Lecture 20:Gaussian Elimination

March 21, 2008

1 LU Factorization

First, let us review some results about lower-triangular matrices.
• If L is lower-triangular then L^{-1} is lower-triangular.
• If L_1 and L_2 are lower-triangular then $L_1 \cdot L_2$ is also lower-triangular.

Let A be an $m \times m$ matrix. We can find a sequence of lower-triangular matrices L_k on the left such that

$$L_{m-1} \cdots L_2 L_1 A = U$$

where U is an upper-triangular matrix. Thus we have $A = L_1^{-1} \cdots L_{m-1}^{-1} U$. Let $L = L_1^{-1} \cdots L_{m-1}^{-1}$. Then $L = L_1^{-1} \cdots L_{m-1}^{-1}$ is lower-triangular from the properties of lower-triangular matrices and $A = LU$ where L is lower-triangular and U is upper-triangular.

Let us recall some results about the row operation of a matrix. In LU decomposition, we only consider the following row operation.
• A multiple of one row is added to another row.
Let us represent it as a multiplication of matrix. Consider the case where we multiply c to $i-th$ row and add it to $j-th$ row.

Let's look at the case of a 3 matrix.
We consider the following two cases.
First Case:
Let $A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$ and B be the matrix obtained by the following two operations:
(1) Multiplying c_1 to the first row and add it to the second row
(2) Multiplying c_2 to the first row and add it to the third row.
So we have $B = \begin{bmatrix} r_1 \\ c_1 r_1 + r_2 \\ c_2 r_1 + r_3 \end{bmatrix}$. We can factor B into two matrix to get

$$\begin{bmatrix} 1 & 0 & 0 \\ c_1 & 1 & 0 \\ c_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ c_1 r_1 + r_2 \\ c_2 r_1 + r_3 \end{bmatrix} = B$$

Let $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ c_1 & 1 & 0 \\ c_2 & 0 & 1 \end{bmatrix}$. Then one can verify easily that $L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -c_1 & 1 & 0 \\ -c_2 & 0 & 1 \end{bmatrix}$.

(We have $\begin{bmatrix} 1 & 0 & 0 \\ -c_1 & 1 & 0 \\ -c_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ c_1 & 1 & 0 \\ c_2 & 0 & 1 \end{bmatrix} = I$.)

Second Case:

Let $A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$ and C be the matrix obtained by the following operation:
Multiplying c_3 to the second row and add it to the third row.

So we have $C = \begin{bmatrix} r_1 \\ r_2 \\ c_3 r_2 + r_3 \end{bmatrix}$. We can factor C into two matrix to get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_3 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ c_3 r_2 + r_3 \end{bmatrix} = C.$$

Let $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_3 & 1 \end{bmatrix}$. Then one can verify easily that $L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c_3 & 1 \end{bmatrix}$.

(We have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c_3 & 1 \end{bmatrix} = I$.)

Note that we also have $L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -c_1 & 1 & 0 \\ -c_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c_3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -c_1 & 1 & 0 \\ -c_2 & -c_3 & 1 \end{bmatrix}$.

Example 1.1. *Let* $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix}$.

We can perform row reduction to A to get a LU decomposition of A. We only consider the following row operation.

• *A multiple of one row is added to another row.*

We will provide two methods to find the LU decomposition.

• **First Method**

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix} \xrightarrow{(-1)r_1 + r_2,\, -2r_1 + r_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow{3r_2 + r_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix}.$$

We can express this process in terms of matrix factorization.

The first step we have $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -3 & 4 \end{bmatrix}$.

The second step we have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix}$.

Combining these two steps, we get

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix}}_{L_1}\underbrace{\begin{bmatrix}1&0&0\\-1&1&0\\-2&0&1\end{bmatrix}}_{L_2}\underbrace{\begin{bmatrix}2&1&1\\2&2&-1\\4&-1&6\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}2&1&1\\0&1&-2\\0&0&-2\end{bmatrix}}_{U}.$$

So $L_1L_2A=U$ *and*

$$A=L_2^{-1}L_1^{-1}U=\underbrace{\begin{bmatrix}1&0&0\\1&1&0\\2&0&1\end{bmatrix}}_{L_2^{-1}}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}}_{L_1^{-1}}\underbrace{\begin{bmatrix}2&1&1\\0&1&-2\\0&0&-2\end{bmatrix}}_{U}=\underbrace{\begin{bmatrix}1&0&0\\1&1&0\\2&-3&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}2&1&1\\0&1&-2\\0&0&-2\end{bmatrix}}_{U}.$$

Thus we get the LU decomposition of A with $A=LU$ *where* $L=\begin{bmatrix}1&0&0\\1&1&0\\2&-3&1\end{bmatrix}$ *is lower-triangular and* $U=\begin{bmatrix}2&1&1\\0&1&-2\\0&0&-2\end{bmatrix}$ *is upper-triangular.*

- **Second Method**

From the row reduction, we have the following.

$$\begin{bmatrix}2&1&1\\2&2&-1\\4&-1&6\end{bmatrix}\ \widetilde{(-1)r_1+r_2,-2r_1+r_3}\ \begin{bmatrix}2&1&1\\0&1&-2\\0&-3&4\end{bmatrix}\ \widetilde{3r_2+r_3}\ \begin{bmatrix}2&1&1\\0&1&-2\\0&0&-2\end{bmatrix}.$$

Now, we collect the first row $\begin{bmatrix}2\\2\\4\end{bmatrix}$ *from the first matrix, the second row* $\begin{bmatrix}1\\1\\-3\end{bmatrix}$ *from the second matrix and the third row* $\begin{bmatrix}1\\-2\\-2\end{bmatrix}$ *from the third matrix.*

First put these three column vectors in a matrix

$$\begin{bmatrix}2&1&1\\2&1&-2\\4&-3&-2\end{bmatrix}.$$

Now take the lower-triangular part of the matrix to get

$$\begin{bmatrix}2&0&0\\2&1&0\\\underbrace{4}_{divideby2}&\underbrace{-3}_{divideby2}&\underbrace{-2}_{divideby-2}\end{bmatrix}.$$

Next divide each column vector by its diagonal entry. In this case, divide the first column by 2, divide the second column by 1 and divide the third column by -2. We get

$$L=\begin{bmatrix}1&0&0\\1&1&0\\2&-3&1\end{bmatrix}.$$

Example 1.2. *Find the LU decomposition of* $A = \begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix}$.

From the row reduction, we have the following.

$$\begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix} (\tfrac{1}{2})r_1 + r_2, \widetilde{-r_1 + r_3}, -(\tfrac{1}{2})r_1 + r_4 \begin{bmatrix} 4 & -2 & 4 & 2 \\ 0 & 9 & 0 & -6 \\ 0 & 0 & 4 & 2 \\ 0 & -6 & 2 & 6 \end{bmatrix}$$

$$\widetilde{(\tfrac{6}{9})r_2 + r4} \begin{bmatrix} 4 & -2 & 4 & 2 \\ 0 & 9 & 0 & -6 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \widetilde{(-\tfrac{2}{4})r_3 + r_4} \begin{bmatrix} 4 & -2 & 4 & 2 \\ 0 & 9 & 0 & -6 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now collect the $i-th$ *vector form* $i-th$ *process.*

We get $\begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 9 & 0 & -6 \\ 4 & 0 & 4 & 2 \\ 2 & -6 & 2 & 1 \end{bmatrix}$. *Now take the lower-triangular part.*

We get $\begin{bmatrix} 4 & 0 & 0 & 0 \\ -2 & 9 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ 2 & -6 & 2 & 1 \end{bmatrix}$. *Now divide each column vector by the diagonal element. We get* $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & -\frac{2}{3} & \frac{1}{2} & 1 \end{bmatrix}$.

Thus $A = LU$ *where* $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & -\frac{2}{3} & \frac{1}{2} & 1 \end{bmatrix}$ *and* $U = \begin{bmatrix} 4 & -2 & 4 & 2 \\ 0 & 9 & 0 & -6 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

2 Uniqueness of LU decomposition

We will show that the LU decomposition a matrix is unique. First, let us review the following important properties of triangular matrices.

• Suppose U_1 and U_2 are upper(lower)-triangular. Then U_1U_2 is also upper(lower)-triangular.

• Suppose U is upper(lower)-triangular. Then U^{-1} is also upper(lower)-triangular.

Definition 2.3. *A lower-triangular matrix* L *is called unit lower-triangular if* $diag(L) = I$, *i.e. its diagonal elements are all* 1.

Lemma 2.4. *Suppose* A *is both upper-triangular and lower-triangular with* $diag(A) = I$. *Then* $A = I$

Theorem 2.5. Uniqueness of LU decomposition *Let A be an invertible matrix. Suppose $A = L_1U_1 = L_2U_2$ where L_1, L_2 are lower-triangular and U_1, U_2 are upper-triangular.*
Proof. *Since $L_1U_1 = L_2U_2$, we have $L_2^{-1}L_1U_1 = U_2$ and $L_2^{-1}L_1 = U_2U_1^{-1}$. Recall that $diag(L_1) = diag(L_2^{-1}) = I$. So $diag(L_2^{-1}L_1) = I$. Moreover, $L_2^{-1}L_1$ is both upper-triangular and lower-triangular. This implies that $L_2^{-1}L_1 = I$. So $L_1 = L_2$. Similarly, we have $U_2U_1^{-1} = I$. So $U_1 = U_2$*

3 Use LU decomposition to solve $Ax = b$

Suppose $A = LU$ is the LU decomposition of a $m \times m$ matrix A with $rank(A) = m$. Then $Ax = b$ can be solved in the following way.
$Ax = b \iff L\underbrace{Ux}_{y} = b \iff Ly = b$ and $Ux = y$

Algorithm:
Solving $Ax = b$ via LU Factorization where A is nonsingular
1. Find the LU Factorization $A = LU$.
2. Solve the triangular system $Ly = b$
3. Solve the triangular system $Ux = y$ to find x.

Example 3.6. *Use LU decomposition to solve $Ax = \begin{bmatrix} 1 \\ 6 \\ -9 \end{bmatrix}$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix}$.*

Solution:
Step1. LU decomposition.
From example 1.1, we have the LU decomposition of $A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix}$.

Step 2. Solve $Ly = b \iff \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -9 \end{bmatrix}$

$$\begin{cases} y_1 = 1 \\ y_1 + y_2 = 6 \\ 2y_1 - 3y_2 + y_3 = -9 \end{cases} \iff \begin{cases} y_1 = 1 \\ y_2 = 6 - y_1 = 6 - 1 = 5 \\ y_3 = -9 - 2y_1 + 3y_2 = -9 - 2 + 3 \cdot 5 = 4 \end{cases} \quad \textit{So } y = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}.$$

Step 3. Solve $Ux = y \iff \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$

$$\begin{cases} -2x_3 = 4 \\ x_2 - 2x_3 = 5 \\ 2x_1 + x_2 + x_3 = 1 \end{cases} \iff \begin{cases} x_3 = \frac{4}{-2} = -2 \\ x_2 = 5 + 2x_3 = 5 - 4 = 1 \\ x_1 = \frac{1 - x_2 - x_3}{2} = \frac{1-(1)-(-2)}{2} = 1 \end{cases}$$

So $x = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Homework 10: Due March 28

1. a. Find the LU decomposition of $A = \begin{bmatrix} 2 & 4 & 2 & 3 \\ -2 & -5 & -3 & -2 \\ 4 & 7 & 6 & 8 \\ 6 & 10 & 1 & 12 \end{bmatrix}$

b. Use LU decomposition to solve $Ax = \begin{bmatrix} -3 \\ 3 \\ -1 \\ -16 \end{bmatrix}$