MATH 2890

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If the augmented matrices of two linear systems are row equivalent, then the two systems has the same solution set. (The solution set is not changed under row operation.)

Strategy for solving a linear system:

Step 1. Express the linear system in terms of the augmented matrix.

Step 2. Use row operations to reduce the augmented matrix of a linear system to echelon form.

Step 3. Use row operations to reduce the echelon form to reduced echelon form.

Step 4. Identify basic variables (corresponding to the the leading coefficient or the pivot vector) and free variables (not basic variables) from reduced echelon form. Express the solution set in terms of the free variables.

Important terms:

• **pivot position**: a position of a leading entry in an echelon form of the matrix

• **pivot**: a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in terms is used to create 0's

• **pivot column**: the column corresponding to the leading coefficient

Free variables correspond to **non pivot columns.**

Example: $\spadesuit$ is the leading coefficient

Echelon form:
$$\begin{bmatrix} \spadesuit & * & * & * & * \\ 0 & \spadesuit & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \textit{pivot column} & \textit{pivot column} & & & \end{bmatrix}$$

Reduced echelon form
$$\begin{bmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \textit{basic} & \textit{basic} & \textit{free} & \textit{free} & \end{bmatrix}$$

Echelon form:

$$\begin{bmatrix} \spadesuit & * & * & * & * & * & * \\ 0 & \spadesuit & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \spadesuit & * & * \\ 0 & 0 & 0 & 0 & 0 & \spadesuit & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ pivot & pivot & & & pivot & pivot & \end{bmatrix}.$$

Reduced Echelon form:

$$\begin{bmatrix} 1 & 0 & * & * & 0 & 0 & * \\ 0 & 1 & * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ basic & basic & free & free & basic & basic & \end{bmatrix}$$

Theorem

1. A linear system is inconsistent (has no solution) if and only if an echelon of the augmented matrix has a row of the form $[0\ 0 \cdots\ b]$ where $b \neq 0$.

2. A linear system is consistent , then the solution contains either

(1) a unique solution if there are no free variables

(2) infinitely many solutions if there are free variables.

Notations for row operations

In the following, we use the following notations:

a. r_1 denotes the first row of the matrix, r_2 denotes the second row of the matrix and r_3 denotes the third row of the matrix, $\cdots$

b. $r_i := r_i + cr_j$ This means that we replace the i-th row of the matrix by adding c times the j-th row to i-th row of the matrix.

c. $r_i := cr_i$ This means that we replace the i-th row of the matrix by multiplying c times the i-th row to the i-th row of the matrix.

d. $r_i \leftrightarrow r_j$ This means that we interchange the i-th row and the j-th row.

Example: Find the general solutions of the system whose augmented matrix are given in the following.

(a) $\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 3 & -2 & 5 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & 4 & 3 & 0 \\ -4 & -6 & -6 & 0 \\ 6 & 13 & 9 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 2 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Solution: (a) The augmented matrix is $\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 3 & -2 & 5 \end{bmatrix}$.

First, we switch the first and the second row to get

$\begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -2 & 3 \end{bmatrix}$.

Next, we multiply -3 to second row and add it to the first row

$$(r_1 := r_1 + (-3)r_2 \quad \begin{array}{rl} r_1 \mapsto & (1 \quad 3 \quad -2 \quad 5) \\ -3r_2 \mapsto & (0 \quad -3 \quad 6 \quad -9). \\ r_1 - 3r_2 \mapsto & (1 \quad 0 \quad 4 \quad -4) \end{array} \;)$$

to get $\begin{bmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & -2 & 3 \end{bmatrix}$.

From here, we know that x_3 is the free variable.

The corresponding system of equations is $\begin{cases} x_1 \quad + 4x_3 = -4 \\ \quad x_2 - 2x_3 = 3 \end{cases}$.

The gives the general solution $\begin{cases} x_1 = -4 - 4x_3 \\ x_2 = 3 + 2x_3 \\ x_3 : \;\; \mathit{free} \end{cases}$

(b) The augmented matrix is $\begin{bmatrix} 2 & 4 & 3 & 0 \\ -4 & -6 & -6 & 0 \\ 6 & 13 & 9 & 0 \end{bmatrix}$.

Next, we multiply 2 to first row and add it to the second row

$$\left(r_2 := r_2 + 2r_1 \quad \begin{array}{rl} 2r_1 \mapsto & (4 \quad 8 \quad 6 \quad 0) \\ r_2 \mapsto & (-4 \quad -6 \quad -6 \quad 0). \\ r_2 + 2r_1 \mapsto & (0 \quad 2 \quad 0 \quad 0) \end{array}\right)$$

to get $\begin{bmatrix} 2 & 4 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 13 & 9 & 0 \end{bmatrix}$.

Next, multiply -3 to first row and add it to the third row

$$\left(r_3 := r_3 - 3r_1 \quad \begin{array}{rl} -3r_1 \mapsto & (-6 \quad -12 \quad -9 \quad 0) \\ r_3 \mapsto & (6 \quad 13 \quad 9 \quad 0). \\ r_3 - 3r_1 \mapsto & (0 \quad 1 \quad 0 \quad 0) \end{array}\right)$$

to get $\begin{bmatrix} 2 & 4 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ which is row equivalent to $\begin{bmatrix} 2 & 4 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

($r_2 := r_2/2$ and $r_3 := r_3 - r_2$).

Next, multiply -4 to second row and add it to the first row

$$\begin{array}{rl} r_1 \mapsto & (2 \quad 4 \quad 3 \quad 0) \\ (r_1 := r_1 - 4r_2 \quad -4r_2 \mapsto & (0 \quad -4 \quad 0 \quad 0).\ ) \\ r_1 - 4r_2 \mapsto & (2 \quad 0 \quad 3 \quad 0) \end{array}$$

to get $\begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which is row equivalent to $\begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $(r_1 := r_1/2)$ From here, we know that x_3 is the free variable.

The corresponding system of equations is

$$\begin{cases} x_1 + \qquad \frac{3}{2}x_3 = 0 \\ \qquad x_2 \qquad = 0 \end{cases}.$$

The gives the general solution $\begin{cases} x_1 = -\dfrac{3}{2}x_3 \\ x_2 = 0 \\ x_3 : \ \textit{free} \end{cases}$

(c) The augmented matrix is $\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is in echelon form

Next, we multiply 2 to third row and add it to the second row

$$\left(r_2 := r_2 + 2r_3, \quad \begin{matrix} r_2 \mapsto & (0 & 1 & 0 & -2 & 4 & -2) \\ 2r_3 \mapsto & (0 & 0 & 0 & 2 & 4 & 2) \\ r_2 + 2r_3 \mapsto & (0 & 1 & 0 & 0 & 8 & 0) \end{matrix} . \right)$$

to get $\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Next, multiply -1 to third row and add it to the first row $(r_1 := r_1 - r_3)$

to get $\begin{bmatrix} 1 & 2 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Next, multiply -2 to second row and add it to the first row

$$\begin{array}{rrrrrrr} & r_1 \mapsto & 1 & 2 & 0 & 0 & -2 & -2 \\ (r_1 := r_1 - 2r_2 & -2r_2 \mapsto & 0 & -2 & 0 & 0 & -16 & 0\ .\ ) \\ & r_1 - 2r_2 \mapsto & 1 & 0 & 0 & 0 & -18 & -2 \end{array}$$

to get $\begin{bmatrix} 1 & 0 & 0 & 0 & -18 & -2 \\ 0 & 1 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is in reduced echelon form.

From here, we know that x_3 and x_5 are the free variables.

The corresponding system of equations is

$$\begin{cases} x_1 \qquad\qquad -18x_5 = -2 \\ \qquad x_2 \qquad\quad +8x_5 = 0 \\ \qquad\qquad x_4 \ +2x_5 = 1 \end{cases}$$

. The gives the general solution

$$\begin{cases} x_1 = -2 + 18x_5 \\ x_2 = -8x_5 \\ x_3 : \ \textit{free} \\ x_4 = 1 - 2x_5, \quad x_5 : \ \textit{free} \end{cases}$$