)$, decreasing on $(-1, 0) \cup (0, 1)$.

x	$(-\infty, -1)$	(-1,0)	(0,1)	$(1, \infty)$
$f'(x)$	$f'(-2) > 0$ +	$f'(-0.5) < 0$ -	$f'(0.5) < 0$ -	$f'(2) > 0$ +
$f(x)$	increasing	decreasing	decreasing	increasing

Note that $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x - 3x^{\frac{1}{3}} = \lim_{x\to-\infty} x(1 - 3\frac{1}{x^{\frac{2}{3}}}) = -\infty$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x - 3x^{\frac{1}{3}} = \lim_{x\to\infty} x(1 - 3\frac{1}{x^{\frac{2}{3}}}) = \infty$. Evaluating f at critical points, we get $f(0) = 0$, $f(-1) = 1-3(-1)^{\frac{1}{3}} = -1+3 = 2$ and $f(1) = 1-3 = -2$. From the graph of f,(see next page) we conclude that f has a local maximum at $x = -1$ with $f(-1) = 1 - 3(-1)^{\frac{1}{3}} = -1 + 3 = 2$ and local minimum at $x = 1$ with $f(1) = 1 - 3 = -2$.

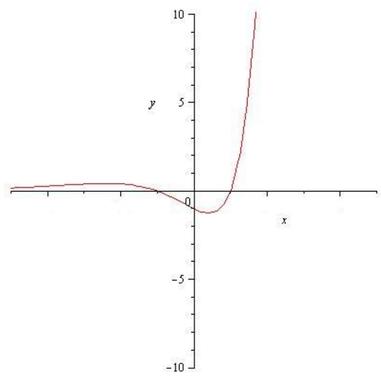


(d) First, note that the domain of $f(x) = (x^2-2)e^{2x}$ is $(-\infty, \infty)$. $f'(x) = (x^2 - 2)'e^{2x} + (x^2 - 2)(e^{2x})' = (2x)e^{2x} + 2(x^2 - 2)e^{2x} = (2x + 2x^2 - 4)e^{2x} = 2(x^2 + x - 2)e^{2x} = 2(x + 2)(x - 1)e^{2x}$. $f'(x)$ exists everywhere. $f'(x) = 0 \iff 2(x + 2)(x - 1)e^{2x} = 0$ and $x = -2$ or $x = 1$. So the critical points of f are $x = -2$ or $x = 1$.
Next we determine where $f' > 0$ and where $f' < 0$. The critical points -2 and 1 divide the domain $(-\infty, \infty)$ into $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$. Take $-3 \in (-\infty, -2), 0 \in (-2, 1)$ and $2 \in (1, \infty)$. Evaluate $f'(-3) = 2(-3 + 2)(-3 - 1)e^{-6} = + \cdot - \cot - \cdot + = + > 0$, $f'(0) = 2(0 + 2)(0 - 1)e^0 =$

$+\cdot+\cot-\cdot+=-<0$, $f'(2)=2(2+2)(2-1)e^4=+\cdot-\cot+\cdot+=+>0$. So $f'(x)>0$ on $(-\infty,-2)\cup(1,\infty)$ and $f'(x)<0$ on $(-2,1)$. This implies that f is increasing on $(-\infty,-2)\cup(1,\infty)$ and decreasing on $(-2,1)$.

x	$(-\infty,-2)$	(-2,1)	$(1,\infty)$
$f'(x)$	$f'(-3)>0$ +	$f'(0)<0$ -	$f'(2)>0$ +
$f(x)$	increasing	decreasing	increasing

Evaluating f at critical points, we get $f(-2)=(4-2)e^{-4}=2e^{-4}>0$, $f(1)=(1^2-2)e^{2\cdot1}=-e^2<0$. Note that $\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}(x^2-2)e^{2x}=0$ and $\lim_{x\to\infty}f(x)=\lim_{x\to-\infty}(x^2-2)e^{2x}=\infty$. From the graph of f,(see next page) we conclude that f has a local maximum at $x=-2$ with $f(-2)=2e^{-4}$ and global minimum at $x=1$ with $f(1)=-e^2$.



9. Find the absolute maximum and minimum values of each function on the given interval.
(a) $f(x)=\frac{x}{x^2+1}$, $-2\le x\le 2$ (b) $f(x)=f(x)=x-3x^{\frac{1}{3}}$, $0\le x\le 27$
(c) $f(x)=f(x)=x-3x^{\frac{1}{3}}$, $-27\le x\le 27$ (d) $f(x)=\frac{1}{x}+\ln(x)$, $\frac{1}{2}\le x\le 4$
(e) $f(x)=xe^{-x}$, $0\le x\le 2$

Solution: (a) First, we find the derivative of f. $f'(x)=(\frac{x}{x^2+1})'=\frac{(x)'(x^2+1)-x(x^2+1)'}{(x^2+1)^2}=\frac{(x^2+1)-x\cdot2x}{(x^2+1)^2}=\frac{(x^2+1)-2x^2}{(x^2+1)^2}=\frac{1-x^2}{(x^2+1)^2}=\frac{(1-x)(1+x)}{(x^2+1)^2}$. f' exists everywhere in $[-2,2]$. Next we solve $f'(x)=0$, i.e. $\frac{(1-x)(1+x)}{(x^2+1)^2}$. Thus $x=\pm1$. So the critical points are ±1. Evaluating at critical points, we get $f(-1)=\frac{-1}{(-1)^2+1}=-\frac{1}{2}$ and $f(1)=\frac{1}{2}$. Next we evaluate f at the end points 2 and -2. $f(2)=\frac{2}{2^2+1}=\frac{2}{5}$ and $f(-2)=\frac{-2}{2^2+1}=-\frac{2}{5}$. Thus f has the absolute maximum at $x=1$ with $f(1)=\frac{1}{2}$ and f has the absolute minimum at $x=-1$ with $f(-1)=-\frac{1}{2}$.

(b) $f(x)=f(x)=x-3x^{\frac{1}{3}}$ $f'(x)=1-x^{-\frac{2}{3}}=1-\frac{1}{\sqrt[3]{x^2}}=\frac{\sqrt[3]{x^2}-1}{\sqrt[3]{x^2}}$. f' doesn't exist when $x=0$. Next we solve $f'(x)=0\Leftrightarrow\frac{\sqrt[3]{x^2}-1}{\sqrt[3]{x^2}}$. So $\sqrt[3]{x^2}=1$, $x^2=1$

and $x = 1$ or $x = -1$. Thus the critical points are $x = -1$, $x = 1$ and $x = 0$ ($f'(0)$ doesn't exist and 0 is in the domain). Evaluating f at critical points, we get $f(-1) = (-1) - 3(-1)^{\frac{1}{3}} = -1 - 3 \cdot (-1) = -1 + 3 = 2$, $f(0) = 0$, $f(1) = 1 - 3 = -2$. The end point of $[0, 27]$ are 0 and 27. We just need to find $f(27) = 27 - 3(27)^{\frac{1}{3}} = 27 - 3 \cdot 3 = 27 - 9 = 18$. The largest of the value from $\{f(-1) = 2, f(0) = 0, f(1) = -3, f(27) = 18\}$ is 18 and the smallest value of $\{f(-1) = 2, f(0) = 0, f(1) = -3, f(27) = 18\}$ is -3. Thus f has the absolute maximum at $x = 27$ with $f(27) = 18$ and f has the absolute minimum at $x = 1$ with $f(1) = -3$.

(c) This problem is similar to (b) except the interval is $[-27, 27]$. We need to evaluate at end points $f(-27) = (-27) - 3(-27)^{\frac{1}{3}} = -27 - 3 \cdot (-3) = -27 + 9 = -18$ The largest of the value from $\{f(-1) = 2, f(0) = 0, f(1) = -3, f(27) = 18, f(-27) = -18\}$ is 18 and the smallest value of $\{f(-1) = 2, f(0) = 0, f(1) = -3, f(27) = 18, f(-27) = -18\}$ is -3. Thus f has the absolute maximum at $x = 27$ with $f(27) = 18$ and f has the absolute minimum at $x = -27$ with $f(-27) = -18$.

(d) $f(x) = \frac{1}{x} + \ln(x)$, $\frac{1}{2} \leq x \leq 4$
$f'(x) = -\frac{1}{x^2} + \frac{1}{x} = -\frac{1}{x^2} + \frac{x}{x^2} = \frac{x-1}{x^2}$. So the critical point of f is $x = 1$. The set of critical points and the end points of $[\frac{1}{2}, 4]$ are $\{1, \frac{1}{2}, 4\}$. Evaluating f at those points, we get $f(1) = 1 + \ln(1) = 1$, $f(\frac{1}{2}) = \frac{1}{\frac{1}{2}} + \ln(\frac{1}{2}) = 2 - \ln 2 \approx 2 - 0.69 \approx 1.31$ $f(4) = \frac{1}{4} + \ln(4) = 0.25 + 1.38 \approx 1.63$. So f has the absolute maximum at $x = 4$ with $f(4) = \frac{1}{4} + \ln(4)$ and the absolute minimum at $x = 1$ with $f(1) = 1$.

(e) $f(x) = xe^{-x}$, $0 \leq x \leq 2$
$f'(x) = (x)'e^{-x} + x(e^{-x})' = e^{-x} - xe^{-x} = (1 - x)e^{-x}$. So the critical point is $x = 1$. The set of critical points and the end points of $[0, 2]$ are $\{1, 0, 2\}$. Evaluating at these points, we get $f(1) = e^{-1}$, $f(0) = 0$ and $f(2) = 2e^{-2}$. From $f'(x) = (1 - x)e^{-x}$, we know that $f' > 0$ if $x < 1$ and $f' < 0$ and $x > 1$. So f is decreasing from 1 to 2. So f has the absolute maximum at $x = 1$ with $f(1) = e^{-1}$ and the absolute minimum at $x = 0$ with $f(0) = 0$