**Solution to Linear Algebra (Math 2890) Review Problems II**

1. (a) What is a subspace in $R^n$?

Solution: A subspace of $R^n$ is any set $H$ in $R^n$ that satisfies the following three properties. (I) The zero vector is in $H$. (II) For each $u$ and $v$ in $H$, then $u + v$ is in $H$. (III) For each $u$ in $H$ and each scalar $c$, the vector $cu$ is in $H$.

(b) Is the set $\{(x, y, z)|x + y + z = 1\}$ a subspace?
Solution: This is not a subspace since the zero vector $(0, 0, 0)$ is not in the set.

(c) Is the set $\{(x, y, z)|x - y - z = 0, x + y - z = 0\}$ a subspace?
Solution: Yes. This is a subspace. This can be regarded as the nullspace of the matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$.

Here $Nul(A) = \{(x, y, z)| \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\}$.

(d) What is a basis for a subspace?
Solution: A basis for a subspace $H$ of $R^n$ is a linearly independent set in $H$ that spans $H$.

(e) What is the dimension of a subspace?
Solution:The dimension of a nonzero subspace $H$ is the number of vectors in any basis for $H$.

(f) What is the column space of a matrix?
Solution: The column space of a matrix $A$ is the set of the span of the column vectors of $A$.

(g) What is the null space of a matrix?
Solution:The null space of a matrix $A$ is the set of all solutions to the homogeneous equation $Ax = 0$, i.e. $Nul(A) = \{x|Ax = 0\}$.

(h) What is the subspace spanned by the vectors $v_1$, $v_2$, $\cdots$, $v_p$? Solution: The subspace spanned by $v_1$, $v_2$, $\cdots$, $v_p$ is the set of all possible linear combination of $v_1$, $v_2$, $\cdots$, $v_p$, i.e. $Span\{v_1, \ \cdots, \ v_n\} = \{c_1v_1 + c_2v_2 + \cdots + c_pv_p|c_1, c_2, \cdots, c_p \ are \ real \ numbers\}$

2. Find the inverses of the following matrices if they exist.

$$A = \begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ -1 & 0 & -1 \end{bmatrix},\ C = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}.$$

Solution: (a) Since $det(A) = -1$, we have $A^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -4 & -7 \end{bmatrix}$

(b)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \overset{r_2 := r_2 - 2r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -1 & -2 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{r_3 := r_3 + r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\overset{r_2 := -r_3,\, r_3 := r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 5 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\overset{r_3 := r_3 - 5r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 1 & 5 \end{array}\right]$$

$$\overset{r_1 := r_1 + r_3,\, r_3 := -r_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 4 & 1 & 5 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -3 & -1 & -5 \end{array}\right]$$

$$\overset{r_1 := r_1 + r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -3 & -1 & -5 \end{array}\right]$$

So $B^{-1} = \begin{bmatrix} 3 & 1 & 4 \\ -1 & 0 & -1 \\ -3 & -1 & -5 \end{bmatrix}$.

(c)

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 5 & 6 & 7 & 0 & 1 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array}\right] \underset{r_2 := r_2 - 2r_1}{\widetilde{\quad}} \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 0 & -1 & -2 & 1 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{r_2 \longleftrightarrow r_1}{\widetilde{\quad}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{r_2 := r_2 - 2r_1, r_3 := r_3 - 8r_1}{\widetilde{\quad}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 3 & -1 & 0 \\ 0 & 9 & 18 & 16 & -8 & 1 \end{array}\right]$$

$$\underset{r_3 := r_3 + (-3)r_2}{\widetilde{\quad}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 3 & -1 & 0 \\ 0 & 0 & 0 & 7 & -5 & 1 \end{array}\right]$$

So $C$ only has one free variable ( or two pivot vectors) and $C$ is not invertible.

3. (a) Let $A$ be an $3 \times 3$ matrix. Suppose $A^3 + 2A^2 - 3A + 4I = 0$. Is $A$ invertible? Express $A^{-1}$ in terms of $A$ if possible.
Solution: From $A^3+2A^2-3A+4I = 0$, we have $A^3+2A^2-3A = -4I$, $A(A^2 + 2A - 3I) = -4I$ and $A \cdot (-\frac{1}{4}(A^2 + 2A - 3I)) = I$. So $A^{-1} = -\frac{1}{4}(A^2 + 2A - 3I)$.
(b) Suppose $A^3 = 0$. Is $A$ invertible?
Solution: If $A$ is invertible then $A^{-2}A^3 = A^{-2}0$ and $A = 0$ which is not invertible. So $A$ is not invertible.

4. Find all values of $a$ and $b$ so that the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} b \\ 1 \\ -a \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ is two-dimensional.

Solution: Consider the matrix $A = \begin{bmatrix} 0 & b & -2 \\ 1 & 1 & 2 \\ 0 & -a & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$\widetilde{\textit{interchange first row and second row}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & b & -2 \\ 0 & -a & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$\widetilde{r_4 := r_1 + r_4} \begin{bmatrix} 1 & 1 & 2 \\ 0 & b & -2 \\ 0 & -a & 0 \\ 0 & 2 & 2 \end{bmatrix}$

$\widetilde{\textit{interchange second row and forth row}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -a & 0 \\ 0 & b & -2 \end{bmatrix}$

$\widetilde{\textit{divide second row by } 2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -a & 0 \\ 0 & b & -2 \end{bmatrix} \widetilde{r_3 := r_3 + ar_2, r_4 := r_4 - br_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & a \\ 0 & 0 & -2-b \end{bmatrix}.$

Now the first and second vectors are pivot vectors. So $rank(A) = 2$ if $a = 0$ and $-2 - b = 0$.

So $a = 0$ and $b = -2$

5. Let $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \}$. You can assume that $\mathcal{B}$ is a basis for $R^3$

(a) Which vector $x$ has the coordinate vector $[x]_B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. So $x = A[x]_B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-3+0 \\ 0-2+0 \\ 0-1+4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$

(b) Find the $\beta$-coordinate vector of $y = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

Solution. We have to solve $Ax = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right] \underset{r_2 := \frac{1}{2}r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 3 \end{array}\right] \underset{r_2 := r_3 - r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\underset{r_3 := \frac{1}{2}r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \underset{r_1 := r_1 - 3r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

So $[y]_B = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$.

6. Let

$$M = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$$

(a) Find bases for $Col(M)$ and $Nul(M)$, and then state the dimensions of these subspaces

Solution: $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \underset{r_2 := -r_1 + r_2, r_3 := -r_2 + r_3}{\sim} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix}$

$$\underset{r_3 := -2r_2 + r_3}{\sim} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{r_1 := -2r_2 + r_3}{\sim} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the first two vectors are pivot vectors and $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \}$ is a basis for $Col(M)$ and $dim(Col(M)) = 2$.

The solution to $Mx = 0$ is $x_1 + x_3 - x_4 = 0$ and $x_2 + 2x_3 + x_4 = 0$. So $x = \begin{bmatrix} -x_3 + x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Hence the basis for $Nul(M)$

is $\{\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\}$ and $dim(Nul(M)) = 2$.

(b) Express the third column vector $M$ as a linear combination of the basis of $Col(M)$. From the row reduced echelon form, we know that column(3) $= 1 \cdot$ column(1) $+ 2 \cdot$ column(2)

So $\begin{bmatrix}3\\5\\7\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\2\\3\end{bmatrix}$.

7. Find a basis for the subspace spanned by the following vectors $\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\5\\7\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\}$.

   What is the dimension of the subspace?

   Solution: Consider the matrix $A = \begin{bmatrix}1 & 1 & 3 & 0\\1 & 2 & 5 & 1\\1 & 3 & 7 & 2\end{bmatrix}$

   From previous example, we know that the first two vectors are pivot vectors and $\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\}$ is a basis. The dimension of the subspace is 2.

8. Determine which sets in the following are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify your answer

   (a) $\begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}$. Solution: Since $\begin{bmatrix}2\\-4\end{bmatrix} = -2\begin{bmatrix}-1\\2\end{bmatrix}$, the set $\{\begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}\}$ is dependent. It is not a basis.

   (b) $\begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}$. Yes. This set forms a basis since they are independent and span $R^3$.

   (c) $\begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$.

   This is not a basis since it doesn't span $R^3$.

(d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This set forms a basis since they are independent and span $R^3$

(e) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. This is not a basis since it is dependent.

9. Find an orthogonal basis for the column space of the following matrices.

(a) $\begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & -1 \\ 1 & 2 & 4 \end{bmatrix}$. (b) $\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$

Solution: (a) We use the Gram-Schmidt process. Let $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$.

Now $v_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$. Compute $u_2 \cdot v_1 = 3$ and $v_1 \cdot v_1 = 3$. Then $v_2 = u_2 - \frac{5}{3} v_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Now $v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$. Compute $u_3 \cdot v_1 = 7$, $u_3 \cdot v_2 = 10$, $v_2 \cdot v_2 = 6$. So $v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix} - \frac{7}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{10}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This means that $u_3$ is a linear combination of $u_1$ and $u_2$ (because $u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2 = 0)$ . So an orthogonal basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(b) Let $u_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$.

The Gram-Schmidt process is $v_1 = u_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$,

$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$. Compute $u_2 \cdot v_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = -36$ and

$v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 12.$

Using the formula

$$\begin{aligned} v_2 &= u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\ &= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{(-36)}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

Now $v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$. Compute $u_3 \cdot v_1 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 6$,

$u_3 \cdot v_2 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 30,\ v_2 \cdot v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 12.$

Using the formula

$$\begin{aligned} v_3 &= u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2 \\ &= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}. \end{aligned}$$

So $\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for the column space.

10. (a) Let $W = Span\{u_1, u_2\}$ where $u_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$. Find an orthogonal basis for $W$.

Solution:(a) We use the Gram-Schmidt process to find the orthogonal basis for $W$.

$v_1 = u_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix},$

$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$. Compute $u_2 \cdot v_1 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = 9$ and $v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = 9$.

Using the formula

$$
\begin{aligned}
v_2 &= u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\
&= \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}
\end{aligned}
$$

Thus $\left\{ v_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for $W$.

(b)Find the closest point to $y = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$ in the subspace $W$.

Solution: The closest point to to $y = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$ in the subspace $W$ is $Proj_W(y) = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2$.

Compute $y \cdot v_1 = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = 1 + 10 - 2 = 9$,

$v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = 1 + 4 + 4 = 9$,

$$y \cdot v_2 = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = -2 + 10 + 1 = 9,$$

$$v_2 \cdot v_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 4 + 4 + 1 = 9.$$

So $Proj_W(y) = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{9}{9} v_1 + \frac{9}{9} v_2 = v_1 + v_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$. Hence the closest point from $y$ to $W$ is $\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

(c) Find the distance between the point $y$ and the subspace $W$. Solution: The distance between $y$ and the subspace $W$ is $||y - Proj_W(y)||$. Compute $y - Proj_W(y) = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ and $||y - Proj_W(y)|| = || \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} || = \sqrt{(-2)^2 + 1^2 + 2^2} = \sqrt{9} = 3$. Hence the distance between the point $y$ and the subspace $W$ is 3.