Theorem 1 p263 (Gagliardo-Nirenberg-Sobolev inequality)
Assume $1 \le p < n$. There exists a constant C, depending only on p and n, such that

$$||u||_{L^{\frac{np}{n-p}}(R^n)} \le C||Du||_{L^p(R^n)}$$

for all $u \in C_0^1(R^n)$.
Remark: From the proof, we my choose $C(n,p) = \frac{p(n-1)}{n-p}$. But this may not be the best constant.

Proof:

1. First, we prove the case $p = 1$. We want to show that

$$||u||_{L^{\frac{n}{n-1}}(R^n)} \le C||Du||_{L^1(R^n)}$$

for all $u \in C_0^1(R^n)$. Since u has compact support, we have for each $1 \le i \le n$ and $x \in R^n$

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) dy_i$$

and

$$\begin{aligned} |u(x)| &\le \int_{-\infty}^{x_i} |u_{x_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| dy_i \\ &\le \int_{-\infty}^{x_i} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| dy_i \\ &\le \underbrace{\int_{-\infty}^{\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| dy_i}_{independent\ of\ x_i} . \end{aligned} \tag{1}$$

Consequently,

$$|u(x)|^{\frac{1}{n-1}} \le (\int_{-\infty}^{\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| dy_i)^{\frac{1}{n-1}}$$

and

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} (\underbrace{\int_{-\infty}^{\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| dy_i)^{\frac{1}{n-1}}}_{independent\ of\ x_i}$$

2. To illustrate the main ideas, we discuss the case when $n = 3$. So we have

$$|u(x)|^{\frac{3}{2}} \leq (\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1)^{\frac{1}{2}} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2)^{\frac{1}{2}} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3)^{\frac{1}{2}}$$

and

$$\begin{aligned}
&\int_{-\infty}^{\infty} |u(x)|^{\frac{3}{2}} dx_1 \\
&\leq \int_{-\infty}^{\infty} \underbrace{(\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1)^{\frac{1}{2}}}_{independent\ of\ x_1} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2)^{\frac{1}{2}} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3)^{\frac{1}{2}} dx_1 \\
&= (\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1)^{\frac{1}{2}} \int_{-\infty}^{\infty} \underbrace{(\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2)^{\frac{1}{2}} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3)^{\frac{1}{2}} dx_1}_{apply\ holder\ inequality\ w.r.t.\ x_1\ \int |fg| dx_1 \leq (\int f^2 dx_1)^{\frac{1}{2}} (\int g^2 dx_1)^{\frac{1}{2}}} \\
&\leq (\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1)^{\frac{1}{2}} \\
&\quad \underbrace{(\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2) dx_1)^{\frac{1}{2}}}_{independent\ of\ x_2} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3) dx_1)^{\frac{1}{2}}
\end{aligned} \tag{2}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^{\frac{3}{2}} dx_1 dx_2 \\
&\leq (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2) dx_1)^{\frac{1}{2}} \\
&\quad \int_{-\infty}^{\infty} \underbrace{(\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1)^{\frac{1}{2}} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3) dx_1)^{\frac{1}{2}} dx_2}_{apply\ holder\ inequality\ w.r.t.\ x_2} \\
&\leq (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2) dx_1)^{\frac{1}{2}} \\
&\quad (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1) dx_2)^{\frac{1}{2}} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3) dx_1) dx_2)^{\frac{1}{2}} \\
&= ||Du||_{L^1(R^3)}^{\frac{1}{2}} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2) dx_1)^{\frac{1}{2}} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1) dx_2)^{\frac{1}{2}}
\end{aligned} \tag{3}$$

Integrating w.r.t x_3, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u(x)|^{\frac{3}{2}}dx_1dx_2dx_3\\
&\le ||Du||_{L^1(R^3)}^{\frac{1}{2}}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}|Du(x_1,y_2,x_3)|dy_2)dx_1)^{\frac{1}{2}}(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}|Du(y_1,x_2,x_3)|dy_1)dx_2)^{\frac{1}{2}}dx_3\\
&\le ||Du||_{L^1(R^3)}^{\frac{1}{2}}(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}|Du(x_1,y_2,x_3)|dy_2)dx_1)dx_3)^{\frac{1}{2}}\\
&\quad(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}|Du(y_1,x_2,x_3)|dy_1)dx_2)dx_3)^{\frac{1}{2}}\\
&= ||Du||_{L^1(R^3)}^{\frac{3}{2}}.
\end{aligned} \tag{4}$$

Thus we have

$$||u||_{L^{\frac{3}{2}}(R^3)}^{\frac{3}{2}} \le ||Du||_{L^1(R^3)}^{\frac{3}{2}}.$$

This implies

$$||u||_{L^{\frac{3}{2}}(R^3)} \le ||Du||_{L^1(R^3)}.$$

3. For the general case, we start with

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n}(\underbrace{\int_{-\infty}^{\infty}|Du(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)|dy_i}_{independent\ of\ x_i})^{\frac{1}{n-1}}.$$

$$\begin{aligned}
&\int_{-\infty}^{\infty}|u(x)|^{\frac{n}{n-1}}dx_1\\
&\le \int_{-\infty}^{\infty}\underbrace{(\int_{-\infty}^{\infty}|Du(y_1,x_2,\cdots x_n)|dy_1)^{\frac{1}{n-1}}}_{independent\ of\ x_1}\prod_{i=2}^{n}(\underbrace{\int_{-\infty}^{\infty}|Du(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)|dy_i}_{independent\ of\ x_i})^{\frac{1}{n-1}}dx_1\\
&= (\int_{-\infty}^{\infty}|Du(y_1,x_2,\cdots x_n)|dy_1)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\underbrace{\prod_{i=2}^{n}(\int_{-\infty}^{\infty}|Du(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)|dy_i)^{\frac{1}{n-1}}dx_1}_{apply\ holder\ inequality\ w.r.t.\ x_1\ \int|f_2\cdots f_n|dx_1\le\prod_{i=2}^{n}(\int f_i^{n-1}dx_1)^{\frac{1}{n-1}}}\\
&\le (\int_{-\infty}^{\infty}|Du(y_1,x_2,\cdots x_n)|dy_1)^{\frac{1}{n-1}}\prod_{i=2}^{n}(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)|dy_idx_1)^{\frac{1}{n-1}}
\end{aligned} \tag{5}$$

Repeating this process, we get

$$\int_{R^n} |u(x)|^{\frac{n}{n-1}} dx \leq (\int_{R^n} |Du| dx)^{\frac{n}{n-1}}.$$

Thus

$$||u||^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(R^n)} \leq ||Du||^{\frac{n}{n-1}}_{L^1(R^n)}.$$

This implies

$$|||u||_{L^{\frac{n}{n-1}}(R^n)} \leq ||Du||_{L^1(R^n)}. \tag{6}$$

4. Now we consider the case $1 < p < n$. Let $v := |u|^\gamma = (u^2)^{\frac{\gamma}{2}}$ where $\gamma > 1$ is to be determined. Then $v_{x_i} = \frac{\gamma}{2}(u^2)^{\frac{\gamma}{2}-1}(2uu_{x_i}) = \gamma(u^2)^{\frac{\gamma}{2}-1}(uu_{x_i})$ and $|Dv| = \gamma|u|^{\gamma-1}|Du|$. We apply estimate 6 to $v := |u|^\gamma$ to get

$$(\int_{R^n} |v(x)|^{\frac{n}{n-1}} dx)^{\frac{n-1}{n}} \leq \ (\int_{R^n} |Dv| dx).$$

Thus we have

$$\begin{aligned} &(\int_{R^n} |u(x)|^{\frac{\gamma n}{n-1}} dx)^{\frac{n-1}{n}} \\ &\leq \gamma \int_{R^n} |u|^{\gamma-1}|Du| dx \\ &\leq \gamma(\int_{R^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx)^{\frac{p-1}{p}} (\int_{R^n} |Du|^p dx)^{\frac{1}{p}} \end{aligned} \tag{7}$$

Choose γ so that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1}$, i.e $\frac{1}{\gamma} = 1 - \frac{n(p-1)}{p(n-1)}$ and $\gamma = \frac{p(n-1)}{n-p} > 1$ (since $1 < p < n$). Now we have $\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1} = \frac{\frac{p(n-1)}{n-p}n}{n-1} = \frac{np}{n-p}$. Hence 7 can be rewritten as $(\int_{R^n} |u(x)|^{\frac{np}{n-p}} dx)^{\frac{n-1}{n}} \leq \frac{p(n-1)}{n-p}(\int_{R^n} |u|^{\frac{np}{n-p}} dx)^{\frac{p-1}{p}} (\int_{R^n} |Du|^p dx)^{\frac{1}{p}}$. Note that $\frac{n-1}{n} - \frac{p-1}{p} = \frac{n-p}{np}$. Thus we have $(\int_{R^n} |u(x)|^{\frac{np}{n-p}} dx)^{\frac{n-p}{np}} \leq \frac{p(n-1)}{n-p}(\int_{R^n} |Du|^p dx)^{\frac{1}{p}}$ which is

$$||u||_{L^{\frac{np}{n-p}}(R^n)} \leq \frac{p(n-1)}{n-p}||Du||_{L^p(R^n)}.$$