Solutions to Linear Algebra Practice Problems 1

1. Show that $A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Solution: $A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix} \sim (r_2 := r_2 + 2r_1) \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}$

$$\sim \quad (r_3 := r_3 + 3r_1) \quad \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 2 & -6 & -2 & 18 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix} \sim \quad (r_4 := r_4 - 3r_1)$$

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 2 & -6 & -2 & 18 \\ 0 & 0 & 0 & 5 & -20 \end{bmatrix} \sim (r_3 := r_3 - 2r_2) \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -20 \end{bmatrix}$$

$$\sim \quad (r_3 \leftrightarrow r_4, r_3 := r_3/5) \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \quad (r_2 := r_2 + r_3)$$

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim (r_1 := r_1 - 2r_2) \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Determine if the following systems are consistent and if so give all solutions in parametric vector form.

(a)

$$\begin{aligned} x_1 \quad -2x_2 &= 3 \\ 2x_1 \quad -7x_2 &= 0 \\ -5x_1 \quad +8x_2 &= 5 \end{aligned}$$

Solution: The augmented matrix is $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -7 & 0 \\ -5 & 8 & 5 \end{bmatrix} \sim (r_2 := r_2 - 2r_1)$
$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & -6 \\ -5 & 8 & 5 \end{bmatrix} \sim (r_3 := r_3 + 5r_1) \begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & -6 \\ 0 & -2 & 20 \end{bmatrix}$
$\sim (r_2 := r_2/-3, r_3 := r_3/-2) \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & -10 \end{bmatrix} \sim (r_3 := r_3 - r_2)$
$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$. The last row implies that $0 \cdot x_2 = -12$ which is impossible. So this system is inconsistent.

(b)

$$\begin{aligned} x_1 \quad +2x_2 \quad -3x_3 \quad +x_4 &= 1 \\ -x_1 \quad -2x_2 \quad +4x_3 \quad -x_4 &= 6 \\ -2x_1 \quad -4x_2 \quad +7x_3 \quad -x_4 &= 1 \end{aligned}$$

The augmented matrix is $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ -1 & -2 & 4 & -1 & 6 \\ -2 & -4 & 7 & -1 & 1 \end{bmatrix} \sim (r_2 := r_2 + r_1)$
$\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ -2 & -4 & 7 & -1 & 1 \end{bmatrix} \sim (r_3 := r_3 + 2r_1) \begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$

$$\sim (r_3 := r_3 - r_2) \begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \sim (r_1 := r_1 - r_3) \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$$

$$\sim (r_1 := r_1 - r_3) \begin{bmatrix} 1 & 2 & -3 & 0 & 5 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \sim (r_1 := r_1 + 3r_2) \begin{bmatrix} 1 & 2 & 0 & 0 & 26 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}.$$

So x_2 is free. The solution is $x_1 = 26 - 2x_2$, $x_3 = 7$, $x_4 = -47$. Its parametric vector form is $\begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 26 - 2x_2 \\ x_2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 26 \\ 0 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 1 & 3 & -4 & 7 \\ 2 & 6 & 5 & 1 \\ 3 & 9 & 4 & 5 \end{bmatrix}$.

   (a) Find all the solutions of the non-homogeneous system $Ax = b$, and write them in parametric form, where $b = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$. Solution:

   Consider the augmented matrix $[A\ b] = \begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 2 & 6 & 5 & 1 & -2 \\ 3 & 9 & 4 & 5 & -3 \end{bmatrix}$.

   Now we perform row operations on the augmented matrix.

$$\begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 2 & 6 & 5 & 1 & -2 \\ 3 & 9 & 4 & 5 & -3 \end{bmatrix} r_2 := (-2)r_1 + \widetilde{r_2, r_3} := (-3)r_1 + r_3 \begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 0 & 0 & 13 & -13 & 0 \\ 0 & 0 & 16 & -16 & 0 \end{bmatrix}$$

$$r_2 := \tfrac{1}{13}r_2, r_3 \widetilde{:=} \tfrac{1}{16}r_3, r_3 := r_3 - r_2 \begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r_1 := \widetilde{4r_2} + r_1 \begin{bmatrix} 1 & 3 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   So the solution is

$$\begin{cases} x_1 + 3x_2 + 3x_4 = -1 \\ x_3 - x_4 = 0 \\ x_2 \text{ and } x_4 \text{ are free.} \end{cases} \tag{1}$$

So

$$\begin{cases} x_1 = -1 - 3x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = x_4 \\ x_4 \text{ is free.} \end{cases} \tag{2}$$

Thus the solution of $Ax = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 - 3x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

where x_2 and x_4 are any numbers.

(b) Find all the solutions of the homogeneous system $Ax = 0$, and write them in parametric form.
Solution:From previous example, we know that the solution of $Ax = 0$ is of the form

$$x = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

where x_2 and x_4 are any numbers.

(c) Are the columns of the matrix A linearly independent? Write down a linear relation between the columns of A if they are dependent.
Solution: Since $Ax = 0$ has nontrivial solution, we know that the columns of the matrix A are linearly dependent. The solution is $x = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Choosing $x_2 = 1$ and $x_3 = 0$, We have

s $x = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. This implies that

$$-3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + 0 \cdot \begin{bmatrix} -4 \\ 5 \\ 4 \end{bmatrix} + 0 \cdot \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

4. Let $S = Span\Big\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -3 \end{bmatrix} \Big\}$.

(a) Find all the vectors $u = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that the u is in S. Write these u in parametric form. Justify your answer.

Solution: Note that $u \in S$ iff the following system is consistent.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & a \\ -2 & 1 & -3 & 1 & b \\ 3 & 1 & 2 & 1 & c \\ 1 & -2 & 3 & -3 & d \end{bmatrix} \sim (r_2 := r_2 + 2r_1) \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b + 2a \\ 3 & 1 & 2 & 1 & c \\ 1 & -2 & 3 & -3 & d \end{bmatrix}$$

$$\sim \ (r_3 := r_3 - 3r_1) \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b + 2a \\ 0 & 1 & -1 & 1 & c - 3a \\ 1 & -2 & 3 & -3 & d \end{bmatrix}$$

$$\sim \ (r_4 := r_4 - 3r_1) \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b + 2a \\ 0 & 1 & -1 & 1 & c - 3a \\ 0 & -2 & 2 & -3 & d - a \end{bmatrix} \sim \ (r_3 := r_3 - r_2)$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b+2\,a \\ 0 & 0 & 0 & 0 & c-5\,a-b \\ 0 & -2 & 2 & -3 & d-a \end{bmatrix} \sim (r_4 := r_4{+}2r_2) \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b+2\,a \\ 0 & 0 & 0 & 0 & c-5\,a-b \\ 0 & 0 & 0 & -1 & d+3\,a+2\,b \end{bmatrix}$$

$$\sim \quad (r_4 \leftrightarrow r_3, r_3 := -r_3) \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b+2\,a \\ 0 & 0 & 0 & 1 & -d-3\,a-2\,b \\ 0 & 0 & 0 & 0 & c-5\,a-b \end{bmatrix}.$$ This system is consistent if $c - 5\,a - b = 0$. So $c = 5a + b$ and $u = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} =$

$$\begin{bmatrix} a \\ b \\ 5a+b \\ d \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 5 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Is $v = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$ in S.

Solution: We have $a = -1$, $b = 3$ and $c = -2$. So $c - 5a - b = -2 + 5 - 3 = 0$ So $v \in S$.

(c) Is $w = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$ in S.

Solution: We have $a = 1$, $b = 3$ and $c = -2$. So $c-5a-b = -2-5-3 = -10 \neq 0$ So v is not in S.

5. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & b \\ 3 & 5 & a & 1 \end{array}\right]$$

(a) For what values of a will the system have a unique solution? What is the solution?(your answer may involve a and b)

(b) For what values of a and b will the system have infinitely many solutions?

(c) For what values of a and b will the system be inconsistent?

Answer:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & b \\ 3 & 5 & a & 1 \end{array}\right] \sim (r_2 := r_2 + (-1)r_1) \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & b-2 \\ 3 & 5 & a & 1 \end{bmatrix}$$

$$\sim (r_3 := r_3 + (-3)r_1) \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & b-2 \\ 0 & 2 & a-3 & -5 \end{bmatrix}$$

$$\sim (r_3 := r_3 + (-2)r_2) \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & b-2 \\ 0 & 0 & a-3 & -1-2\,b \end{bmatrix}.$$

(a) If $a - 3 \neq 0$ then previous augmented matrix is row equivalent to ($r_3 := \frac{1}{a-3} r_3$) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & b-2 \\ 0 & 0 & 1 & \frac{-1-2\,b}{a-3} \end{bmatrix} \sim (r_1 := r_1 - r_3)$

$$\begin{bmatrix} 1 & 1 & 0 & 2 + \frac{1+2\,b}{a-3} \\ 0 & 1 & 0 & b-2 \\ 0 & 0 & 1 & \frac{-1-2\,b}{a-3} \end{bmatrix}$$

$$\sim (r_1 := r_1 - r_2) \begin{bmatrix} 1 & 0 & 0 & 4 + \frac{1+2\,b}{a-3} - b \\ 0 & 1 & 0 & b-2 \\ 0 & 0 & 1 & \frac{-1-2\,b}{a-3} \end{bmatrix}$$

The system will have a unique solution when $a \neq 3$. The solution is $x_1 = 4 + \frac{1+2\,b}{a-3} - b$, $x_2 = b - 2$ and $x_2 = \frac{-1-2\,b}{a-3}$.

(b) The system will have infinitely many solutions if $a - 3 = 0$ and $-1 - 2b = 0$, i.e $a = 3$ and $b = -\frac{1}{2}$.

(c) The system will be inconsistent if $a - 3 = 0$ and $-1 - 2b \neq 0$, i.e $a = 3$ and $b \neq -\frac{1}{2}$.

6. (a) $\begin{bmatrix} -2 & 1 \\ 4 & -2 \\ 0 & 0 \\ -6 & 3 \end{bmatrix} = [v_1 \ v_2]$. We have $v_1 = -2v_2$. So the set of column vectors is linearly dependent.

(b) $\begin{bmatrix} -2 & 1 \\ 4 & -2 \\ 2 & 2 \end{bmatrix}$. The first column vector is not a multiple of the second column vector. So the set of column vectors is linearly independent.

(c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} r_3 := r_3 - r_1, r_4 := r_4 - r_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$\sim r_2 := -r_4, r_4 := r_2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim r_3 := r_3 + r_2, r_4 := r_4 - r_2$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim r_4 := r_4 - r_3, r_1 := r_1 - r_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So $x_1 = x_2 = x_3 = 0$. Thus the columns of the matrix is linearly independent.

This matrix has three pivot vectors. So the columns of the matrix form a linearly independent set.

(d)

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \underset{\substack{r_4 := r_4 - r_1 \\ r_3 := r_3 - r_1}}{\sim} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\underset{\substack{r_3 := r_3 + 2r_2 \\ r_4 := r_4 + r_2}}{\sim} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_1 := r_1 - r_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix has only two pivot vectors. So the columns of the matrix form a linearly dependent set.

(e)

The column vectors of

$$\begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}$$

form a dependent set since we have five column vectors in R^4. We will have at least one free variable for the solution of $Ax = 0$.

7. (a)

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & a+1 & 3 \\ 1 & a & a+1 \end{bmatrix} r_2 := r_2 + (-1)\underset{}{\overset{\sim}{r_1, r_3}} := r_3 + (-1)r_1 \begin{bmatrix} 1 & 1 & 2 \\ 0 & a & 1 \\ 0 & a-1 & a-1 \end{bmatrix}$$

$$r_3 := \frac{1}{a-1} r_3 \overset{\sim}{if a - 1} \neq 0, r_2 \leftrightarrow r_3 \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & a & 1 \end{bmatrix} r_3 := \overset{\sim}{r_3 + (-a)} r_2 \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1-a \end{bmatrix}$$

Thus the column vectors are independent if $a \neq 1$.
(b) The column vectors are dependent if $a = 1$.

8. First, note that

$$T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + T\left(2\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + T\left(3\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + 2T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + 3T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T(e_1) + 2T(e_2) + 3T(e_3).$$

We need to find $T(e_1)$, $T(e_2)$ and $T(e_3)$.

Since T is linear, we have $T(e_1 + e_2) = T(e_1) + T(e_2)$, $T(e_1 - e_2) = T(e_1) - T(e_2)$ and $T(e_1+e_2+e_3) = T(e_1)+T(e_2)+T(e_3)$. The conditions $T(e_1 + e_2) = \begin{bmatrix}1\\-1\end{bmatrix}$, $T(e_1 - e_2) = \begin{bmatrix}2\\3\end{bmatrix}$ and $T(e_1 + e_2 + e_3) = \begin{bmatrix}1\\-2\end{bmatrix}$ can be written as

$$\begin{cases} T(e_1) + T(e_2) = \begin{bmatrix}1\\-1\end{bmatrix} \\ T(e_1) - T(e_2) = \begin{bmatrix}2\\3\end{bmatrix} \\ T(e_1) + T(e_2) + T(e_3) = \begin{bmatrix}1\\-2\end{bmatrix}. \end{cases} \tag{3}$$

Adding $T(e_1)+T(e_2) = \begin{bmatrix}1\\-1\end{bmatrix}$ and $T(e_1)-T(e_2) = \begin{bmatrix}2\\3\end{bmatrix}$, we get $2T(e_1) = \begin{bmatrix}3\\2\end{bmatrix}$ and $T(e_1) = \begin{bmatrix}\frac{3}{2}\\1\end{bmatrix}$. Using $T(e_1) + T(e_2) = \begin{bmatrix}1\\-1\end{bmatrix}$, we have $T(e_2) = \begin{bmatrix}1\\-1\end{bmatrix} - T(e_1) = \begin{bmatrix}1\\-1\end{bmatrix} - \begin{bmatrix}\frac{3}{2}\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\-2\end{bmatrix}$. From $T(e_1) + T(e_2) + T(e_3) = \begin{bmatrix}1\\-2\end{bmatrix}$, we have $T(e_3) = \begin{bmatrix}1\\-2\end{bmatrix} - (T(e_1)+T(e_2)) = \begin{bmatrix}1\\-2\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$.

Hence $T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = T(e_1)+2T(e_2)+3T(e_3) = \begin{bmatrix}\frac{3}{2}\\1\end{bmatrix} + 2\cdot\begin{bmatrix}-\frac{1}{2}\\-2\end{bmatrix} + 3\cdot\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-6\end{bmatrix}$.