MATH 2890

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Step 4. Identify basic variables (corresponding to the the leading coefficient or the pivot vector) and free variables (not basic variables) from reduced echelon form. Express the solution set in terms of the free variables.

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Free variables correspond to **non pivot columns.**

Example: $\spadesuit$ is the leading coefficient

Echelon form: $\begin{bmatrix} \spadesuit & * & * & * & * \\ 0 & \spadesuit & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \textit{pivot column} & \textit{pivot column} & & & \end{bmatrix}$

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Reduced echelon form $\begin{bmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \textit{basic} & \textit{basic} & \textit{free} & \textit{free} & \end{bmatrix}$

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Reduced Echelon form:

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Theorem

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d. $r_i \leftrightarrow r_j$ This means that we interchange the i-th row and the j-th row.

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to get $\begin{bmatrix} 2 & 4 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ which is row equivalent to $\begin{bmatrix} 2 & 4 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

($r_2 := r_2/2$ and $r_3 := r_3 - r_2$).

Next, multiply -4 to second row and add it to the first row

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The corresponding system of equations is

$$\begin{cases} x_1 + \qquad \frac{3}{2}x_3 = 0 \\ \qquad x_2 \qquad\quad = 0 \end{cases}.$$

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The gives the general solution $\begin{cases} x_1 = -\dfrac{3}{2}x_3 \\ x_2 = 0 \\ x_3 : \ \textit{free} \end{cases}$

(c) The augmented matrix is $\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is in echelon form

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Next, we multiply 2 to third row and add it to the second row

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Next, we multiply 2 to third row and add it to the second row

$$\left(r_2 := r_2 + 2r_3, \quad \begin{array}{rl} r_2 \mapsto & \begin{pmatrix} 0 & 1 & 0 & -2 & 4 & -2 \end{pmatrix} \\ 2r_3 \mapsto & \begin{pmatrix} 0 & 0 & 0 & 2 & 4 & 2 \end{pmatrix} \\ r_2 + 2r_3 \mapsto & \begin{pmatrix} 0 & 1 & 0 & 0 & 8 & 0 \end{pmatrix} \end{array} . \right)$$

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$$\left(r_2 := r_2 + 2r_3, \quad \begin{array}{r} r_2 \mapsto \\ 2r_3 \mapsto \\ r_2 + 2r_3 \mapsto \end{array} \begin{array}{l} (0 \quad 1 \quad 0 \quad -2 \quad 4 \quad -2) \\ (0 \quad 0 \quad 0 \quad 2 \quad 4 \quad 2) \\ (0 \quad 1 \quad 0 \quad 0 \quad 8 \quad 0) \end{array} . \right)$$

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Next, multiply -1 to third row and add it to the first row

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to get $\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Next, multiply -1 to third row and add it to the first row

$$(r_1 := r_1 - r_3)$$

(c) The augmented matrix is $\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is in echelon form

Next, we multiply 2 to third row and add it to the second row

$$\left(r_2 := r_2 + 2r_3, \quad \begin{matrix} r_2 \mapsto & (0 & 1 & 0 & -2 & 4 & -2) \\ 2r_3 \mapsto & (0 & 0 & 0 & 2 & 4 & 2) \\ r_2 + 2r_3 \mapsto & (0 & 1 & 0 & 0 & 8 & 0) \end{matrix}. \right)$$

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$$\left(r_2 := r_2 + 2r_3, \quad \begin{matrix} r_2 \mapsto & (0 & 1 & 0 & -2 & 4 & -2) \\ 2r_3 \mapsto & (0 & 0 & 0 & 2 & 4 & 2) \\ r_2 + 2r_3 \mapsto & (0 & 1 & 0 & 0 & 8 & 0) \end{matrix}. \right)$$

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Next, multiply -2 to second row and add it to the first row

$$\left(r_1 := r_1 - 2r_2 \quad \begin{matrix} r_1 \mapsto & 1 & 2 & 0 & 0 & -2 & -3 \\ -2r_2 \mapsto & 0 & -2 & 0 & 0 & -16 & 0 \\ r_1 - 2r_2 \mapsto & 1 & 0 & 0 & 0 & -18 & -3 \end{matrix} . \right)$$

Next, multiply -2 to second row and add it to the first row

$$\left(r_1 := r_1 - 2r_2 \quad \begin{array}{rcccccc} r_1 \mapsto & 1 & 2 & 0 & 0 & -2 & -3 \\ -2r_2 \mapsto & 0 & -2 & 0 & 0 & -16 & 0 \\ r_1 - 2r_2 \mapsto & 1 & 0 & 0 & 0 & -18 & -3 \end{array}.\right)$$

to get $\begin{bmatrix} 1 & 0 & 0 & 0 & -18 & -3 \\ 0 & 1 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Next, multiply -2 to second row and add it to the first row

$$\begin{array}{rrrrrrr} r_1 \mapsto & 1 & 2 & 0 & 0 & -2 & -3 \\ (r_1 := r_1 - 2r_2 \quad -2r_2 \mapsto & 0 & -2 & 0 & 0 & -16 & 0\ .\ ) \\ r_1 - 2r_2 \mapsto & 1 & 0 & 0 & 0 & -18 & -3 \end{array}$$

to get $\begin{bmatrix} 1 & 0 & 0 & 0 & -18 & -3 \\ 0 & 1 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is in reduced echelon form.

Next, multiply -2 to second row and add it to the first row

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From here, we know that x_3 and x_5 are the free variables.

Next, multiply -2 to second row and add it to the first row

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From here, we know that x_3 and x_5 are the free variables.

The corresponding system of equations is

$$\begin{cases} x_1 \quad -18x_5 = -3 \\ x_2 \quad +8x_5 = 0 \\ x_4 \quad +2x_5 = 1 \end{cases}.$$

Next, multiply -2 to second row and add it to the first row

$$\begin{array}{rrrrrrrr} & r_1 \mapsto & 1 & 2 & 0 & 0 & -2 & -3 \\ (r_1 := r_1 - 2r_2 & -2r_2 \mapsto & 0 & -2 & 0 & 0 & -16 & 0\ .\ ) \\ & r_1 - 2r_2 \mapsto & 1 & 0 & 0 & 0 & -18 & -3 \end{array}$$

to get $\begin{bmatrix} 1 & 0 & 0 & 0 & -18 & -3 \\ 0 & 1 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is in reduced echelon form.

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The corresponding system of equations is

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. The gives the general solution

$$\begin{cases} x_1 = -3 + 18x_5 \\ x_2 = -8x_5 \\ x_3 : \ \textit{free} \\ x_4 = 1 - 2x_5, \quad x_5 : \ \textit{free} \end{cases}$$