Math 6(8)540 Homework 5 Due date: December 5

(1) Integration by parts to prove the interpolation inequality

$$\int_\Omega |Du|^2 dx \le C(\int_\Omega u^2 dx)^{\frac{1}{2}}(\int_\Omega |D^2u|^2 dx)^{\frac{1}{2}}$$

for all $u \in C_0^\infty(\Omega)$. By approximation, prove this inequality if $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.
$(Hint: \int_\Omega div(u\nabla u)dx = \int_\Omega(|Du|^2 + u\triangle u)dx$. Note that $|D^2u|^2 = \sum_{i,j=1}^n |u_{ij}|^2$.)

(2) Integration by parts to prove

$$\int_\Omega |Du|^p dx \le C(\int_\Omega u^p dx)^{\frac{1}{2}}(\int_\Omega |D^2u|^p dx)^{\frac{1}{2}}$$

for $2 \le p < \infty$ and all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.
$(Hint: \int_\Omega |Du|^p dx = \sum_{i=1}^n \int_\Omega u_{x_i} u_{x_i} |Du|^{p-2})dx$.

(3) Suppose Ω is bounded and $u \in W^{1,p}(\Omega)$ satisfies $Du = 0$ a.e. in Ω. Prove that u is constant *a.e.* in Ω.