1 (16 points) Consider the ODE

$$2x^2 y \frac{dy}{dx} = y^2 + 1 \qquad \text{with} \qquad x, y > 0. \tag{3}$$

(a) (5 points) Solve the ODE (3).

(b) (1 point) Write (3) in the form

$$\frac{dy}{dx} = f(x, y).$$

(c) (5 points) Can we guarantee that the solution of (3) exists and is unique if $x, y > 0$? Justify your answer using the **Existence and Uniqueness Theorem.**

(d) (5 points) If the initial condition for (3) is $y(0) = 2$, can we assure that the initial value problem has only one solution? Use the **Existence and Uniqueness Theorem** to explain briefly your answer.

a) $\dfrac{2y}{y^2+1}\,dy = \dfrac{dx}{x^2}$ Separable ODE

$\ln|y^2+1| = \ln|x^2| + c \Rightarrow y^2+1 = ce^{\ln x^2}$

$y^2+1 = cx^2$

b) $\dfrac{dy}{dx} = \dfrac{y^2+1}{2x^2y} = f(x,y)$

c) If $x, y > 0$ then $f(x,y) = \dfrac{y^2+1}{2x^2y}$ is continuous

and $\dfrac{\partial f}{\partial y} = \dfrac{2y(2x^2y) - 2x^2(y^2+1)}{4x^4y^2}$ is also

continuous

∴ Then, the existence and uniqueness theorem guarant that the solution exist and is unique

d) if $x=0,\ y=2$ then $\dfrac{\partial f}{\partial y}$ is not continuous (neither $f(x,y)$). Thus, the existence and uniqueness theorem fails to hold. Then, we cannot derive any conclusion concerning existence or uniqueness of solution

5 (17 points) Compute the inverse Laplace Transform of

(a) (4 points) $\frac{s-1}{s^2-4s+13}$.

(b) (7 points) $e^{-2s}\frac{1}{(s^2+1)(s-4)}$.

~~(c)~~ (6 points) $G(s)\frac{1}{s-2}$. ← Do not solve

a) $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2+1}{(s-2)^2+9}\right\} =$

$\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+9}\right\} = e^{2t}(\cos 3t + \sin 3t)$

b) $\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{(s^2+1)(s-4)}\right\} = u(t-2)\left\{\mathcal{L}^{-1}\left\{\frac{As+B}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{C}{s-4}\right\}\right\}_{t=t-2}$

$= u(t-2)\left[A\cos(t-2) + B\sin(t-2) + e^{4(t-2)}\right]$

c) $\mathcal{L}^{-1}\left\{G(s)\frac{1}{s-2}\right\} = \mathcal{L}^{-1}\{G(s)\} \underset{\substack{\uparrow \\ \text{convolution}}}{*} \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$

$= g(t) * e^{2t}$

$= \int_0^t g(t-\tau)e^{2\tau}\,d\tau$ or

$\int_0^t g(t)\,e^{2(t-\tau)}\,d\tau$

$g(t) = \mathcal{L}^{-1}\{G(s)\}$

6 (16 points) Consider the following piecewise continuous function

$$g(t) = \begin{cases} 0 & t < 1 \\ t & 1 \le t < 2 \\ 1 & t \geq 2 \end{cases}$$

(a) (2 points) Graph the function $g(t)$.

(b) (4 points) Write the function using the appropriate Heaviside functions.

(c) (5 points) Compute the Laplace Transform of $g(t)$ using (b) and the t-translation property.
If you can't solve (b) use $g(t) = (t-1)[u(t-3) - u(t-4)]$

(d) (5 points) Compute the Laplace Transform of $g(t)$ by definition.

a)

b) $g(t) = t\left[u(t-1) - u(t-2)\right] + u(t-2)$

c) $\mathcal{L}\{t.(u(t-1) - u(t-2)) + u(t-2)\}$

$= \mathcal{L}\{(t-1+1)u(t-1) - (t-1)u(t-2)\}$

$= \mathcal{L}\{(t-1)u(t-1)\} + \mathcal{L}\{u(t-1)\} - \mathcal{L}\{(t-2+1)u(t-2)\}$

$= e^{-s}\mathcal{L}\{t\} + \dfrac{e^{-s}}{s} - \mathcal{L}\{(t-2)u(t-2)\} - \mathcal{L}\{u(t-2)\}$

$= \dfrac{e^{-s}}{s^2} + \dfrac{e^{-s}}{s} - e^{-2s}\mathcal{L}\{t\} - \dfrac{e^{-2s}}{s} = e^{-s}\left(\dfrac{1+s}{s^2}\right) - e^{-2s}\left(\dfrac{1+s}{s^2}\right)$

$= \dfrac{1+s}{s^2}\left(e^{-s} - e^{-2s}\right), \quad s \neq 0$

d) $\mathcal{L}\{g(t)\} = \int_0^{+\infty} e^{-st} g(t)\,dt = \int_1^2 t e^{-st}\,dt + \int_2^{+\infty} e^{-st}\,dt \quad s \neq 0$

$= -e^{-st}\left(\dfrac{t}{s} + \dfrac{1}{s^2}\right)\Big|_1^2 - \dfrac{1}{s}\lim_{a\to+\infty} e^{-st}\Big|_2^{a} = $ * cont. $\longrightarrow$ **[illegible]**

$$= -e^{-2s}\left(\frac{2}{s}+\frac{1}{s^2}\right)+e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)-\frac{1}{s}\lim_{a\to+\infty} e^{-as}$$

$$+\frac{1}{s}e^{-2s} \quad ; \quad s\neq 0$$

a.c.

$$\lim_{a\to+\infty} e^{-as} = \begin{cases} 0 & \text{if } s>0 \\ \infty & \text{if } s<0 \end{cases}$$

$$\mathcal{L}\{g(t)\} = e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)+\frac{e^{-2s}}{s} - e^{-2s}\left(\frac{2}{s}+\frac{1}{s^2}\right)$$

whenever $s>0$

$$= e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)+e^{-2s}\left(-\frac{1}{s^2}-\frac{1}{s}\right)$$

$$= e^{-s}\left(\frac{s+1}{s^2}\right) - e^{-2s}\left(\frac{s+1}{s^2}\right) = \frac{s+1}{s^2}\left(e^{-s}-e^{-2s}\right)$$

7 (16 points) Compute

do not solve ← ~~(a)~~ (6 points) the Laplace transform of $f(t) = \dfrac{\int_0^t e^{-\tau}\, d\tau}{e^{-t}}$;

(b) (5 points) the inverse Laplace transform of $F(s) = \dfrac{1}{s(s-1)(s^2-4)}$;
If you need to use partial fraction decomposition you can give the solution without computing the coefficients

(c) (5 points) the solution of the following initial value problem

$$x'' - 4x = f(t) \qquad x(0) = 0 = x'(0),$$

where $f(t)$ is given in (a).
HINT: In (a) and (b) you did ALMOST all the work needed to answer this question.

a) $\mathcal{L}\{f(t)\} = ?$

$$f(t) = e^t \int_0^t e^{-\tau}\, d\tau = \int_0^t e^{t-\tau}\, d\tau = \underbrace{1 * e^t}_{\text{convolution}}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 * e^t\}$$

$$= \mathcal{L}\{1\}\, \mathcal{L}\{e^t\} \quad \text{by convolution theorem}$$

$$= \frac{1}{s}\, \frac{1}{s-1} = \frac{1}{s(s-1)}$$

b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s-1)(s^2-4)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{A}{s}\right\} + \mathcal{L}^{-1}\left\{\dfrac{B}{s-1}\right\} + \mathcal{L}^{-1}\left\{\dfrac{Cs+D}{s^2-4}\right\}$

$$= A + Be^t + C\cosh 2t + D\sinh 2t$$

c) $\mathcal{L}\{x'' - 4x\} = \mathcal{L}\{f(t)\}$, $\mathcal{L}\{x(t)\} = X(s)$

$$s^2 X(s) - s\overset{0}{x(0)} - \overset{0}{x'(0)} - 4X(s) = \frac{1}{s(s-1)} \leftarrow \text{computed in a)}$$

$$X(s)\,(s^2-4) = \frac{1}{s(s-1)}$$

$$X(s) = \frac{1}{s(s-1)(s^2-4)} \Rightarrow x(t) = A + Be^t + C\cosh 2t + D\sinh 2t$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} \quad \text{computed in b)}$$