ODE Exam I - sol.

#1 $\frac{1}{y}\frac{dy}{dx} = \frac{2}{1+2x} \Rightarrow \int \frac{dy}{y} = \int \frac{2}{1+2x}dx + c$

$\Rightarrow \ln|y| = \ln|1+2x| + c$

$\ln\left|\frac{y}{1+2x}\right| = \ln c \Rightarrow \frac{y}{1+2x} = c$

$y = (1+2x)\,c$

$\begin{cases} y(0) = e \\ y = c(1+2x) \end{cases} \Rightarrow c = e$

solution to the IVP: $y = e(2x+1)$

$\lim_{x\to+\infty} y(x) = \lim_{x\to+\infty} e(2x+1) = +\infty$

#2 $\frac{dy}{dt} + \frac{1}{t}y = \frac{2}{\sqrt{1+t^2}}$ $t \neq 0$

a) First order linear (in y) ODE

b) $\mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln t} = t$

$\frac{d}{dt}(ty) = \frac{2t}{\sqrt{1+t^2}} \Rightarrow ty = \int 2t(1+t^2)^{-1/2}dt + c$

$ty = 2\sqrt{1+t^2} + c \Rightarrow y = \frac{2}{t}\sqrt{1+t^2} + \frac{c}{t}$

c) $y(-1) = 0 = \frac{2}{-1}\sqrt{1+1} + \frac{c}{-1} \Rightarrow c = -2\sqrt{2}$

solution to IVP: $y = \frac{2}{t}\left(\sqrt{1+t^2} - \sqrt{2}\right)$

~~d)~~

#2 d) $P(t) = \frac{1}{t}$ defined if $t \neq 0$

$Q(t) = \frac{2}{\sqrt{1+t^2}}$ defined for all t in $\mathbb{R}$

~~Time~~ interval of definition needs to contain $t=-1$, which is the initial time, and cannot contain $t=0$. Thus, the interval of existence is $(-\infty, 0)$

#3 $\frac{dx}{dy} = \frac{1+2x}{2y} \Rightarrow \frac{dx}{dy} + \left(-\frac{1}{y}\right)x = \frac{1}{2y}$

thus ODE can be written in the form $\frac{dx}{dt} + P(y)x = Q(x)$. Thus, it is linear in X

#4 a) $\frac{dy}{dx} = \frac{y^2+1}{2x^2 y}$

b) $f(x,y) = \frac{y^2+1}{2x^2 y}$ is continuous whenever $x,y > 0$

$\frac{\partial f}{\partial y} = \frac{2y(2x^2y) - 2x^2(y^2+1)}{4x^4y^2}$ is continuous if $x,y > 0$

Therefore, by the existence and uniqueness theorem there exists only one solution to the IVP when ~~as long as~~ $x,y > 0$

4 c) $f(x,y)$ is discontinuous at $x = 0$ which is the value of our initial x ($y(0) = 2$). Therefore, the thm can't assume that the IVP has only one solution

#5 The solution needs to satisfy the initial condition $y(0)=1$

let's check if it does!

Is $y \stackrel{?}{=} 3(e^0+1)$? No: $1 \neq 6$

Therefore, the function $y = 3(e^{-x^2}+1)$ is not a solution of the IVP.

#6

a) Write the ODE as: $(3t+\cos y)\,dy + (3y+e^t)\,dx = 0$

$$\frac{\partial}{\partial t}(3t+\cos y) = 3$$

$$\frac{\partial}{\partial y}(3y+e^t) = 3$$

since $\frac{\partial}{\partial t}(3t+\cos y) = \frac{\partial}{\partial y}(3y+e^t)$ the ODE is exact

b) $$\frac{\partial \psi}{\partial y} = 3t+\cos y$$

$$\frac{\partial \psi}{\partial t} = 3y+e^t \Rightarrow 3yt+e^t+h(y) = \psi$$

solve

$$\frac{\partial}{\partial y}\left(3yt+e^t+h(y)\right) = 3t+\cos y$$

$$\cancel{3t}+h'(y) = \cancel{3t}\ \cos y \Rightarrow h'(y) = \cos y$$

$$h(y) = +\sin y$$

$\psi(t,x) = 3yt+e^t+\sin y$

solution of the ODE: $3yt+e^t+\sin y = c$

$y(0) = \frac{\pi}{2} \quad \left((t, x) = (0, \frac{\pi}{2}) \right)$

Substitute into the solution of the ODE leads:

$$0 + 1 + \sin \frac{\pi}{2} = c \implies c = 2$$

solution of IVP:

$$3yt + e^t + \sin y = 2$$