Linear Algebra (Math 2890) Review Problems for Final Exam

Final exam on Dec 14, Monday, 12:30pm-2:30pm.
Regular office hours:
UH2080B M 2:00-3:30pm W 11-12 a.m., 3-4 pm, F 2-3 pm
Office hour before the final exam:
Monday (Dec 7) 11-12, 2-3:30, Wednesday (Dec 9) 11-12, 3-4 and Friday (Dec 11)11-12, 2-3 p.m.
Monday (Dec 14) 10:30-12.
Topics in the final exam. The final exam is compressive. It coves 1.1 1.5, 1.7, 1.8, 2.1 2.3, 2.8, 2.9, 3.1, 3.2, 5.1 5.3, 6.1 6.6, 7.1, 7.2.

1. Let A be the matrix
$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$
   (a) Prove that $det(A - \lambda I) = (1 - \lambda)^2(4 - \lambda)$.
   (b) Orthogonally diagonalizes the matrix A, giving an orthogonal matrix P and a diagonal matrix D such that $A = PDP^t$.
   (c) Write the quadratic form associated with A using variables x_1, x_2, and x_3?
   (d) Find A^{-1}, A^{10} and e^A.
   (e) What's $A^{-5}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix})$?
   (f) What is $lim_{n\to\infty} A^{-n}$?

2. Classify the quadratic forms for the following quadratic forms. Make a change of variable $x = Py$, that transforms the quadratic form into one with no cross term. Also write the new quadratic form in new variables y_1, y_2.
   (a) $9x_1^2 - 8x_1x_2 + 3x_2^2$.
   (b) $-5x_1^2 + 4x_1x_2 - 2x_2^2$.
   (c) $8x_1^2 + 6x_1x_2$.

3. (a) Find a 3×3 matrix A which is not diagonalizable?
(b) Give an example of a 2×2 matrix which is diagonalizable but not orthogonally diagonalizable?

4. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$.

(a) Find the condition on $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ such that $Ax = b$ is consistent.

(b) What is the column space of A?

(c) Describe the subspace $col(A)^{\perp}$ and find an basis for $col(A)^{\perp}$. What's the dimension of $col(A)^{\perp}$?

(d) Use Gram-Schmidt process to find an orthogonal basis for the column space of A.

(e) Find an orthonormal basis for the column of the matrix A.

(f) Find the orthogonal projection of $y = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix}$ onto the column space of A and write $y = \widehat{y} + z$ where $\widehat{y} \in col(A)$ and $z \in col(A)^{\perp}$. Also find the shortest distance from y to $Col(A)$.

(g) Using previous result to explain why $Ax = y$ has no solution.

(h) Use orthogonal projection to find the least square solution of $Ax = y$.

(i) Use normal equation to find the least square solution of $Ax = y$.

5. Find the equation $y = a + mx$ of the least square line that best fits the given data points. $(0, 1)$, $(1, 1)$, $(3, 2)$.

6. (a) Show that the set of vectors

$$B = \left\{ u_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0 \right),\ u_2 = \left(\frac{4}{5}, \frac{3}{5}, 0 \right),\ u_3 = (0, 0, 1) \right\}$$

is an **orthonormal basis** of $\mathbb{R}^3$.
(b) Find the coordinates of the vector $(1, -1, 2)$ with respect to the basis in (a).

7. (a) Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$. Find the inverse matrix of A if possible.
(b) Find the coordinates of the vector $(1, -1, 2)$ with respect to the basis B obtained from the column vectors of A.

8. Let $H = \left\{ \begin{bmatrix} a + 2b - c \\ a - b - 4c \\ a + b - 2c \end{bmatrix} : a, b, c \text{any} \ \ \text{real} \ \ \text{numbers} \right\}$.
a. Explain why H is a a subspace of R^3.
b. Find a set of vectors that spans H.
c. Find a basis for H.
d. What is the dimension of the subspace?
e. Find an orthogonal basis for H.

9. Determine if the following systems are consistent and if so give all solutions in parametric vector form.
(a)

$$\begin{aligned} x_1 \ -2x_2 &= 3 \\ 2x_1 \ -7x_2 &= 0 \\ -5x_1 \ +8x_2 &= 5 \end{aligned}$$

(b)

$$\begin{aligned} x_1 \ +2x_2 \ -3x_3 \ +x_4 &= 1 \\ -x_1 \ -2x_2 \ +4x_3 \ -x_4 &= 6 \\ -2x_1 \ -4x_2 \ +7x_3 \ -x_4 &= 1 \end{aligned}$$

10. Let $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{bmatrix}$ which is row reduced to $\begin{bmatrix} 1 & -3 & -2 & -20 & -3 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for the column space of A
(b) Find a basis for the nullspace of A
(c) Find the rank of the matrix A
(d) Find the dimension of the nullspace of A.
(e) Is $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ in the range of A?
(e) Does $Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$ have any solution? Find a solution if it's solvable.

11. Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}.$$

12. Let A be a 12×5 matrix. You may assume that $Nul(A^TA) = Nul(A)$. (This relation holds form any matrix A.)
a. What is the size of A^TA?
b. Use the Rank Theorem to obtain an equation involving $rankA$. Find another equation involving $rank(A^TA)$. What is the connection between these two ranks?
c. Suppose the columns of A are linearly independent. Explain why A^TA is invertible.