5.1,5.2,5.3 Eigenvalues, Eigenvectors and Diagonalization

Definition 0.1 *Let A be a $n \times n$ matrix. A scalar λ such that $Ax = \lambda x$ for some $x \neq 0$ is called an eigenvalue and the corresponding vector is called an eigenvector.*

From the definition, we know that λ is an eigenvalue if and only if there is $x \neq 0$ such that $(A - \lambda I)x = 0$. The set of solutions is called the eigenspace corresponding to eigenvalue λ. We know that eigenspace corresponding to eigenvalue $\lambda = Nul(A - \lambda I) = \{x|(A - \lambda I)x = 0\}$.

Since $(A - \lambda I)x = 0$ has nonzero solution, we know that $A - \lambda I$ is not invertible. Therefore $det(A - \lambda I) = 0$. So we have the following.

Theorem 0.1 *λ is an eigenvalue of A iff $det(A - \lambda I) = 0$ (this is called the characteristic polynomial).*

In the following, we will discuss the diagonalization of a matrix.

Definition 0.2 *A $n \times n$ matrix A is diagonizable if $A = PDP^{-1}$ where P is invertible and D is diagonal.*

If a matrix is diagonizable then we can find the power of A easily.

First, we can show that if D is diagonal, i.e. $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

then $D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$.

Example 1 *Let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $D^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}$*

Let $E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}$. Then $E^k = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & (-5)^k \end{bmatrix}$

If A is diagonizable then we have $A = PDP^{-1}$.
So $A^2 = AA = PDP^{-1}PDP^{-1} = PDIDP^{-1} = PD^2P^{-1}$ and $A^k = PD^kP^{-1}$.
Then we have the following result.

Theorem 0.2 *Suppose* $A = PDP^{-1}$. *Then* $A^k = PD^kP^{-1}$.

Next we will discuss the relation between eigenvalues, eigenvectors and diagonalization of a matrix.

Suppose we have n independent eigenvectors $v_1, v_2, \cdots, v_n$ corresponding to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. This implies that $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$, $\cdots$, $Av_n = \lambda_n v_n$. Let P be a $n \times n$ matrix with columns $v_1, v_2, \cdots, v_n$, i.e. $P = [v_1 \; v_2 \; \cdots \; v_n]$ and D be the diagonal matrix with diagonal entries λ_1, $\lambda_2, \cdots, \lambda_n$, i.e. $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$.

Then $AP = A[v_1 \; v_2 \cdots \; v_n] = [Av_1 \; Av_2 \cdots \; Av_n] = [\lambda_1 v_1 \; \lambda_2 v_2 \; \cdots \; \lambda_n v_n]$ and $PD = [v_1 \; v_2 \; \cdots \; v_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1 v_1 \; \lambda_2 v_2 \; \cdots \; \lambda_n v_n]$.

This implies that $AP = PD$. Since P is invertible (because we assume v_1, $v_2, \cdots, v_n$ are independent), we have $APP^{-1} = PDP^{-1}$, $AI = PDP^{-1}$ and $A = PDP^{-1}$. Thus we have proved the following theorem.

Theorem 0.3 *A* $n \times n$ *matrix is diagonizable if it has n independent eigenvectors. More precisely, Suppose* $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$, $\cdots$, $Av_n = \lambda_n v_n$. *Let* P *be a* $n \times n$ *matrix with columns* $v_1, v_2, \cdots, v_n$, *i.e.* $P = [v_1 \; v_2 \; \cdots \; v_n]$ *and* D *be the diagonal matrix with diagonal entries* $\lambda_1, \lambda_2, \cdots, \lambda_n$, *i.e.* $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$. *Then we have* $A = PDP^{-1}$

To find eigenvalue and eigenvector:

1. Compute $A - \lambda I$ and $det(A - \lambda I)$.
2. Solve the characteristic polynomial $det(A - \lambda I) = 0$.
3. For each eigenvalue, use row reduction to find a basis for $Null(A - \lambda I) = \{x | (A - \lambda I)x = 0\}$. These vectors are the eigenvectors corresponding to eigenvalue λ.

To diagonalize a $n \times n$ **matrix** A**.**

1. Find eigenvalues and eigenvectors.

2. If there are n independent eigenvectors $v_1, \cdots, v_n$ with eigenvalues $\lambda_1, \cdots, \lambda_n$, then $A = PDP^{-1}$ where $P = [v_1\ v_2\ \cdots\ v_n]$ and $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$.

3. If we don't have n independent eigenvectors then A is not diagonizable.

Example 2 *a. Find the characteristic polynomial, eigenvalues and eigenvectors of* $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.
b. Diagonalize the matrix A if possible.
c. Find a formula for A^k.

Solution: 1^0 Compute $A - \lambda I = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}$.

2^0 Compute $det(A - \lambda I) = det(\begin{bmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}) = (1-\lambda)^2 - 4 = (1-\lambda)^2 - (-2)^2 = \lambda^2 - 2\lambda + 1 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$.

3^0 Solve $det(A - \lambda I) = 0$,i.e. $(\lambda - 3)(\lambda + 1) = 0$. So the eigenvalues are 3 and -1.

4^0When $\lambda = 3$, $A - \lambda I = A - 3I = \begin{bmatrix} 1-3 & -2 \\ -2 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ So the solution of $(A - 3I)x = 0$ is $x_1 + x_2 = 0$ and x_2 is free. Hence $x_1 = -x_2$. So $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 3$.

5^0When $\lambda = -1$, $A - \lambda I = A - (-1)I = A + I = \begin{bmatrix} 1+1 & -2 \\ -2 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ So the solution of $(A - (-1)I)x = 0$ is $x_1 - x_2 = 0$ and x_2 is free. Hence $x_1 = x_2$. So $x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -1$.

6^0 So we have found that $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 3$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -1$. Let

$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$. Then we have $A = PDP^{-1}$.

7^0 $A^k = PD^kP^{-1} = P(\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix})^k P^{-1} = P \begin{bmatrix} 3^k & 0 \\ 0 & (-1)^k \end{bmatrix} P^{-1}$. Since $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, we have $det(P) = -2$ and $P^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{-2} \\ \frac{1}{-2} & \frac{1}{-2} \end{bmatrix}$.

Hence $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{-2} \\ \frac{1}{-2} & \frac{1}{-2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3^k}{2} & -\frac{3^k}{2} \\ -\frac{(-1)^k}{2} & -\frac{(-1)^k}{2} \end{bmatrix} = \begin{bmatrix} -\frac{3^k}{2} - \frac{(-1)^k}{2} & \frac{3^k}{2} - \frac{(-1)^k}{2} \\ \frac{3^k}{2} - \frac{(-1)^k}{2} & -\frac{3^k}{2} - \frac{(-1)^k}{2} \end{bmatrix}$.

Example 3 $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$.

a. Show that $det(A - \lambda I) = (1 - \lambda)(2 - \lambda)(3 - \lambda)$.

b. Find the eigenvalues and eigenvectors of $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$.

c. Diagonalize the matrix A if possible.

d. Find a formula for A^k.

Solution: 1^0 $A - \lambda I = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}$.

2^0 Compute $det(A - \lambda I) = det(\begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}) = (-1-\lambda)(4-\lambda)(3-\lambda) + 0 + (-2)(-3) \cdot 1 - (-2)(4-\lambda)(-3) - 4(-3)(3-\lambda) - 0 = -12 - 5\lambda + 6\lambda^2 - \lambda^3 + 6 - 24 + 6\lambda + 36 - 12\lambda = 6 - 11\lambda + 6\lambda^2 - \lambda^3$. Expanding $(1-\lambda)(2-\lambda)(3-\lambda)$, we get $(1-\lambda)(2-\lambda)(3-\lambda) = 6 - 11\lambda + 6\lambda^2 - \lambda^3$. So $det(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda)$.

3^0 Solve $det(A - \lambda I) = 0$,i.e. $(1-\lambda)(2-\lambda)(3-\lambda) = 0$. So the eigenvalues are 1 , 2 and 3.

4^0When $\lambda = 1$, $A - \lambda I = A - I = \begin{bmatrix} -1-1 & 4 & -2 \\ -3 & 4-1 & 0 \\ -3 & 1 & 3-1 \end{bmatrix} = \begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix} \sim$

$(r_1 := r_1/(-2), r_2 := r_2/(-3)) \begin{bmatrix} 1 & -2 & 1 \\ 1 & -1 & 0 \\ -3 & 1 & 2 \end{bmatrix} \sim (r_2 := r_2 - r_1, r_3 := r_3 + 3r_1) \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -5 & 5 \end{bmatrix} \sim (r_3 := r_3 + 5r_2, r_1 := r_1 + 2r_2) \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

So the solution of $(A-I)x = 0$ is $x_1 - x_3 = 0$,$x_2 - x_3 = 0$ and x_3 are free. Hence $x_1 = x_3$, $x_2 = x_3$, $x_3 = x_3$. So $x = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 1$.

5^0 When $\lambda = 2$, $A - \lambda I = A - 2I = \begin{bmatrix} -1-2 & 4 & -2 \\ -3 & 4-2 & 0 \\ -3 & 1 & 3-2 \end{bmatrix} = \begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix} \sim (r_2 := r_2 - r_1, r_3 := r_3 - r_1) \begin{bmatrix} -3 & 4 & -2 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{bmatrix} \sim (r_2 := r_2/(-2), r_3 := r_3 + 3r_2) \begin{bmatrix} -3 & 4 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim (r_1 := r_1 - 4r_2, r_1 := r_1 + 2r_2) \begin{bmatrix} -3 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim (r_1 := r_1/(-3)) \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ So the solution of $(A-2I)x = 0$ is $x_1 = \frac{2}{3}x_3$, $x_2 = x_3$ and x_3 is free. So $x = \begin{bmatrix} \frac{2}{3}x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$. We can choose $x_3 = 3$ So $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 2$.

6^0 When $\lambda = 3$, $A - \lambda I = A - 3I = \begin{bmatrix} -1-3 & 4 & -2 \\ -3 & 4-3 & 0 \\ -3 & 1 & 3-3 \end{bmatrix} = \begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \sim (r_1 := r_1/(-4), r_3 := r_2/(-3), r_3 := r_3 - r_2) \begin{bmatrix} 1 & -1 & 0.5 \\ 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim (r_2 := r_2 -$

$r_1)\begin{bmatrix}1 & -1 & 0.5\\ 0 & \frac{2}{3} & -0.5\\ 0 & 0 & 0\end{bmatrix} \sim (r_1 := r_1 - \frac{3}{2}r_2)\begin{bmatrix}1 & -1 & 0.5\\ 0 & 1 & -0.75\\ 0 & 0 & 0\end{bmatrix} \sim (r_1 := r_1 +$ $r_2 \begin{bmatrix}1 & 0 & -0.25\\ 0 & 1 & -0.75\\ 0 & 0 & 0\end{bmatrix}$ So the solution of $(A-3I)x = 0$ is $x_1 = 0.25x_3$, $x_2 = 0.75x_3$ and x_3 is free. So $x = \begin{bmatrix}0.25x_3\\ 0.75x_3\\ x_3\end{bmatrix} = x_3\begin{bmatrix}0.25\\ 0.75\\ 1\end{bmatrix}$. We can choose $x_3 = 4$

So $4\cdot\begin{bmatrix}0.25\\ 0.75\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 3\\ 4\end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 3$.

7^0 So we have found that $\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$, $\begin{bmatrix}2\\ 3\\ 3\end{bmatrix}$ and $\begin{bmatrix}1\\ 3\\ 4\end{bmatrix}$ are eigenvectors corresponding to eigenvalue $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$

Let $P = \begin{bmatrix}1 & 2 & 1\\ 1 & 3 & 3\\ 1 & 3 & 4\end{bmatrix}$ and $D = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$. Then we have $A = PDP^{-1}$.

7^0 $A^k = PD^kP^{-1} = P(\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix})^kP^{-1} = P\begin{bmatrix}1 & 0 & 0\\ 0 & 2^k & 0\\ 0 & 0 & 3^k\end{bmatrix}P^{-1}$. We can find the formula for P^{-1} to simplify this expression. But let us just stop here.

Example 4 *a. Find the characteristic polynomial, eigenvalues and eigenvectors of* $\begin{bmatrix}2 & 0 & 0\\ -1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$.
b. Diagonalize the matrix A if possible.
c. Find a formula for A^k.

Solution: 1^0 Compute $A-\lambda I = \begin{bmatrix}2 & 0 & 0\\ -1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}2-\lambda & 0 & 0\\ -1 & 3-\lambda & 1\\ -1 & 1 & 3-\lambda\end{bmatrix}$.

2^0 Compute $det(A-\lambda I) = det(\begin{bmatrix}2-\lambda & 0 & 0\\ -1 & 3-\lambda & 1\\ -1 & 1 & 3-\lambda\end{bmatrix}) = (2-\lambda)(3-\lambda)^2 - (2-\lambda) = (2-\lambda)((3-\lambda)^2-1) = (2-\lambda)((3-\lambda)-1)((3-\lambda)+1) =$

$(2-\lambda)(2-\lambda)(4-\lambda) = (2-\lambda)^2(4-\lambda)$.
3^0 Solve $det(A-\lambda I) = 0$,i.e. $(2-\lambda)^2(4-\lambda) = 0$. So the eigenvalues are 2 and 4.

4^0When $\lambda = 2$, $A - \lambda I = A - 2I = \begin{bmatrix} 2-2 & 0 & 0 \\ -1 & 3-2 & 1 \\ -1 & 1 & 3-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ So the solution of $(A-2I)x = 0$ is $x_1 - x_2 - x_3 = 0$,x_2 and x_3 are free. Hence $x_1 = x_2 + x_3$. So $x = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors corresponding to eigenvalue $\lambda = 2$.

5^0 When $\lambda = 4$, $A - \lambda I = A - 4I = \begin{bmatrix} 2-4 & 0 & 0 \\ -1 & 3-4 & 1 \\ -1 & 1 & 3-4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \sim$ $(r_1 := r_1/2) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \sim (r_2 := r_2 + r_1, r_3 := r_3 + r_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim$ $(r_2 := -r_2, r_3 := r_3 + r_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ So the solution of $(A-4I)x = 0$ is $x_1 = 0$,$x_2 - x_3 = 0$ and x_3 is free. Hence $x_1 = 0$, $x_2 = x_3$. So $x = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 4$.

6^0 So we have found that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors corresponding to eigenvalue $\lambda = 2$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue

$\lambda = 4$ Let $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Then we have $A = PDP^{-1}$.

7^0 $A^k = PD^kP^{-1} = P(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix})^k P^{-1} = P \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} P^{-1}$. We can find the formula for P^{-1} to simplify this expression. But let us just stop here.

Not every matrix is diagonizable. The following is an example.

Example 5 *a. Find the characteristic polynomial, eigenvalues and eigenvectors of* $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$.

b. Diagonalize the matrix A if possible.

Solution: 1^0 Compute $A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix}$.

2^0 Compute $det(A - \lambda I) = det(\begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix}) = (2-\lambda)^2(4-\lambda)$.

3^0 Solve $det(A - \lambda I) = 0$,i.e. $(2-\lambda)^2(4-\lambda) = 0$. So the eigenvalues are 2 and 4.

4^0When $\lambda = 2$, $A - \lambda I = A - 2I = \begin{bmatrix} 2-2 & 1 & 1 \\ 0 & 2-2 & 1 \\ 0 & 0 & 4-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ So the solution of $(A - 2I)x = 0$ is $x_2 = 0$,$x_3 = 0$ and x_1 is free. So $x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 2$.

5^0 When $\lambda = 4$, $A - \lambda I = A - 4I = \begin{bmatrix} 2-4 & 1 & 1 \\ 0 & 2-4 & 1 \\ 0 & 0 & 4-4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim$

$(r_2 := -r_2/2) = \begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \sim (r_1 := r_2 - r_1) = \begin{bmatrix} -2 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \sim (r_1 := -r_1/2) = \begin{bmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ So the solution of $(A - 4I)x = 0$ is $x_1 - \frac{3}{4}x_3 = 0$,$x_2 - \frac{1}{2}x_3 = 0$ and x_3 is free. Hence $x_1 = \frac{3}{4}x_3$, $x_2 = \frac{1}{2}x_3$. So $x = \begin{bmatrix} \frac{3}{4}x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$. So $\begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 4$.

6^0 So we have found that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 2$ and $\begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 4$. Therefore we have only two eigenvectors for A. Thus A is not diagonizable. (We need 3 eigenvectors to diagonalize a 3×3 matrix.

1 Exponential of a matrix and characteristic polynomial

Recall that $e^x = 1 + x + \frac{x^2}{2!} + \cdots$. We can define the exponential f a matrix by the following.

Definition 1.1 *The exponential of a $n \times n$ matrix A is denoted by e^A which is defined by $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$. We use the convention that $A^0 = I$ and $0! = 1$.*

If D is an diagonal matrix with $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ then

$$
\begin{aligned}
& e^D \\
=& I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots \\
=& \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^2}{2!} & 0 & \cdots & 0 \\ 0 & \frac{\lambda_2^2}{2!} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{\lambda_n^2}{2!} \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^3}{3!} & 0 & \cdots & 0 \\ 0 & \frac{\lambda_2^3}{3!} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{\lambda_n^3}{3!} \end{bmatrix} + \cdots \\
=& \begin{bmatrix} 1 + \lambda_1 + \frac{\lambda_1^2}{2!} + \frac{\lambda_1^3}{3!} + \cdots & 0 & \cdots & 0 \\ 0 & 1 + \lambda_2 + \frac{\lambda_2^2}{2!} + \frac{\lambda_2^3}{3!} + \cdots & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 + \lambda_n + \frac{\lambda_n^2}{2!} + \frac{\lambda_n^3}{3!} + \cdots \end{bmatrix} \\
=& \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix}
\end{aligned}
\tag{1.1}
$$

Thus we have the following theorem.

Theorem 1.1 *Suppose* $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$. *Then* $e^D = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix}$

If A is diagonizable with $A = PDP^{-1}$ then $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = I + PDP^{-1} + \frac{PD^2P^{-1}}{2!} + \frac{PD^3P^{-1}}{3!} + \cdots = P(I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots)P^{-1} = Pe^DP^{-1}$. Thus we have the following theorem.

Theorem 1.2 *Suppose* $A = PDP^{-1}$ *where* $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$. *Then*

$$e^A = P \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix} P^{-1}.$$

Example 6 *Use the result in example 3 to find* e^A *where* $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

Solution: From example 3, we have $A = PDP^{-1}$ where $P = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. So $e^A = Pe^D P^{-1} = P \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^4 \end{bmatrix} P^{-1}$.

The last thing that we want to discuss is the characteristic polynomial gives us a nice equation for A. Given a polynomial $f(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$. We define $f(A) = a_nA^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I$. We have the following interesting result.

Theorem 1.3 *Let* $f(\lambda) = det(A - \lambda I)$. *Then* $f(A) = 0$.

Example 7 *Let* $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ *Recall that* $det(A - \lambda I) = (\lambda - 3)(\lambda + 1) = \lambda^2 - 2\lambda - 3$. *One can verify that* $A^2 - 2A - 3I = 0$