Exam 2 - Solutions

1. (10 points) Consider the differential equation

$$y'' + 2y' + y = 0 \tag{1}$$

(a) (2 points) Write the associated characteristic equation

(b) (2 points) Determine two linearly independent solutions

(c) (3 points) Determine the real-valued general solution

(d) (3 points) Solve the initial value problem consisting of (1) and the initial condition

$$y(0) = 1 \qquad \text{and} \qquad y'(0) = 0$$

a) $r^2 + 2r + 1 = 0$

b) $(r+1)^2 = 0 \Rightarrow r = -1$ double

linearly independent solutions:

$y_1 = e^{-t}$; $y_2 = t e^{-t}$

c) $y = c_1 e^{-t} + c_2 t e^{-t}$

d) $y' = -c_1 e^{-t} + c_2 e^{-t}(1 - t)$

$y'(0) = 0 = -c_1 + c_2$

$y(0) = 1 = c_1 + 0$

$\Rightarrow \begin{cases} c_1 = 1 \\ c_2 = c_1 = 1 \end{cases}$

$y(t) = +e^{-t} + t e^{-t} = e^{-t}(1+t)$

2. (5 points) The roots of the characteristic equation associated to a homogeneous linear differential equation with constant coefficients are

$$r_1 = 3, r_2 = -3$$

Determine the corresponding differential equation.

characteristic equation $(r - r_1)(r - r_2) = 0$

$\Rightarrow (r-3)(r+3) = 0 \quad \Rightarrow \quad r^2 - 9 = 0$

ODE: $y'' - 9y = 0$

3. (5 points) Compute the real valued solution of the following differential equation

$$y'' + 2y' + 5y = 0$$

$r^2 + 2r + 5 = 0 \quad \Rightarrow (r+1)^2 + 4 = 0$

$r = -1 \pm 2i$

$y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

4. (5 points) Determine the form of a solution to

$$y'' - 4y' + 8y = e^{2t}\cos(2t)$$

$y'' - 4y' + 8y = 0$; $\quad r^2 - 4r + 8 = 0$

$(r-2)^2 + 4 = 0$

$r = 2 \pm 2i$

$y_c = c_1 e^{2t}\cos 2t + c_2 e^{2t}\cos 2t$

$y_p = (A e^{2t}\cos 2t + B e^{2t}\sin 2t)\, t$ duplication!!

$= A t e^{2t}\cos 2t + B t e^{2t}\sin 2t$

$= t e^{2t}(A\cos 2t + B\sin 2t)$

$$y(0) = \frac{3}{4} = c_1 + c_2 - \frac{1}{3}$$
$$y'(0) = \frac{1}{5} = 4c_2 - \frac{7}{8} - \frac{1}{3}$$
$$\Rightarrow \begin{cases} c_1 = \frac{3}{4} + \frac{1}{3} - \frac{7}{32} - \frac{1}{12} - \frac{1}{5} \\ c_2 = \frac{7}{32} + \frac{1}{12} + \frac{1}{5} \end{cases}$$

5. (18 points) Consider the differential equation

$$y'' - 4y' = 2x + 3 + e^x \tag{2}$$

(a) (2 points) Compute the general solution of the associated homogeneous differential equation

(b) (10 points) Determine a solution of (2)

(c) (2 points) Write the general solution of (2)
(**Hint:** using your answer in (a) and (b))

(d) (4 points) Solve the initial value problem consisting of (2) and the initial condition $y(0) = 3/4$ and $y'(0) = 1/5$

a) $y'' - 4y' = 0$

$r^2 - 4r = 0 \Rightarrow r(r-4) = 0 \Rightarrow r_1 = 0,\ r_2 = 4$

$y_c = c_1 + c_2 e^{4x}$

b) $y_{p1} = Ax + B$ $\qquad y'' - 4y' = 2x + 3$

y_{p1} and y_c have duplicate terms. Thus,

$y_{p1} = Ax^2 + Bx$

$y_{p2} = Ce^x$ $\qquad y'' - 4y' = e^x$

y_{p2} and y_c have no duplicate terms so y_{p2} is the right guess

$y_p = y_{p1} + y_{p2} = Ax^2 + Bx + Ce^x$. To find A, B, C

$y' = 2Ax + B + Ce^x$; $y''_p = 2A + Ce^x$

substitution into (2) yield

$2A + Ce^x - 4(2Ax + B + Ce^x) = 2x + 3 + e^x$

$2A - 4B - 8Ax - 3Ce^x = 2x + 3 + e^x$

Hence
$$\begin{cases} 2A - 4B = 3 \\ -8A = 2 \\ -3C = 1 \end{cases} \Rightarrow \begin{cases} A = -1/4 \\ C = -1/3 \\ B = -7/8 \end{cases} \qquad \boxed{y_p = -\frac{x^2}{4} - \frac{7}{8}x - \frac{1}{3}e^x}$$

c) $y(x) = c_1 + c_2 e^{4x} - \frac{x^2}{4} - \frac{7}{8}x - \frac{e^x}{3}$

d) $y'(x) = 4c_2 e^{4x} - \frac{1}{2}x - \frac{7}{8} - \frac{e^x}{3}$;

6. (10 points) Given that two solutions of the homogeneous linear equation

$$t^2y'' - 2y = 0, \quad t > 0$$

are

$$y_1(t) = t^2 \quad \text{and} \quad y_2(t) = t^{-1},$$

determine the general solution of the following differential equation using the variation of parameter method

$$t^2y'' - 2y = 3t^2 - 1, \quad t > 0$$

$$y_p = u_1 y_1 + u_2 y_2$$

solve the system: $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(t) \end{cases}$

$$y_1 = t^2 \Rightarrow y_1' = 2t$$

$$y_2 = t^{-1} \Rightarrow y_2' = -t^{-2} \qquad g(t) = 3t^2 - 1$$

Thus the system becomes

$$\begin{cases} u_1' t^2 + u_2' t^{-1} = 0 \\ u_1'(2t) + u_2'(-t^{-2}) = 3t^2 - 1 \end{cases} \Rightarrow \begin{cases} u_1' = -\frac{1}{t^3} u_2' \\ 2t\left(-\frac{1}{t^3} u_2'\right) - u_2' \cdot \frac{1}{t^2} = 3t^2 - 1 \end{cases}$$

$$\begin{cases} - \\ -\frac{2}{t^2} u_2' - \frac{1}{t^2} u_2' = 3t^2 - 1 \end{cases} \quad \begin{cases} - \\ u_2' = -\frac{1}{3}\left(3t^4 - t^2\right) \end{cases} \Rightarrow \begin{cases} u_1' = \frac{1}{3t^3}\left(3t^4 - t^2\right) \\ - \end{cases}$$

$$u_1' = \frac{1}{3}\left(3t - \frac{1}{t}\right) \Rightarrow u_1 = \frac{1}{3}\left(\frac{3t^2}{2} - \ln t\right)$$

$$u_2' = -\frac{1}{3}\left(3t^4 - t^2\right) \qquad u_2 = -\frac{1}{3}\left(\frac{3t^5}{5} - \frac{t^3}{3}\right)$$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{1}{3}\left(\frac{3t^2}{2} - \ln t\right)t^2 - \frac{1}{3}\left(\frac{3t^5}{5} - \frac{t^3}{3}\right)\frac{1}{t}$$

$$y_p = \frac{1}{3}\left(\frac{3}{2} t^4 - t^2 \ln t - \frac{3t^4}{5} + \frac{t^2}{3}\right)$$

$$y_p = \frac{t^4}{2} - \frac{t^2}{3}\ln t - \frac{t^4}{5} + \frac{t^2}{9}$$

$$y = c_1 t^2 + c_2 t^{-1} + \frac{t^4}{2} - \frac{t^2}{3}\ln t - \frac{t^4}{5} + \frac{t^2}{9}$$

7. (5 points) Compute the Laplace transform of $f(t) = e^t \cos 2t + 5e^{3t}$.

$$\mathcal{L}\{e^t \cos 2t + 5e^{3t}\} \underset{\text{Linearity}}{=} \mathcal{L}\{e^t \cos 2t\} + 5\mathcal{L}\{e^{3t}\}$$

2 2

Table

$$= \frac{s-1}{(s-1)^2+4} + \frac{5}{s-3}$$

2 1

8. (10 points) Consider the following function

$$f(t) = \begin{cases} e^t & 0 \le t \le 2 \\ 3 & t > 2 \end{cases}$$

Compute the Laplace transform of $f(t)$ **using the definition.**

$$\mathcal{L}\{f(t)\} = \int_0^2 e^{-st} e^t\,dt + \int_2^{+\infty} 3e^{-st}\,dt \quad 1$$

$$= \int_0^2 e^{(1-s)t}\,dt + 3\lim_{a\to+\infty}\int_2^a e^{-st}\,dt \quad 1 \quad 1$$

$$= \left.\frac{e^{(1-s)t}}{1-s}\right|_0^2 + \frac{3}{-s}\lim_{a\to+\infty} \left. e^{-st}\right|_2^a \quad 1 \quad 1$$

$s \neq 1$ $s \neq 0$

$$= \frac{e^{2(1-s)}-1}{1-s} - \frac{3}{s}\lim_{a\to+\infty}\left(e^{-sa} - e^{-2s}\right) \quad s\neq 0,1 \quad 1 \quad 1$$

$$= \frac{1-e^{2(1-s)}}{s-1} - \frac{3}{s}\begin{cases} -e^{-2s} & \text{if } s>0 \quad 1 \\ \infty & \text{if } s<0 \quad 1 \end{cases}$$

$$= \frac{1-e^{2(1-s)}}{s-1} + \frac{3e^{-2s}}{s} \quad s>0,\ s\neq 1 \quad 1$$