Lecture 23:Cholesky Factorization

March 28, 2008

1 Hermitian Positive Definite Matrix

Definition 1.1. *A complex matrix $A \in C^{m\times m}$ is hermitian if $A^* = A$ ($\overline{A}^T = A$ or $a_{ij} = \overline{a_{ji}}$). A is said to be hermitian positive definite if $x^*Ax > 0$ for all $x \neq 0$.*

Remark:

• A is hermitian positive definite if and only if it's eigenvalues are all positive.

• If A is hermitian positive definite and $A = LU$ is the LU decomposition of A then $u_{11} > 0$, $u_{22} > 0$, $\cdots$, $u_{mm} > 0$.

This can be proved by the following steps.

1. A is hermitian positive definite then $det(A_k) > 0$ for $k = 1, \cdots n$ where A_k is the $k \times k$ diagonal matrix of A, i.e. $A_k = (a_{ij})_{1\le i\le k, 1\le j\le k}$. If we write $L = \begin{bmatrix} L_k & 0 \\ P & Q \end{bmatrix}$ and $U = \begin{bmatrix} U_k & R \\ 0 & S \end{bmatrix}$ where $L_k = (l_{ij})_{1\le i\le k,1\le j\le k}$ and $U_k = (u_{ij})_{1\le i\le k,1\le j\le k}$ then $A_k = L_kU_k$. Note that L_k is unit lower-triangular and $det(L_k) = 1$. Similarly, U_k is upper-triangular and $det(U_k) = u_{11}u_{22}\cdots u_{kk}$. Therefore $det(A_k) = det(L_k)det(U_k) = u_{11}u_{22}\cdots u_{kk} > 0$ for $1 \le\le m$. So $u_{11} > 0$, $u_{11}u_{22} > 0$, $\cdots$, $u_{11}u_{22}\cdots u_{mm} > 0$. This implies that $u_{11} > 0$, $u_{22} > 0$, $\cdots$, $u_{mm} > 0$.

2 Cholesky Factorization

Definition 2.2. *A complex matrix $A \in C^{m\times m}$ is has a Cholesky factorization if $A = R^*R$ where R is a upper-triangular matrix*

Theorem 2.3. *Every hermitian positive definite matrix A has a unique Cholesky factorization.*

Proof: From the remark of previous section, we know that $A = LU$ where L is unit lower-triangular and U is upper-triangular with $u_{11} > 0$, $u_{22} > 0$, $\cdots$, $u_{mm} > 0$. First, we factor U as

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m-1} & u_{1m} \\ 0 & u_{22} & \cdots & u_{2m-1} & u_{2m} \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u_{m-1m-1} & u_{m-1m} \\ 0 & 0 & \cdots & 0 & u_{mm} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & 0 & \cdots & 0 & 0 \\ 0 & u_{22} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u_{m-1m-1} & 0 \\ 0 & 0 & \cdots & 0 & u_{mm} \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{12}}{u_{11}} & \cdots & \frac{u_{1m-1}}{u_{11}} & \frac{u_{1m}}{u_{11}} \\ 0 & 1 & \cdots & \frac{u_{2m-1}}{u_{22}} & \frac{u_{2m}}{u_{22}} \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \frac{u_{m-1m}}{u_{m-1m-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \Lambda W.$$

Since $u_{11} > 0, \cdots, u_{mm} > 0$, we can write $\Lambda = D^2$

where $D = \begin{bmatrix} \sqrt{u_{11}} & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{u_{22}} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sqrt{u_{m-1m-1}} & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{u_{mm}} \end{bmatrix}$.

So we have $A = LU = L\Lambda W = LD^2W$ where L is unit-triangular and W is unit upper-triangular. Since $A^* = A$, we have $LD^2W = (LD^2W)^* = W^*(D^2)^*L^* = W^*(D^2)L^*$. Note that W^* is unit lower-triangular. By the uniqueness of LU factorization, we have $L = W^*$. So $A = LDDL^* = (LD)(LD)^*$. Let $R = DL^*$. Then R is upper-triangular and $A = R^*R$.

Lemma 2.4. *Suppose A^*A is invertible. Then $A^*A = R^*R$ where R is uupper-triangular.*

Proof: One can check easily that A^*A is hermitian(b/c $(A^*A)^* = A^*(A^*)^* = A^*A$). Since $x^*(A^*A)x = (Ax)^*(Ax) = ||Ax||^2 \geq 0$ and $x^*(A^*A)x = 0$ if $Ax = 0$. Note that A^*A is invertible. So $Ax = 0$ implies $A^*Ax = 0$ and $x = 0$.

Algorithm for Cholesky Factorization for a Hermitian positive definite matrix

Step1. Find a LU decomposition of $A = LU$.

Step2. Factor $U = D^2W$ where W is a unit upper-triangular matrix and D is a diagonal matrix.

Step3. $A = R^*R$ where $R = DW$.

Example 2.5. *Determine if the following matrix is hermitian positive definite. Also find its Cholessky factorization if possible.*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 8 & 0 \\ 2 & 0 & 24 \end{bmatrix}.$$

Solution:

(1) From the row reduction, we have the following.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 2 \end{bmatrix} -2r_1 + \widetilde{r_2, -}r_1 + r_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} 1 \cdot \widetilde{r_2} + r_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Now $u_{22} = -1 < 0$. So A is not positive definite.

(2) From the row reduction, we have the following.

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 8 & 0 \\ 2 & 0 & 24 \end{bmatrix} \overset{-2r_1 + r_2,\, -2r_1 + r_3}{\sim} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & -4 \\ 0 & -4 & 20 \end{bmatrix} \overset{1 \cdot r_2 + r_3}{\sim} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & -4 \\ 0 & 0 & 16 \end{bmatrix}.$$

So $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & -4 \\ 0 & 0 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}^2 \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. *Since* $u_{11} = 1 > 0$, $u_{22} = 4 > 0$ *and* $u_{33} = 16 > 0$, *we know that* B *is positive definite.*

So $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ *and* $B = R^*R$

3 Least square via Cholesky factorization

Recall that the solution of the least square problem $Ax = b$ is the solution to $A^*Ax = A^*b$. Assume that A has full rank. Then A^*A is hermitian and positive definite. Then $A^*A = R^*R$ (the Cholesky factorization of A^*A) where R is upper-triangular. Then $A^*Ax = A^*b \iff R^* \underbrace{Rx}_{y} = A^*b$

$\iff R^*y = A^*b$ and $Rx = y$.

Algorithm: Least Squares via Cholesky factorization

1. Compute A^*A
2. Find the Cholesky factorization of $A^*A = R^*R$.
2. Solve the lower-triangular system $R^*y = A^*b$
3. Solve the upper-triangular system $Rx = y$ for x.

Example 3.6. *Use Cholesky factorization to find the solution to the least square problem* $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix}$

Solution:

1. Compute $A^*A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}$.

2. From the row reduction, we have the following.

$$A^*A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix} \widetilde{r_1 + r_2} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix} \widetilde{1 \cdot r_2 + r_3} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$\text{So } A^*A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}^2 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{So } R = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} \text{ and}$$

$A^*A = R^*R$.

$$\text{Compute } A^* \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}.$$

$$\text{Now solve } R^*y = A^* \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\iff \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{2} & \sqrt{2} & 0 \\ 0 & -\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$\iff y_1 = \frac{1}{\sqrt{2}}$, $y_2 = \frac{-2+\sqrt{2}y_1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$ *and* $y_3 = \frac{4+\sqrt{2}y_2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

Last, we solve $Rx = y$

$$\iff \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix}$$

$\iff x_3 = \frac{3}{2}$, $x_2 = \frac{-\frac{1}{\sqrt{2}}+\sqrt{2}x_3}{\sqrt{2}} = 1$ *and* $x_1 = \frac{\frac{1}{\sqrt{2}}+\sqrt{2}x_2}{\sqrt{2}} = \frac{3}{2}$.

$$\text{Hence the solution is } x = \begin{bmatrix} \frac{3}{2} \\ 1 \\ \frac{3}{2} \end{bmatrix}.$$

Homework 11: Due April 4

1. Determine if the following matrix is hermitian positive definite. Also find its Cholessky factorization if possible.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 8 & -14 \\ 1 & -14 & 28 \end{bmatrix}, \ B = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 8 & -14 \\ 1 & -14 & 46 \end{bmatrix}.$$

b.Use Cholesky factorization to find the solution to the least square problem

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$