MATH 2650 - Exam 2 Solutions

1 (6 points) Determine the equation for the plane tangent to the surface $x^2 + xy^2 + xyz = 10$ at $(2, 1, 2)$

$\nabla f = \langle 2x + y^2 + yz,\ 2xy + xz,\ xy \rangle$

$\nabla f|_{2,1,2} = \langle 4+1+2,\ 4+4,\ 2 \rangle = \langle 7, 8, 2 \rangle$

Eq. of the tangent plane: $7(x-2) + 8(y-1) + 2(z-2) = 0$

$7x + 8y + 2z = 26$

2. (10 points) Consider the function $T(x, y) = x^2 + 2y^2 - x$ defined on the region $x^2 + y^2 \leq 1$ Determine maximum, minimum, and sladdle points of T in the interior region What are the global maximum and global minimum?

<u>interior region</u> $T_x = 0 = T_y$ give critical points

$T_x = 2x - 1 = 0 \Rightarrow x = 1/2$ critical point: $(\frac{1}{2}, 0)$

$T_y = 4y = 0 \quad y = 0$

2nd Derivative test: Hessian $= T_{xx}T_{yy} - T_{xy}^2 = 2(4) - 0 = 8 > 0$

$\therefore$ The critical point $(\frac{1}{2}, 0)$ gives a minimum. $T(\frac{1}{2}, 0) = -\frac{1}{4}$

no saddles

<u>Global extrema</u> From previous study we have a potential global extrema at $(\frac{1}{2}, 0)$: $T(\frac{1}{2}, 0) = -\frac{1}{4}$

$\longrightarrow$ cont.

3 (10 points) Use Lagrange multipliers method to compute the maximum and minimum values of the function

$$f(x, y, z) = x^2 - y^2 \Rightarrow \nabla f = 2x\,i - 2y\,j$$

subject to th econstraint $x^2 + y^2 = 2 \Rightarrow g(x,y) = x^2 + y^2 - 2$; $\nabla g = 2x\,i + 2y\,j$

Lagrange multipliers: $\begin{cases} \nabla f = \lambda \nabla g \\ x^2 + y^2 = 2 \end{cases}$ $\begin{cases} \cancel{2}x = \lambda \cancel{2}x & (a) \\ -2y = \lambda 2y & (b) \\ x^2 + y^2 = 2 & (c) \end{cases}$

from (a) $x - \lambda x = 0 \Rightarrow x = 0$ or $\lambda = 1$

$\lambda = 1$: from (b) $y = 0$. Then from (c) $x^2 + 0 = 2 \Rightarrow x = \pm\sqrt{2}$

so far: $(+\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$

$x = 0$; from (b) $\lambda = -1$ and from (c) $y^2 = 2 \Rightarrow y = \pm\sqrt{2}$

list of points candidates for max and min: $(\pm\sqrt{2}, 0)$, $(0, \pm\sqrt{2})$

$f(\pm\sqrt{2}, 0) = 2 = f(-\sqrt{2}, 0)$, $f(0, -\sqrt{2}) = -2 = f(0, +\sqrt{2})$

max: $f(\sqrt{2}, 0) = f(-\sqrt{2}, 0) = 2$ min: $f(0, -\sqrt{2}) = f(0, \sqrt{2}) = -2$

2 cont.

study the boundary points.

we need to study $T(x,y) = x^2 + 2y^2 - x$ with the constrain: $x^2 + y^2 = 1$ (a)

solve (a), for example, for y^2: $y^2 = 1 - x^2$ (b)

substitute into $T(x,y)$: $T(x) = x^2 + 2(1-x^2) - x$

$= -x^2 - x + 2, \quad -1 \le x \le 1$

on the interval $[-1,1]$, $T(x)$ has a critical point:

$T_x(x) = -2x - 1 = 0 \Rightarrow x = -\frac{1}{2}$

substituting into (b): $y = \pm \frac{\sqrt{3}}{2}$

critical points: $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the end points of the interval $x = -1, y = 0$; $x = 1$ $y = 0$

list of points at which $T(x,y)$ have extreme values:

$(-1,0), (1,0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, 0\right)$

Now we evaluate T at thus points:

$T(-1,0) = 1 + 0 - (-1) = 2$; $T(1,0) = 1 + 0 - 1 = 0$

$T\left(\frac{1}{2}, 0\right) = -\frac{1}{4}$; $T\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = \frac{1}{4} + 2\frac{(3)}{4} - \left(-\frac{1}{2}\right) = \frac{9}{4}$

Thus global maxi ~~ma~~ are $T\left(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\right) = \frac{9}{4} = T\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

global minimum is $T\left(\frac{1}{2}, 0\right) = -\frac{1}{4}$

4 (6 points) Compute the following integral

$$\int_0^1 \int_x^1 \int_0^{y-x} \cancel{dydxdz}\; dz\,dy\,dx$$

$$= \int_0^1 \int_x^1 z\Big|_0^{y-x} dy\,dx = \int_0^1 \int_x^1 (y-x)\,dy\,dx = \int_0^1 \left[\frac{y^2}{2} - yx\right]_x^1 dx$$

$$= \int_0^1 \left[\frac{1}{2} - x - \left(\frac{x^2}{2} - x^2\right)\right] dx = \int_0^1 \left(\frac{1}{2} + \frac{x^2}{2} - x\right) dx$$

$$= \left[\frac{1}{2}x + \frac{x^3}{6} - \frac{x^2}{2}\right]_0^1 = \cancel{\frac{1}{2}} + \frac{1}{6} - \cancel{\frac{1}{2}} = \frac{1}{6}$$

5. (10 points) Evaluate the following iterated integral:

$$I = \underbrace{\int_{-3}^{0} \int_0^{\sqrt{9-y^2}}}_{R} \sqrt{x^2+y^2}\,dx\,dy$$

change to polar coordinates

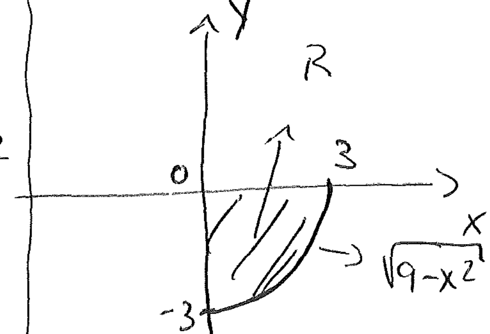
$x = R\cos\theta$, $y = R\sin\theta$, $R^2 = x^2+y^2$

$x = \sqrt{9-y^2} \Rightarrow x^2+y^2 = 9 \Rightarrow \underset{R>0}{R = 3}$

$$I = \int_{\frac{3\pi}{2}}^{2\pi} \int_0^3 R\,(R\,dR)\,d\theta$$

$$= \int_{3\pi/2}^{2\pi} \frac{R^3}{3}\Big|_0^3 d\theta = \int_{3\pi/2}^{2\pi} 9\,d\theta$$

$$= 9\,\theta\Big|_{3\pi/2}^{2\pi} = 9\,\frac{\pi}{2}$$



$\frac{3\pi}{2} \le \theta \le 2\pi$

$0 \le R \le 3$

6 (18 points) Consider the region in the following figure

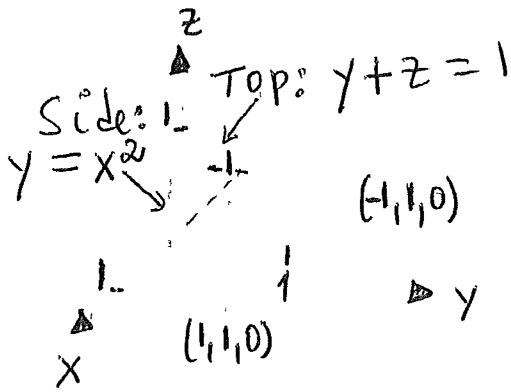


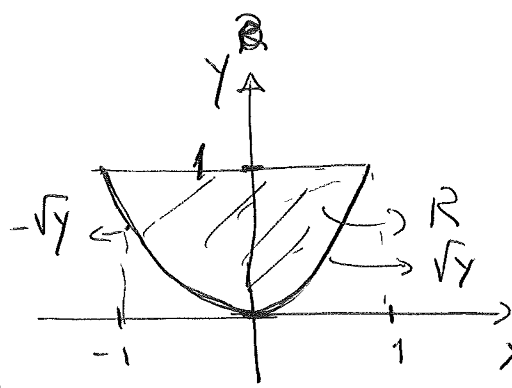
Figure 1

(a) (8 points) Write an iterated triple integral in the order $dzdxdy$ that gives the volume of the region in Figure 1 (**DO NOT Solve the integral**).

(b) (10 points) Suppose that the solid shown in Figure 1 has constant density ρ. Write iterated triple integrals that allow the computation of the nonzero first moments about the coordinate planes (**DO NOT Solve the integrals**)

a) $$V = \underbrace{\int_{y_1}^{y_2}\int_{x_1}^{x_2}}_{R}\int_{z_1}^{z_2} dz\,dx\,dy$$

$$V = \int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{1-y} dz\,dx\,dy$$



$y = x^2 \Rightarrow x = \pm\sqrt{y}$

$y+z=1 \Rightarrow z = 1-y$

b) $$M_{xy} = \iiint_D \rho z\,dV = \int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{1-y} \rho z\,dz\,dx\,dy$$

$$M_{xz} = \iiint_D \rho y\,dV = \int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{1-y} \rho y\,dz\,dx\,dy$$

$$M_{yz} = \iiint_D \rho x\,dV = 0 \leftarrow \text{symmetry in } x\text{-axis}$$