Solution to Linear Algebra (Math 2890) Review Problems II

1. (a) What is a subspace in R^n?

Solution: A subspace of R^n is any set H in R^n that satisfies the following three properties. (I) The zero vector is in H. (II) For each u and v in H, then $u + v$ is in H. (III) For each u in H and each scalar c, the vector cu is in H.

(b) Is the set $\{(x, y, z)|x + y + z = 1\}$ a subspace?
Solution: This is not a subspace since the zero vector $(0, 0, 0)$ is not in the set.

(c) Is the set $\{(x, y, z)|x - y - z = 0, x + y - z = 0\}$ a subspace?
Solution: Yes. This is a subspace. This can be regarded as the nullspace of the matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$.

Here $Nul(A) = \{(x, y, z)| \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\}$.

(d) What is a basis for a subspace?
Solution: A basis for a subspace H of R^n is a linearly independent set in H that spans H.

(e) What is the dimension of a subspace?
Solution:The dimension of a nonzero subspace H is the number of vectors in any basis for H.

(f) What is the column space of a matrix?
Solution: The column space of a matrix A is the set of the span of the column vectors of A.

(g) What is the null space of a matrix?
Solution:The null space of a matrix A is the set of all solutions to the homogeneous equation $Ax = 0$, i.e. $Nul(A) = \{x|Ax = 0\}$.

(h) What is an eigenvalue of a matrix A?
Solution: Let A be a $n \times n$ matrix. A scalar λ such that $Ax = \lambda x$ for some $x \neq 0$ is called an eigenvalue and the corresponding vector is called an eigenvector.

(i) What is an eigenvector of a matrix A?
Solution: Let A be a $n \times n$ matrix. A scalar λ such that $Ax = \lambda x$ for some $x \neq 0$ is called an eigenvalue and the corresponding vector is called an eigenvector.

(j) What is the characteristic polynomial of a matrix A ?
Solution: The polynomial $det(A - \lambda I)$ is the the characteristic polynomial of a matrix A.

(k) What is the subspace spanned by the vectors $v_1,\ v_2,\ \cdots,\ v_p$? Solution: The subspace spanned by $v_1,\ v_2,\ \cdots,\ v_p$ is the set of all possible linear combination of $v_1,\ v_2,\ \cdots,\ v_p$, i.e. $Span\{v_1,\ \cdots,\ v_n\} = \{c_1v_1 + c_2v_2 + \cdots + c_pv_p | c_1, c_2, \cdots, c_p \textit{ are real numbers}\}$

2. Find the inverses of the following matrices if they exist.

$$A = \begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ -1 & 0 & -1 \end{bmatrix},\ C = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}.$$

Solution: (a) Since $det(A) = -1$, we have $A^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -4 & -7 \end{bmatrix}$

(b)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \overset{r_2 := r_2 - 2r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -1 & -2 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{r_3 := r_3 + r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\overset{r_2 := -r_3, r_3 := r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 5 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\overset{r_3 := r_3 - 5r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 1 & 5 \end{array}\right]$$

$$\overset{r_1 := r_1 + r_3, r_3 := -r_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 4 & 1 & 5 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -3 & -1 & -5 \end{array}\right]$$

$$\overset{r_1 := r_1 + r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -3 & -1 & -5 \end{array}\right]$$

So $B^{-1} = \begin{bmatrix} 3 & 1 & 4 \\ -1 & 0 & -1 \\ -3 & -1 & -5 \end{bmatrix}$.

(c)

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 5 & 6 & 7 & 0 & 1 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array}\right] \overset{r_2 := r_2 - 2r_1}{\sim} \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 0 & -1 & -2 & 1 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{r_2 \longleftrightarrow r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{r_2 := r_2 - 2r_1, r_3 := r_3 - 8r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 3 & -1 & 0 \\ 0 & 9 & 18 & 16 & -8 & 1 \end{array}\right]$$

$$\overset{r_3 := r_3 + (-3)r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 3 & -1 & 0 \\ 0 & 0 & 0 & 7 & -5 & 1 \end{array}\right]$$

So C only has one free variable (or two pivot vectors) and C is not invertible.

3. (a) Let A be an 3×3 matrix. Suppose $A^3 + 2A^2 - 3A + 4I = 0$. Is A invertible? Express A^{-1} in terms of A if possible.
Solution: From $A^3 + 2A^2 - 3A + 4I = 0$, we have $A^3 + 2A^2 - 3A = -4I$, $A(A^2 + 2A - 3I) = -4I$ and $A \cdot (-\frac{1}{4}(A^2 + 2A - 3I)) = I$. So $A^{-1} = -\frac{1}{4}(A^2 + 2A - 3I)$.
(b) Suppose $A^3 = 0$. Is A invertible?
Solution: If A is invertible then $A^{-2}A^3 = A^{-2}0$ and $A = 0$ which is not invertible. So A is not invertible.

4. Find all values of a and b so that the subspace of $\mathbb{R}^4$ spanned by

$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} b \\ 1 \\ -a \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ is two-dimensional.

Solution: Consider the matrix $A = \begin{bmatrix} 0 & b & -2 \\ 1 & 1 & 2 \\ 0 & -a & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$\underbrace{\text{interchange first row and second row}}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & b & -2 \\ 0 & -a & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$\widetilde{r_4 := r_1 + r_4}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & b & -2 \\ 0 & -a & 0 \\ 0 & 2 & 2 \end{bmatrix}$

$\widetilde{\textit{interchange second row and forth row}}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -a & 0 \\ 0 & b & -2 \end{bmatrix}$

$\widetilde{\textit{divide second row by 2}}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -a & 0 \\ 0 & b & -2 \end{bmatrix}$ $\widetilde{r_3 := r_3 + ar_2, r_4 := r_4 - br_2}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & a \\ 0 & 0 & -2-b \end{bmatrix}$.

Now the first and second vectors are pivot vectors. So $rank(A) = 2$ if $a = 0$ and $-2 - b = 0$.

So $a = 0$ and $b = -2$

5. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$. You can assume that $\mathcal{B}$ is a basis for R^3

(a) Which vector x has the coordinate vector $[x]_B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. So $x = A[x]_B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-3+0 \\ 0-2+0 \\ 0-1+4 \end{bmatrix} =$
$\begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$

(b) Find the β-coordinate vector of $y = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

Solution. We have to solve $Ax = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right] \widetilde{r_2 := \tfrac{1}{2} r_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 3 \end{array}\right] \widetilde{r_2 := r_3 - r_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right]$$
$$\widetilde{r_3 := \tfrac{1}{2} r_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \widetilde{r_1 := r_1 - 3r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

So $[y]_B = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$.

6. Let

$$M = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$$

(a) Find bases for $Col(M)$ and $Nul(M)$, and then state the dimensions of these subspaces

Solution: $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \widetilde{r_2 := -r_1 + r_2, r_3 := -r_2 + r_3} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix}$

$$\widetilde{r_3 := -2r_2 + r_3} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \widetilde{r_1 := -2r_2 + r_3} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the first two vectors are pivot vectors and $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \}$ is a basis for $Col(M)$ and $dim(Col(M)) = 2$.

The solution to $Mx = 0$ is $x_1 + x_3 - x_4 = 0$ and $x_2 + 2x_3 + x_4 = 0$. So $x = \begin{bmatrix} -x_3 + x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Hence the basis for $Nul(M)$ is $\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \}$ and $dim(Nul(M)) = 2$.

(b) Express the third column vector M as a linear combination of the basis of $Col(M)$. From the row reduced echelon form, we know that column(3) = 1 · column(1) + 2 · column(2)

So $\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

7. Find a basis for the subspace spanned by the following vectors $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \}$.

   What is the dimension of the subspace?

   Solution: Consider the matrix $A = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$

   From previous example, we know that the first two vectors are pivot vectors and $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \}$ is a basis. The dimension of the subspace is 2.

8. Determine which sets in the following are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify your answer

   (a) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$. Solution: Since $\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, the set $\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \}$ is dependent. It is not a basis.

(b) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Yes. This set forms a basis since they are independent and span R^3.

(c) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

This is not a basis since it doesn't span R^3.

(d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This set forms a basis since they are independent and span R^3

(e) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. This is not a basis since it is dependent.

9. Let A be the matrix

$$A = \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}.$$

Find a polynomial $f(A)$ in A such that $f(A) = 0$. Verify your answer.

Solution: 1. $A - \lambda I = \begin{bmatrix} -3-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix}$.

So $det(A - \lambda I) = (-3-\lambda)(3-\lambda) - 16 = \lambda^2 - 9 - 16 = \lambda^2 - 25$

2. Let $f(A) = A^2 - 25I$. Then $A^2 - 25I = 0$. One can verify this by checking $A^2 = \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = 25 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 25I$.

Hence $A^2 - 25I = 0$.

10. Let A be the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Prove that $det(A - \lambda I) = -(\lambda - 1)^2(\lambda - 4)$.

(b) Find the eigenvalues and a basis of eigenvectors for A.

(c) Diagonalize the matrix A if possible.

(d) Find an expression for A^k.

(e) Find the matrix exponential e^A. Solution.

a. 1. $A-\lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}$.

So $det(A-\lambda I) = (2-\lambda)^3+1+1-(2-\lambda)-(2-\lambda)-(2-\lambda) = (4-4\lambda+\lambda^2)(2-\lambda)+2-6+3\lambda = 8-8\lambda+2\lambda^2-4\lambda+4\lambda^2-\lambda^3-4+3\lambda$ $= -\lambda^3+6\lambda^2-9\lambda+4 = -(\lambda-1)^2(\lambda-4)$. So the characteristic equation is $-(\lambda-1)^2(\lambda-4)=0$.

2. Solving the characteristic equation $-(\lambda-1)^2(\lambda-4)=0$, we get that the eigenvalues are $\lambda=1$ and $\lambda=4$.

3. When $\lambda=1$, we have

$$A-\lambda I = \begin{bmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \underbrace{r_1 := r_2 - r_1, r_3 := r_3 - r_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution of $(A-I)x=0$ is $x_1+x_2+x_3=0$ and $x_1=-x_2-x_3$ So

$$Null(A-I) = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}.$$

The basis for the eigenspace corresponding to eigenvalue 1 is $\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$

4. When $\lambda=4$, we have

$$A-\lambda I = \begin{bmatrix} 2-4 & 1 & 1 \\ 1 & 2-4 & 1 \\ 1 & 1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\underbrace{interchange\ 1st\ row\ and\ 2nd\ row} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\underbrace{r_2 := r_2 + 2r_1, r_3 := r_3 - r_1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \underbrace{r_2 := r_2/3, r_2 : r_2 + r_3} =$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underbrace{r_2 := r_1 + 2r_2} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

.

The solution of $(A-4I)x=0$ is $x_1-x_3=0$ and $x_2-x_3=0$. This implies that $x_1=x_3$, $x_2=x_3$ and x_3 is free. So $Null(A-I)=\{\begin{bmatrix}x_3\\x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\1\\1\end{bmatrix}\}$.

The basis for the eigenspace corresponding to eigenvalue 4 is $\{\begin{bmatrix}1\\1\\1\end{bmatrix}\}$

So A is diagonalizable with $A=PDP^{-1}$ where $P=\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}$ and $D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&4\end{bmatrix}$.

Hence $A^k=P\begin{bmatrix}1&0&0\\0&1&0\\0&0&4^k\end{bmatrix}P^-1$.

Also $e^A=Pe^DP^{-1}=P\begin{bmatrix}e&0&0\\0&e&0\\0&0&e^4\end{bmatrix}P^{-1}$.

11. Let B be the matrix $\begin{bmatrix}2&1&1\\0&2&1\\0&0&1\end{bmatrix}$.

(a) Find the characteristic equation of A.

Solution: $B-\lambda I=\begin{bmatrix}2-\lambda&1&1\\0&2-\lambda&1\\0&0&1-\lambda\end{bmatrix}$.

So $det(B-\lambda I)=(2-\lambda)^2(1-\lambda)$. The characteristic equation of A is $(2-\lambda)^2(1-\lambda)=0$.

(b) Find the eigenvalues and a basis of eigenvectors for B.

Solving $(2-\lambda)^2(1-\lambda) = 0$, we know that the eigenvalues of B are $\lambda = 2$ and $\lambda = 1$.

When $\lambda = 2$, we have

$$B - \lambda I = \begin{bmatrix} 2-2 & 1 & 1 \\ 0 & 2-2 & 1 \\ 0 & 0 & 1-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$r_2 := r_2 + \widetilde{r_3, r_1} := r_1 + r_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution of $(B-2I)x = 0$ is $x_2 = 0$, $x_3 = 0$ and x_1 is free. So $Null(B-2I) = \{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$.

The basis for the eigenspace corresponding to eigenvalue 2 is $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$.

When $\lambda = 1$, we have

$$B - \lambda I = \begin{bmatrix} 2-1 & 1 & 1 \\ 0 & 2-1 & 1 \\ 0 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r_1 \widetilde{:= r_1} - r_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution of $(B-I)x = 0$ is $x_1 = 0$ and $x_2 + x_3 = 0$ So $x_1 = 0$, $x_2 = -x_3$ and x_3 is free. $Null(B-I) = \{ \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \}$.

The basis for the eigenspace corresponding to eigenvalue 1 is $\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

So $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue 2 and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue 1

(c) Diagonalize the matrix B if possible.

From (b), we know that B has only two independent eigenvectors and B is not diagonzalizable.

12. Find an basis for $W^\perp$ for the following W. Verify your answer.

(a) $W = Span\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\}$.

Solution: $W^\perp = \{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 0\}$

So $-x_1 + 2x_2 + x_3 = 0$ and $x_1 = 2x_2 + x_3$. Hence $x = \begin{bmatrix} 2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. So $\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis for $W^\perp$.

We can check that $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -2 + 2 = 0$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -1 + 1 = 0$.

(b)$W = Span\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\}$.

Solution: $W^\perp = \{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 0, x \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 0\}$

So $-x_1 + 2x_2 + x_3 = 0$ and $2x_1 - 3x_2 + x_3 = 0$. Hence $W^\perp = Nul(A)$ where $A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -3 & 1 \end{bmatrix}$.

Perform row reduction on $\begin{bmatrix} -1 & 2 & 1 & 0 \\ 2 & -3 & 1 & 0 \end{bmatrix}$

$\sim (r_1 = -r_1, r_2 = r_2 + 2r_2) \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \sim (r_1 = r_1 + 2r_2) \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$

So $x_1 + 5x_3 = 0$ and $x_2 + 3x_3 = 0$. Hence $x_1 = -5x_3$ and $x_2 = -3x_3$.

$x = \begin{bmatrix} -5x_3 \\ -3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$. So $\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \}$ is a basis for $W^\perp$.

We can check that $\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 5-6+1 = 0$ and $\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = -10+9+1 = 0$.

13. (a) Let $W = Span\{u_1, u_2\}$ where $u_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. Show that $\{u_1, u_2\}$ is an orthogonal basis for W.

Solution: Compute $u_1 \cdot u_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = -2+4-2 = 0$. So $\{u_1, u_2\}$ is an orthogonal basis for W.

(b)Find the closest point to $y = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$ in the subspace W.

Solution: The closest point to to $y = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$ in the subspace W is $Proj_W(y) = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$.

Compute $y \cdot u_1 = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = 1 + 10 - 2 = 9$,

$u_1 \cdot u_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = 1 + 4 + 4 = 9$,

$y \cdot u_2 = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = -2 + 10 + 1 = 9$,

$u_2 \cdot u_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 4 + 4 + 1 = 9$.

So $Proj_W(y) = \frac{y\cdot u_1}{u_1\cdot u_1}u_1 + \frac{y\cdot u_2}{u_2\cdot u_2}u_2 = \frac{9}{9}u_1 + \frac{9}{9}u_2 = u_1 + u_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

(c) Find the distance between the point y and the subspace W.

The distance between y and the subspace W is $||y - Proj_W(y)||$. Compute $y - Proj_W(y) = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ and $||y - Proj_W(y)|| = ||\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}|| = \sqrt{(-2)^2 + 1^2 + 2^2} = \sqrt{9} = 3$. Hence the distance between the point y and the subspace W is 3.