# Exam 1 - Solutions

1. (8 points) Compute the following limit if it exists. Otherwise show that it does not exist

$$\lim_{(x,y)\to(1,2)} \frac{-1+x+xy-y}{x-1} = \frac{-1+1+2-2}{1-1} = 0/0$$

simplify expression: $\frac{x-1+y(x-1)}{x-1} = \frac{\cancel{(x-1)}(1+y)}{\cancel{x-1}} = 1+y$

$$\lim_{(x,y)\to(1,2)} \frac{-1+x+xy-y}{x-1} = \lim_{(x,y)\to(1,2)} (1+y) = 3$$

2 (8 points) Compute the equation for the tangent line to the curve:

$$r(t) = (t + \cos t)\mathbf{i} + \sin t\mathbf{j} + e^t\mathbf{k}$$

at $t = 0$.

$R'(t) = (1-\sin t)\hat{i} + \cos t\,\hat{j} + e^t\,\hat{k}$

$R'(0) = (1-\sin 0)\hat{i} + \cos 0\,\hat{j} + e^0\,\hat{k} = \hat{i}+\hat{j}+\hat{k}$

$R'(0) = \langle 1,1,1\rangle$ ; $R_0 = \langle 1,0,1\rangle$

$t=0 \Rightarrow \begin{cases} x = 0+\cos 0 = 1 \Rightarrow \\ y = \sin 0 = 0 \\ z = e^0 = 1 \end{cases}$

eq tangent line: $T(t) = \langle 1,0,1\rangle + t\langle 1,1,1\rangle$

3. (10 points) Consider the following function:

$$g(r, s, t) = rst + \ln(r + 3t)$$

(a) (2 ponts) Determine its domain.

(b) (8 points) Compute $\frac{\partial^2 g}{\partial t \partial r}$.

a) all points $(r,s,t)$ in the space such that $r+3t>0$

b) $\frac{\partial^2 g}{\partial t\,\partial r} = \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial r}\right) = \frac{\partial}{\partial t}\left[st + \frac{1}{r+3t}\right]$

$= s - \frac{3}{(r+3t)^2} = \frac{s(r+3t)^2 - 3}{(r+3t)^2}$

4. (8 points) Compute the derivative $\frac{\partial x}{\partial z}$ of

$$z^2 - yz + xz - xy = 5 \quad \text{at} \quad (1, -1, 1).$$

$; \; F(x,y,z) = z^2 - yz + xz - xy - 5$

$\frac{\partial x}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial x}$ ;

$\frac{\partial F}{\partial z} = 2z - y + x$

$\frac{\partial F}{\partial x} = z - y$

$\frac{\partial x}{\partial z} = -\frac{2z - y + x}{z - y}$

$\frac{\partial x}{\partial z} = \frac{2z - y + x}{y - z}$

$\boxed{\frac{\partial x}{\partial z}\Big|_{(1,-1,1)} = \frac{2+1+1}{-1-1} = -2}$

5. (12 points) The temperature at $(x, y)$ is $T(x, y) = \sqrt{x^2 + y^2}$.

(a) (6 points) What is the rate of change of $T$ at $(4, 3)$ in the direction $\langle -4, 3 \rangle$.

(b) (3 points) Give one possible direction in which a heat-seeking bug should move from the point $(4, 3)$ to maintain its temperature? → see next page

(c) (3 points) What is the magnitude of the bug's greatest temperature increment? at $(4,3)$

a) $(D_u T)_{(4,5)} = (\nabla T)_{(4,3)} \cdot \frac{\langle -4, 3 \rangle}{\sqrt{16+9}}$

$V = \langle -4, 3 \rangle$

$|V| = \sqrt{16+3}$

$u = \frac{V}{|V|}$

$(\nabla T)_{(4,3)} = \frac{\partial T}{\partial x}\Big|_{(4,3)} \hat{i} + \frac{\partial T}{\partial y}\Big|_{(4,3)} \hat{j}$

$(\nabla T)_{(4,3)} = \frac{1}{\cancel{2}} \frac{\cancel{2}x}{\sqrt{x^2+y^2}}\Big|_{(4,3)} \hat{i} + \frac{1}{\cancel{2}} \frac{\cancel{2}y}{\sqrt{x^2+y^2}}\Big|_{(4,3)} \hat{j} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$

$(D_u T)_{(4,5)} = \langle \frac{4}{5}, \frac{3}{5} \rangle \cdot \langle -\frac{4}{5}, \frac{3}{5} \rangle = -\frac{16}{25} + \frac{9}{25} = -\frac{7}{25}$

6. (8 points) Compute the derivative $\frac{\partial u}{\partial t}$ of the following function:

$$u = e^x \cos y; \qquad x = s^2 t^3; \qquad y = \frac{s}{t}.$$

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = e^x \cos y (3s^2t^2) - e^x \sin y \left(-\frac{s}{t^2}\right)$

$= 3e^x \cos y \, s^2 t^2 + \frac{s}{t^2} e^x \sin y$

$= 3e^{s^2t^3} \cos\left(\frac{s}{t}\right) s^2 t^2 + \frac{s}{t^2} e^{s^2t^3} \sin\left(\frac{s}{t}\right)$

$= e^{s^2t^3} s \left[3s\cancel{t}^2 \cos\left(\frac{s}{t}\right) + \frac{1}{t^2} \sin\left(\frac{s}{t}\right)\right]$

Scratch Paper

#5 b) to maintain the temperature means that the rate of change of T is zero.

Thus $(\nabla T)_{(3,4)} \cdot n = 0$, where $n$ gives the direction of movement

Since $(\nabla T)_{(3,4)} = \langle \frac{4}{5}, \frac{3}{5} \rangle$

$\langle \frac{4}{3}, \frac{3}{5} \rangle \cdot n = 0$. One possible direction is $n = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

#5 c) The greatest increment of temperature is

$$|(\nabla f)_{(3,4)}| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \frac{1}{5}\sqrt{16+9}$$

$$= 1$$

**Extra-credit problem** (12 points)
Consider the function

$$F(t) = \int_{t^2}^{0} \sqrt{t^3 + x^2}\, dx$$

Compute $F'(t)$.

$$F(t) = -\int_0^{t^2} \sqrt{t^3+x^2}\ dx$$

$$F'(t) = -\left[ \sqrt{t^3+(t^2)^2}\, 2t + \int_0^{t^2} \frac{1}{2} \frac{3t^2}{\sqrt{t^3+x^2}}\, dx \right]$$

$$F'(t) = -2t\sqrt{t^3+t^4} \quad \frac{3}{2} \int_0^{t^2} \frac{t^2}{\sqrt{t^3+x^2}}\, dx$$