1. (6 points) Given the differential equation

$$y'' + 6y' + 9y = 0, \qquad (1)$$

(a) (2 points) Determine the characteristic equation and two linearly independent solutions.

$R^2 + 6R + 9 = 0 \leftarrow$ characteristic equation

$(R+3)^2 = 0 \Rightarrow R = -3$ double

l. i. solutions: $y_1 = e^{-3x}$; $y_2 = xe^{-3x}$

(b) (3 points) Solve the initial value problem consisting of equation (1) and initial conditions: $y(0) = 1$, $y'(0) = 1$.

$y_c = c_1 e^{-3x} + c_2 x e^{-3x}$

$y_c(0) = 1 = c_1$

$y_c' = -3c_1 e^{-3x} + c_2 (e^{-3x} - 3xe^{-3x})$

$y_c'(0) = 1 = -3c_1 + c_2$

$\begin{cases} c_1 = 1 \\ c_2 = +3(1)+1 = 4 \end{cases}$

$y_c = e^{-3x} + 4x e^{-3x}$

2. (6 points) Determine the real-valued general solution of the following differential equations:

$$y'' + 4y' + 8y = 0.$$

$R^2 + 4R + 8 = 0 \Rightarrow (R+2)^2 + 4 = 0$

$R = -2 \pm 2i$

$y_c = c_1 e^{-2x} \cos 2x + c_2 e^{-2x} \sin 2x$

3. (15 points) Consider the differential equation

$$y'' - 4y' = 5x + 3xe^{4x} + \cos x \tag{2}$$

(a) (2 points) Write its associated homogeneous differential equation.

$y'' - 4y' = 0$

(b) (3 points) Determine the general solution of the differential equation written in (a).

$R^2 - 4R = 0 \Rightarrow R = 0 \quad R = 4$

$y_c = c_1 e^{4x} + c_2$

(c) (4 points) Using undetermined coefficients ~~method determine a~~ solution of

$$y'' - 4y' = \cos x$$

$\boxed{y_p = -\frac{1}{17}\cos x - \frac{4}{17}\sin x}$

$y_p = A\cos x + B\sin x$ (no duplication!)

$y_p' = -A\sin x + B\cos x$; $y_p'' = -A\cos x - B\sin x$

$-A\cos x - B\sin x + 4A\sin x - 4B\cos x = \cos x \Rightarrow \begin{cases} -A - 4B = 1 \\ 4A - B = 0 \end{cases}$

(d) (3 points) Solve the initial value problem

$$y'' - 4y' = \cos x; \qquad y(0) = 0 \text{ and } y'(0) = -\frac{4}{17}.$$

$y = \frac{1}{17} - \frac{1}{17}\cos x - \frac{4}{17}\sin x$

$y = c_1 e^{4x} + c_2 - \frac{1}{17}\cos x - \frac{4}{17}\sin x$

$y' = 4c_1 e^{4x} + \frac{1}{17}\sin x - \frac{4}{17}\cos x$ | $y(0) = c_1 + c_2 - \frac{1}{17} = 0$, $y'(0) = -\cancel{\frac{4}{17}} = 4c_1 - \cancel{\frac{4}{17}} \Rightarrow \begin{cases} c_2 = \frac{1}{17} \\ c_1 = 0 \end{cases}$

(e) (6 points) Determine the *form* of the general solution of equation (2).

$y = y_c + y_p$ $\qquad y_p = y_{p1} + y_{p2} + y_{p3}$

$f_1(x) = 5x$

$y_{p1} = (A_1 x + B_1)x$ since there is duplication

$= A_1 x^2 + B_1 x$

$f_2(x) = 3xe^{4x}$

$y_{p2} = x(A_2 x + B_2)e^{4x}$ since there is duplication

$y_{p2} = A_2 x^2 + B_2 x$

$f_3(x) = -\frac{1}{17}\cos x - \frac{4}{17}\sin x$ (computed in c))

$y = c_1 e^{4x} + c_2 + A_1 x^2 + B_1 x + A_2 x^2 + B_2 x - \frac{1}{17}\cos x - \frac{4}{17}\sin x$

$$y_p = \frac{1}{2} e^{t} e^{2t} - \frac{1}{10} e^{5t} e^{-2t} = \frac{1}{2} e^{3t} - \frac{1}{10} e^{3t}$$
$$= \frac{2}{5} e^{3t}$$

4. (6 points) Use variation of parameter method to determine a particular solution for the differential equation $y'' - 4y = 2e^{3t}$.

$y_p = u_1 y_1 + u_2 y_2$

$y'' - 4y = 0$, $R^2 - 4 = 0$, $R = \pm 2$

$y_1 = e^{2t}$; $y_2 = e^{-2t}$

$y_p = u_1 e^{2t} + u_2 e^{-2t}$

$$\begin{cases} u_1' e^{2t} + u_2' e^{-2t} = 0 \\ u_1' 2e^{2t} - 2u_2' e^{-2t} = 2e^{3t} \end{cases}$$

$u_1 = \int_{t_0}^{t} \frac{1}{2} e^{\tau} d\tau = \frac{1}{2} e^{t} + c_1$; $u_2 = -\frac{1}{2}\int_{t_0}^{t} e^{5\tau} d\tau = -\frac{1}{10} e^{5t} + c_2$

5. (5 points) Use the definition of Laplace transform to compute $F(s)$, where

$$f(t) = \begin{cases} 0 & 0 < t \le 1 \\ 1 & t > 1. \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\, dt = \int_0^1 e^{-st} \cdot \cancel{0}\, dt + \int_1^{+\infty} e^{-st} dt$$

$$= -\frac{1}{s} \lim_{M \to +\infty} \left(e^{-sM} - e^{-s}\right) = +\frac{1}{s} \begin{cases} 0 + e^{-s}, & s > 0 \\ +\infty, & s < 0 \end{cases}$$

$s \neq 0$ $\quad$ $s \neq 0$

$$\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s} \quad s > 0$$

6. (6 points) Compute the Laplace transforms of each of the following functions

(a) (3 points) $2t^2 - t^2e^t + 2 - e^{-t} + 2\cos(3t) + \sin(6t)$. $= f(t)$

$\mathcal{L}\{f(t)\}$

$$= 2\mathcal{L}\{t^2\} - \mathcal{L}\{t^2 e^t\} + 2\mathcal{L}\{1\} - \mathcal{L}\{e^{-t}\} + 2\mathcal{L}\{\cos 3t\} + \mathcal{L}\{\sin 6t\}$$

by linearity

$$= 2\frac{2!}{s^3} - \frac{2!}{(s-1)^3} + \frac{2}{s} - \frac{1}{s+1} + 2\frac{s}{s^2+9} + \frac{6}{s^2+36}$$

(b) (3 points) $2e^{3t}t$, using s-translation property.

$$\mathcal{L}\{2e^{3t}t\} = 2\mathcal{L}\{e^{3t}t\} = 2\mathcal{L}\{t\}\Big|_{s=s-3}$$

s-translation

$$= 2\frac{1}{s^2}\Big|_{s=s-3} = \frac{2}{(s-3)^2}$$

7. (10 points) Compute the inverse Laplace transform of each of the following:

If you need to use partial fraction decomposition, compute the inverse without determining the values of the coefficients

(a) (5 points) $\frac{2s+1}{s^2(s-1)}$. $= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-1}$

$$\mathcal{L}^{-1}\left\{\frac{A}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{B}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{C}{s-1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2(s-1)}\right\} = At + B + Ce^{+t}$$

(b) (5 points) $\frac{5s+2}{s^2-2s+5}$. $\mathcal{L}^{-1}\left\{\frac{5s+2}{s^2-2s+5}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{5(s-1+1)+2}{(s-1)^2+4}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + \cancel{7\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}}$$

$$+7\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = 5e^{t}\cos 2t + \frac{7}{2}e^{t}\sin 2t$$

8. (8 points) Solve the following initial value problem

$$y' - \int_0^t y(\tau)d\tau = 1, \qquad y(0) = -1$$

$$\mathcal{L}\{y'\} - \mathcal{L}\{y(t) * 1\} = \mathcal{L}\{1\}$$

$$sY(s) - \overset{-1}{y(0)} - Y(s)\frac{1}{s} = \frac{1}{s} \qquad s \neq 0$$

$$Y(s)\left(s - \frac{1}{s}\right) = \frac{1}{s} - 1$$

$$Y(s)\left(\frac{s^2-1}{\cancel{s}}\right) = \frac{1-s}{\cancel{s}}$$

$$Y(s) = \frac{1-s}{s^2-1} = \frac{1-s}{(s-1)(s+1)} = -\frac{1}{s+1} =$$

$$\mathcal{L}^{-1}\{Y(s)\} = -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = -e^{-t}$$