The Hitchhiker's Guide to Linear Algebra

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Contents

Comments and examples included in these notes are not sufficient for a complete course on linear algebra. They do not replace the lecture notes nor the textbook. They are mere elaborations on some of the concepts, ideas, techniques, examples and exercises that are discussed in the text and lectures. Students are expected to attend all the lectures and take notes and study them and read the textbook and do all the recommended exercise.

1 Systems of Linear Equations

1.0.1. Linear Equations vs nonlinear Equations Which of the following equations is linear and which is nonlinear? Why?

$$3x_1 - 4x_2 + \sqrt{2}x_3 = 4 \tag{1}$$
$$-x_1 + 2\pi x_2 + 5x_3 = 12 \tag{2}$$
$$3x_1 - 4x_2 + 2\sqrt{x_3} = 4 \tag{3}$$
$$4x_1x_3 - 4x_2 + 8x_3 = -3 \tag{4}$$

A linear equation takes the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where $a_1, a_2, \cdots, a_n$ and c are constants.

1. The constants $a_1, a_2, \cdots a_n$ and c are called the coefficients.
2. On the other hand $x_1, x_2, \cdots$ and x_n are the variables that we want to solve for.
3. A solution of a linear equation is a collection of real numbers that when substituted for $x_1, x_2, \cdots$ and x_n we obtain the value c on the right hand side.

1.0.2 Example. 2×2 **systems.** Solve the system

$$\begin{cases} x & -2y & = 5 \\ 2x & +6y & = 8 \end{cases} \tag{1.1}$$

Solution.

Notice that these are the equations of two straight lines. The slope

$$m = -\frac{\text{coefficient of } y}{\text{coefficient of } x}, \qquad m_1 = 2, \qquad m_2 = 3, \qquad 2 \neq 3$$

The two line are not parallel because the slopes are different. Therefore, the must intersect at a unique point. The components of this point give us a unique solution for the system. Now we try to find this point.

$$\xrightarrow{[-2Eq(1)+Eq(1)]\mapsto Eq(1)} \begin{cases} x & -2y & = 5 \\ & +10y & = -2 \end{cases} \longrightarrow \begin{cases} x & -2y & = 5 \\ & +y & = -1/5 \end{cases} \longrightarrow \begin{cases} x & = 23/5 \\ y & = -1/5 \end{cases}$$

In this example the system has a unique solution. Geometrically, the two lines intersect at the point $(23/5, -1/5)$.

1.0.3 Example. Solve the system

$$\begin{cases} x & +3y & = 5 \\ 2x & +6y & = 8 \end{cases} \tag{1.2}$$

- In this case the two lines have the same slope $m = 3$.

- This means they are parallel.
- But they have different y-intercepts: 5/3 and 4.
- We conclude from this observation that the two lines do not intersect.
- Thus, the system does not have a solution.
- A system that has no solution is said to be **inconsistent**.

Let's see what happens if we try to solve the system:

$$\xrightarrow{[-2Eq(1)+Eq(1)]\mapsto Eq(1)} \begin{cases} x & +3y & = 5 \\ & 0 & = -2 \end{cases}$$

The last equation is **inconsistent** with what we know. Namely, that $2 \neq 0$. This way we can conclude algebraically that the system does not have any solutions.

1.1 Gauss-Jordan Elimination

1.1.1 Example (Sometimes the linear system has a unique solution)**.** Solve the following system and describe the solution set.

$$\begin{cases} x_1 & -3x_2 & +4x_3 & = 8 \\ x_1 & -x_2 & -2x_3 & = 2 \\ 2x_1 & -5x_2 & +3x_3 & = 15 \end{cases} \tag{1.3}$$

Answer. First use Eq(1) to eliminate x_1 from Eq(2) and Eq(3):

$$\xrightarrow[{[-2Eq(1)+Eq(3)]\mapsto Eq(3)}]{[-Eq(1)+Eq(2)]\mapsto Eq(2)} \begin{cases} x_1 & -3x_2 & +4x_3 & = 8 \\ & 2x_2 & -6x_3 & = -6 \\ & x_2 & -5x_3 & = -1 \end{cases} \xrightarrow{Eq(2)\leftrightarrow Eq(3)} \begin{cases} x_1 & -3x_2 & +4x_3 & = 8 \\ & x_2 & -5x_3 & = -1 \\ & 2x_2 & -6x_3 & = -6 \end{cases}$$

Use Eq(2) to eliminate x_2 from Eq(3):

$$\xrightarrow{[-2Eq(2)+Eq(3)]\mapsto Eq(3)} \begin{cases} x_1 & -3x_2 & +4x_3 & = 8 \\ & x_2 & -5x_3 & = -1 \\ & & x_3 & = -1 \end{cases}$$

Use Eq(3) to eliminate x_3 from Eq(1) and Eq(2):

$$\begin{cases} x_1 & -3x_2 & & = 12 \\ & x_2 & & = -6 \\ & & x_3 & = -1 \end{cases}$$

Use Eq(2) to eliminate x_2 from Eq(1):

$$\begin{cases} x_1 & & & = -6 \\ & x_2 & & = -6 \\ & & x_3 & = -1 \end{cases}$$

Thus the solution set of (1.3) is a unique point:

$$\mathbf{S}: \qquad \{\mathbf{x} = \begin{pmatrix} -6 \\ -6 \\ -1 \end{pmatrix}\}$$

1.1.2. What does solving the system (1.3) mean? It means that if we substitute $x_1 = -6, x_2 = -6$ and $x_3 = -1$ in the 3 equations of (1.3), we obtain the values 8, 2 and -1 respectively.

In vector form, the system (1.3) can be written in the equivalent vector form

$$x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ -1 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 15 \end{pmatrix} \tag{1.4}$$

Finding a solution $\mathbf{x} = (-3, -4, -1)^\intercal$ to (1.4), equivalently to the system (1.3), means that

$$-6 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 6 \begin{pmatrix} -3 \\ -1 \\ -5 \end{pmatrix} - 1 \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 15 \end{pmatrix} \tag{1.5}$$

In other words (1.5) shows that we can express the vector $(8, 2, 15)^\intercal$ as a linear combination of the three vectors $(1, 1, 2)^\intercal$, $(-3, -1, -5)^\intercal$ and $(4, -2, 3)^\intercal$ in only one way with coefficients $-6, -6$ and -1 respectively.

1.1.3 Example (Sometimes a linear system has no solution)**.** Solve the following system

$$\begin{cases} x_1 & -3x_2 & +4x_3 & = 2 \\ x_1 & -x_2 & -2x_3 & = 3 \\ -3x_1 & +10x_2 & -15x_3 & = -7 \end{cases} \tag{1.6}$$

Answer. Use Eq(....) to eliminate x_1 from Eq(2) and Eq(....):

$$\xrightarrow[(3Eq_1+Eq_3)\mapsto Eq_3]{(-Eq_1+Eq_2)\mapsto Eq_2} \begin{cases} x_1 & -3x_2 & +4x_3 & = 2 \\ & +2x_2 & -6x_3 & = 1 \\ & x_2 & -3x_3 & = -1 \end{cases}$$

Switch Eq(....) with Eq(....):

$$\xrightarrow{Eq......\leftrightarrow Eq......} \begin{cases} x_1 & -3x_2 & +4x_3 & = 2 \\ & +x_2 & -3x_3 & = -1 \\ & +2x_2 & -6x_3 & = 1 \end{cases}$$

Use Eq(....) to eliminate x_2 from Eq(3):

$$\xrightarrow{............+...........\mapsto Eq......} \begin{cases} x_1 & -3x_2 & +4x_3 & = 2 \\ & +x_2 & -3x_3 & = -1 \\ & & 0 & = 3 \end{cases}$$

The Third equation is $0 = 3$ which is known to be false. Thus the system (1.6) has no solution. In this case we call the system (1.6) **inconsistent**.

1.1.4. Inconsistent systems.

A linear system that has **no solution** is called **inconsistent**

1.1.5 Example (Sometimes a linear system has infinitely many solutions.)**.** Solve the following system and describe the solution set.

$$\begin{cases} & 2x_2 & -4x_3 & = -2 \\ x_1 & -3x_2 & +10x_3 & = 5 \\ x_1 & -x_2 & +6x_3 & = 3 \end{cases} \tag{1.7}$$

Solution.

Switch the the two equations so that we have x_1 with coefficient 1 in the upper left corner:

$$\xrightarrow{Eq(1)\leftrightarrow Eq(2)} \begin{cases} x_1 & -3x_2 & +10x_3 & = 5 \\ & 2x_2 & -4x_3 & = -2 \\ x_1 & -x_2 & +6x_3 & = 3 \end{cases}$$

Divide Eq(2) by 2:

$$\xrightarrow{Eq(2)/2} \begin{cases} x_1 & -3x_2 & +10x_3 & = 5 \\ & x_2 & -2x_3 & = -1 \\ x_1 & -x_2 & +6x_3 & = 3 \end{cases}$$

Use Eq(1) to eliminate x_1 from Eq(3):

$$\xrightarrow{(-Eq(1)+Eq(3))\mapsto Eq(3)} \begin{cases} x_1 & -3x_2 & +10x_3 & = 5 \\ & x_2 & -2x_3 & = -1 \\ & 2x_2 & -4x_3 & = -2 \end{cases}$$

Use Eq(2) to eliminate x_2 from Eq(1) and Eq(3):

$$\xrightarrow[(-2Eq(1)+Eq(3))\mapsto Eq(3)]{(3Eq(2)+Eq(1))\mapsto Eq(1)} \begin{cases} x_1 & & +4x_3 & = 2 \\ & x_2 & -2x_3 & = -1 \\ & & 0 & = 0 \end{cases} \tag{1.8}$$

Remarks. We make a few remarks before we proceed with the solution.

1. The third equation is correct but useless. But we can see now that we have reduced the original *three equations with 3 unknowns* of the linear system (1.7) to a system of only *two equations with the same 3 unknowns* in (1.8). This means that the original 3 equations are not completely independent from each other. There is some redundancy among them.

2. Notice that the linear system (1.8) cannot be reduced to one equation. That is we can not reduce the second equation to the trivial equation $0 = 0$.

   Exercise Explain.

3. Notice that if we specify a value for x_3, say $x_3 = 2$, then we have solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 - 4x_3 \\ -1 + 2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}$$

   If instead we choose $x_3 = -3$ we obtain a different solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 - 4x_3 \\ -1 + 2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 14 \\ -7 \\ -3 \end{pmatrix}$$

4. There is nothing special about the real numbers 2 and -3. We can take x_3 to be any real number, say $x_3 = r$ and obtain a solution.

Answer.

- We can see that x_3 is a free variable that can take any arbitrary value $x_3 = r \in \mathbb{R}$.
- Thus the **solution set** (the collection of all possible solutions) of the system (1.7) can be written in **parametric form** as

$$\mathbf{S}: \quad \begin{aligned} x_1 &= 2 - 4r \\ x_2 &= -1 + 2r, \qquad -\infty < r < \infty \\ x_3 &= r \end{aligned} \tag{1.9}$$

  We also write the solution set in **parametric vector form** as

$$\mathbf{S}: \qquad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + r \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}, \qquad -\infty < r < \infty \tag{1.10}$$

- **Description of solution set.** Equation (1.10) is the equation of a straight line in the 3-dimensional space $\mathbb{R}^3$: It is the straight line through the point $(2, -1, 0)^\intercal$ and parallel to the vector $(-4, 2, 1)^\intercal$.

1.1.6. Parametric equation of a line. The equation of a line that passes through a point $\mathbf{x}_o$ and parallel the the vector $\mathbf{v}$ is

$$\mathbf{x} = \mathbf{x}_o + t\mathbf{v}, \qquad -\infty < t < \infty$$

1.1.7 Example. Solve the following system and describe the solution set and find its dimension.

$$\begin{pmatrix} 3 \\ -6 \\ 15 \\ 12 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ -3 \\ 4 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 4 \end{pmatrix} \tag{1.11}$$

<u>Solution</u>.

First we write the vector equation as a linear system:

$$\left\{ \begin{array}{rrrrcr} x_1 & -3\,x_2 & +x_3 & +x_4 & = & 3 \\ 2\,x_1 & -6\,x_2 & -x_3 & +x_4 & = & -6 \\ -3\,x_1 & +9\,x_2 & +3\,x_3 & -x_4 & = & 15 \\ 4\,x_1 & -12\,x_2 & +4\,x_3 & +4x_4 & = & 12 \end{array} \right. \tag{1.12}$$

Use Eq(1) to eliminate x_1 from the other three equations

$$\left\{ \begin{array}{ll} (-2Eq_1 + Eq_2) & \mapsto Eq_2 \\ (3Eq_1 + Eq_3) & \mapsto Eq_3 \\ (-4Eq_1 + Eq_4) & \mapsto Eq_4 \end{array} \right. \Longrightarrow \left\{ \begin{array}{lllll} x_1 & -3x_2 & +x_3 & +x_4 & = 3 \\ & & -3x_3 & -x_4 & = -12 \\ & & 6x_3 & +2x_4 & = 24 \\ & & & 0 & = 0 \end{array} \right. \tag{1.13}$$

Notice that the fourth equation became irrelevant.
Use Eq(....) to eliminate $\cdots$ from Eq(3) and Eq(...); and then divide the second by -3 and we have:

$$\xrightarrow[(......+.......)\mapsto Eq....]{(......+.......)\mapsto Eq....} \left\{ \begin{array}{lllll} x_1 & -3x_2 & & +(2/3)x_4 & = -1 \\ & & -3x_3 & -x_4 & = -12 \\ & & & 0 & = 0 \\ & & & 0 & = 0 \end{array} \right. \tag{1.14}$$

$$\xrightarrow{Eq_2/-3} \left\{ \begin{array}{lllll} x_1 & -3x_2 & & +(2/3)x_4 & = -1 \\ & & +x_3 & +(1/3)x_4 & = 4 \\ & & & 0 & = 0 \\ & & & 0 & = 0 \end{array} \right. \tag{1.15}$$

We let x_2 and x_4 be free variables (parameters) that take the values $x_2 = r$ and $x_4 = t$. Then we express the variables x_1 and x_2 in terms of the parameters r and t:

$$\left\{ \begin{array}{lllll} x_1 & & & & = -1 + 3r - (2/3)t \\ & x_2 & & & = r \\ & & x_3 & & = 4 - (1/3)t \\ & & & x_4 & = t \end{array} \right. \tag{1.16}$$

<u>Answer</u>.

The solution set to the system (1.12) is

$$\mathbf{S}: \quad \mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 4 \\ 0 \end{pmatrix} + r \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2/3 \\ 0 \\ -1/3 \\ 1 \end{pmatrix}, \qquad -\infty < r, t < \infty \tag{1.17}$$

Description of **S**: It is the plane in $\mathbb{R}^4$ passing through the point $(-1, 0, 4, 0)^\intercal$ and generated by the two vectors $\mathbf{v}_1 = (3, 1, 0, 0)^\intercal$ and $\mathbf{v}_2 = (2/3, 0, -1/3, 1)^\intercal$.
Dimension of **S**. Notice that the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel. Thus dimension $\mathbf{S} = 2$.

1.1.8. Remark. The solution (1.17) means that we can write $(3, -6, 15, 12)^\intercal$ as a linear combination of the 4 vectors on the right hand side of (1.11) in infinitely many ways, namely, for any $r \in \mathbb{R}$ and any $t \in \mathbb{R}$ we can write $\mathbf{b}$ as the linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_4$ given by

$$\begin{pmatrix} 3 \\ -6 \\ 15 \\ 12 \end{pmatrix} = [-1 + 3r - (2/3)t] \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + r \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \end{pmatrix} + [4 - (1/3)t] \begin{pmatrix} 1 \\ -1 \\ -3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 4 \end{pmatrix} \tag{1.18}$$

For example if we take $r = 3$ and $t = -2$ we have $\mathbf{b}$ as the linear combination

$$\begin{pmatrix} 3 \\ -6 \\ 15 \\ 12 \end{pmatrix} = [\cdots] \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + 3 \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \end{pmatrix} + [\cdots] \begin{pmatrix} 1 \\ -1 \\ -3 \\ 4 \end{pmatrix} + [\cdots] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 4 \end{pmatrix}$$

1.1.9 Definition (Parametric form and vector parametric form of solution sets)**.** The solution set **S** of a linear system such as (1.11) can be represented in several forms:

- We call (1.16) the **solution set in parametric form** of the linear system (1.11).
- We call (1.16) the **solution set in parametric vector form** of the linear system (1.11).
- We refer to r and t as **parameter** and they take any values $-\infty < r, t < \infty$.

1.1.10 Definition (**The Gauss-Jordan method**)**.** The Gauss-Jordan method for solving a linear system consists of reducing the given linear system to a simpler one using the following and only the following **elementary row operations**:

- Interchanging two rows (equations).
- Multiplying (or dividing) a row (equation) by a nonzero number.
- Adding a multiple of one row (equation) to another row (equation).

1.1.11 Definition (**Row-equivalent linear systems**)**.** If a linear system is reduced to another linear system using Gauss-Jordan elimination, the two systems are said to be **row-equivalent** or just **equivalent**.

1.1.12 Theorem. *1. Elementary row operations are reversible.*

2. Any two row-equivalent linear systems have the same solution sets.

1.2 Exercises

1. Show that for any (r, t), $\mathbf{x} = (r, 7 - 2r + 5t, t)^\intercal$ is a solution of the equation

$$2x_1 + x_2 - 5x_3 = 7$$

2. In each of the following systems:

   (a) Find the solution set of the system in parametric form and in vector parametric form.

   (b) Determine the free variables and the basic variables.

   (c) What is the dimension of the solution set?

   (d) Describe the solution set geometrically.

$$(a) \quad \begin{cases} x_1 - 3x_2 + 4x_3 & = 5 \\ x_2 - 2x_3 & = 7 \end{cases}, \qquad (b) \quad \begin{cases} x_1 - 3x_2 + 4x_3 & = 5 \\ x_2 - 2x_3 & = 7 \end{cases}$$

$$(c) \quad \begin{cases} 2x_1 + 4x_3 - x_3 & = 6 \\ x_1 - 3x_2 - 5x_3 & = 7 \end{cases}, \qquad (d) \quad \begin{cases} x_1 & -3x_2 & +x_3 & +x_4 & = 3 \\ 2x_1 & -6x_2 & -x_3 & +x_4 & = -6 \\ 3x_1 & -9x_2 & -3x_3 & +x_4 & = -10 \\ 4x_1 & -12x_2 & +4x_3 & +4x_4 & = 12 \end{cases}$$

1.3 Reduced Echelon Form, Leading Variables and Free variables

1.3.1. Echelon forms. A system is said to be in **echelon form** if it satisfies the following:

- All equations of the form $0 = 0$ are at the bottom.
- The first variable with nonzero coefficient in a row (equation) is to the right of the first variable with nonzero coefficient in the row (equation) above it.

1.3.2. Leading (basic) variables and free (independent) variables. When the system is reduced to an echelon form using the Gauss-Jordan method, we define the following:

- The first variable that appear in a row (i.e. has a nonzero coefficient) is called a **leading variable** (LV) or a **basic variable** (BV).
- The rest of the variables are called **free variables** (FV) or **independent variables** (IV).

1.3.3. Reduced echelon form. A system is said to be in **reduced echelon form** if it satisfies the following:

- It is already in the echelon form.

- Each LV appears only in the equation that it leads, and not in any other equation above it or below it.
- The coefficient of each LV is 1.

1.3.4 Rule. We need to express the solution of a linear system in the simplest possible way and with absolutely no redundancy. To achieve this goal we have the following rule:

> We have to reduce the system
> all the way to its <u>reduced echelon form</u>
> before we write down the solution set.

1.3.5 Theorem. *1. A linear system may have more than one echelon form.*

2. But the reduced echelon form is unique.

3. A linear system is equivalent to (i.e. has the same solution set as) its reduced echelon form.

4. In fact, a linear system is equivalent to (i.e. has the same solution set as) any linear system that we obtain from it by elementary row operations.

1.3.6 Example. 1. Equations (1.13) and (1.14) are in echelon form but not in the reduced echelon form. (Why?)

2. Equation (1.15) is in the reduced echelon form.

1.3.7 Example. Solve the following system and describe the solution set.

$$\begin{cases} 2x_1 - 4x_2 + 5x_3 &= 3 \\ x_1 - 3x_2 + 4x_3 &= 1 \end{cases} \quad \Leftrightarrow \quad \left(\begin{array}{ccc|c} 2 & -4 & 5 & 3 \\ 1 & -3 & 4 & 1 \end{array}\right) \tag{1.19}$$

<u>**Solution**</u>.

1. First change the order of the equations

$$\xrightarrow{eq_1 \leftrightarrow eq_2} \begin{cases} x_1 - 3x_2 + 4x_3 &= 1 \\ 2x_1 - 4x_2 + 5x_3 &= 3 \end{cases} \quad \Leftrightarrow \quad \left(\begin{array}{ccc|c} 1 & -3 & 4 & 1 \\ 2 & -4 & 5 & 3 \end{array}\right) \tag{1.20}$$

2. Then we use Eq(1) to eliminate x_1 from the second equation:

$$\xrightarrow{(-2eq_1+eq_2)\mapsto eq_2} \begin{cases} x_1 & -3x_2 & +4x_3 & = 1 \\ & 2x_2 & -3x_3 & = 1 \end{cases} \quad \Leftrightarrow \quad \left(\begin{array}{ccc|c} 1 & -3 & 4 & 1 \\ 0 & 2 & -3 & 1 \end{array}\right)$$

Notice that we left the first equation unchanged.

3. Divide Eq(2) by 2:

$$\xrightarrow{eq_2/2} \begin{cases} x_1 & -3x_2 & +4x_3 & = 1 \\ & x_2 & -(3/2)x_3 & = 1/2 \end{cases} \quad \Leftrightarrow \quad \left(\begin{array}{ccc|c} 1 & -3 & 4 & 1 \\ 0 & 1 & -3/2 & 1/2 \end{array}\right)$$

4. Eliminate x_2 from Eq(1):

$$\xrightarrow{(3eq_2+eq_1)\mapsto eq_1} \begin{cases} x_1 \qquad -(1/2)x_3 = 1 \\ \qquad x_2 \; -(3/2)x_3 = 5/2 \end{cases} \qquad \Leftrightarrow \qquad \left(\begin{array}{ccc|c} 1 & 0 & -1/2 & 1 \\ 0 & 1 & -3/2 & 5/2 \end{array}\right)$$

5. The free variable (parameter) is x_3, it takes any arbitrary value $x_3 = r \in \mathbb{R}$. The basic variables are x_1 and x_2.

Answer.

- The solution set is

$$\mathbf{S}: \quad \begin{aligned} x_1 &= 1 + (1/2)r \\ x_2 &= (5/2) + (3/2)r, \qquad -\infty < r < \infty \\ x_3 &= r \end{aligned} \tag{1.21}$$

  In parametric vector form, the solution is

$$\mathbf{S}: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 5/2 \\ 0 \end{pmatrix} + r \begin{pmatrix} 1/2 \\ 3/2 \\ 1 \end{pmatrix}, \qquad -\infty < r < \infty \tag{1.22}$$

- Description of the solution set: The solution set $\mathbf{S}$ is a straight line in $\mathbb{R}^3$ through the point $(1, 5/2, 0)^\intercal$, parallel to the vector $(1/2, 3/2, 1)$.

1.3.8 Example. Solve the following equation and describe the solution set.

$$2x_1 + 4x_2 - x_3 = 5 \tag{*}$$

Solution.

In this example we have three variables (unknowns) but only one equation. And we expect to have two free variables (parameters). So, let's take $x_1 = r$ and $x_2 = t$.

Answer.

1. Thus, the solution set is

$$\mathbf{S}: \quad \begin{aligned} x_1 &= r \\ x_2 &= t, \qquad -\infty < r, t < \infty \\ x_3 &= 2r + 4t - 5 \end{aligned}$$

   In parametric vector form

$$\begin{aligned} \mathbf{S}: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} r \\ t \\ 2r + 4t - 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} + r \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \\ & r \in \mathbb{R}, t \in \mathbb{R} \end{aligned}$$

2. Description of solution set: The solution set **S** is a plane through the point $(0, 0, -5)^{\mathsf{T}}$ and generated by the two vectors $(1, 0, 2)$ and $(0, 1, 4)$.

1.3.9. Remarks.

1. The solution set **S** has two free parameters r and t that we cannot control and are independent of each other. But once we make a choice $(x_1, x_2) = (r, t)$, x_3 is completely and uniquely determined in terms of the ordered pair (r, t).

2. Therefore, it makes sense to say that **S has two dimensions** or that it is **two dimensional**.

3. **Question:** Why do we always refer to (r, t) as an ordered pair?

1.3.10 Example. Solve the following system and describe the solution set geometrically and explain the meaning of the solution you found.

$$\left\{\begin{array}{llll} x_1 & -3x_2 & +4x_3 & = 5 \\ x_1 & -x_2 & -2x_3 & = 3 \\ 2x_1 & -5x_2 & +5x_3 & = 9 \end{array}\right. \tag{1.23}$$

Answer. Use Eq(1) to eliminate x_1 from Eq(2) and Eq(3):

$$\xrightarrow[(-2eq_1+eq_3)\mapsto eq_3]{(-eq_1+eq_2)\mapsto eq_2} \left\{\begin{array}{llll} x_1 & -3x_2 & +4x_3 & = 5 \\ & 2x_2 & -6x_3 & = -2 \\ & x_2 & -3x_3 & = -1 \end{array}\right.$$

Switch Eq(2) with Eq(3):

$$\xrightarrow{eq_2 \leftrightarrow eq_3} \left\{\begin{array}{llll} x_1 & -3x_2 & +4x_3 & = 5 \\ & x_2 & -3x_3 & = -1 \\ & 2x_2 & -6x_3 & = -2 \end{array}\right.$$

Use Eq(2) to eliminate x_2 from Eq(3):

$$\xrightarrow{(-2eq_2+eq_3)\mapsto eq_3} \left\{\begin{array}{llll} x_1 & -3x_2 & +4x_3 & = 5 \\ & x_2 & -3x_3 & = -1 \\ & & 0 & = 0 \end{array}\right.$$

Use Eq(2) to eliminate x_2 from Eq(1):

$$\xrightarrow{(3eq_2+eq_1)\mapsto eq_1} \left\{\begin{array}{llll} x_1 & & -5x_3 & = 2 \\ & x_2 & -3x_3 & = -1 \\ & & 0 & = 0 \end{array}\right.$$

Now the system is in the reduced echelon form.
Take $x_3 = r$ to be a free variable. Then x_1 and x_2 are basic variables and

$$\left\{\begin{array}{ll} x_1 & = 2 + 5r \\ x_2 & = -1 + 3r, \qquad -\infty < r < \infty \\ x_3 & = r \end{array}\right.$$

And the solution set is

$$\mathbf{S}: \qquad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + r \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}, \qquad -\infty < r < \infty \tag{1.24}$$

The solution set is the straight line that passes through the point $(2, -1, 0)$ and parallel to the vector $(5, 3, 1)$.

What does solving the system (1.23) mean? It means that if we pick any value for r, say $r = 2$ than $\mathbf{x} = (12, 5, 2)$ is a solution of the system (1.23). That is if substitute $x_1 = 12, x_2 = 5$ and $x_3 = 2$ in the 3 equations of (1.23), we obtain 5, 3 and -1 respectively. In vector form we have

$$12 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} \tag{1.25}$$

This means that if we choose $r = 2$ we can write the vector $(5, 3, -1)$ as a linear combination of the 3 vectors $(1, 1, 0)$, $(3, -1, 1)$ and $(4, -2, 2)$. Nothing special about $r = 2$. In fact for any $r \in \mathbb{R}$ we can write the vector $(5, 3, -1)$

$$\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} = (2 + 5r) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (-1 + 3r) \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + r \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}, \qquad -\infty < r < \infty$$

1.3.11 Observation.

A linear system has either
no solution,
a unique solution,
or
infinitely many solutions

1.3.12 Observation.

#(free variables) + #(basic variables) = #(all variables)
dim(solution set) = #(.......... variables)

1.3.13 Observations.

A linear system is **inconsistent** if it has an echelon form with an equation of the form $0 = c$ where $c \neq 0$.
A linear system is **consistent** iff its echelon form does not have an equation of the form $0 = c$ where $c \neq 0$.
A linear system is **consistent** iff its echelon form has a LV in each row (equation).
A linear system **has a unique solution** if it is consistent and each variable is a leading variable. That is, consistent and has no free variables.
A linear system **has infinitely many solutions** if it is consistent and has at least one free variables. In this case dim(solution set) = #(.......... variables)

1.4 Homogenous systems(General = particular + homogenous)

A linear system of the form
$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$
is called a **homogenous linear system**
or a **coefficient linear system**.

A homogenous system is always consistent
because it always has the zero solution (or trivial solution)
$\mathbf{x} = 0$.

The system

$$(\mathrm{H})\begin{cases} x_1 & -3x_2 & +4x_3 & = 0 \\ x_1 & -x_2 & -2x_3 & = 0 \\ 2x_1 & -5x_2 & +5x_3 & = 0 \end{cases}$$

is called a **homogenous system**
or a **coefficient system**.

The system

$$(\mathrm{NH})\ \begin{cases} x_1 & -3x_2 & +4x_3 & = 5 \\ x_1 & -x_2 & -2x_3 & = 3 \\ 2x_1 & -5x_2 & +5x_3 & = 9 \end{cases} \tag{1.23}$$

is called a **non-homogenous system**
or an **augmented system**.

Let us find whether the homogenous system (H) has any nontrivial (non zero) other solutions:

$$(\mathrm{H}) \quad \longrightarrow \begin{cases} x_1 & -3x_2 & +4x_3 & = 0 \\ & 2x_2 & -6x_3 & = 0 \\ & x_2 & -3x_3 & = 0 \end{cases} \longrightarrow \begin{cases} x_1 & -3x_2 & +4x_3 & = 0 \\ & x_2 & -3x_3 & = 0 \\ & 2x_2 & -6x_3 & = 0 \end{cases}$$

$$\longrightarrow \begin{cases} x_1 & -3x_2 & +4x_3 & = 0 \\ & x_2 & -3x_3 & = 0 \\ & & 0 & = 0 \end{cases} \longrightarrow \begin{cases} x_1 & & -5x_3 & = 0 \\ & x_2 & -3x_3 & = 0 \\ & & 0 & = 0 \end{cases}$$

Now the system is in the reduced echelon form. Take $x_3 = r$ to be a free variable. Then x_1 and x_2 are basic variables. And the general solution (solution set) of (H) is

$$\mathbf{S}: \qquad \mathbf{x}_h = r\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}, \qquad -\infty < r < \infty \tag{1.26}$$

1. We used a subscript h in $\mathbf{x}_h$ to indicate that this is a solution to the homogenous equations.

2. Notice that the coefficient side of the homogenous system (H) is the same as that of the system (1.23).

3. In fact, we used the same row operations to solve the two systems because row operations depend only on the coefficient side of the equation.

4. Observe that the general solution (1.24) of the *non-homogenous system* (*augmented system*) (1.23) is
$$\mathbf{x} = \mathbf{x}_o + \mathbf{x}_h \tag{1.27}$$
where $\mathbf{x}_h$ is the general solution (1.26) of the *homogenous system* (H) and $\mathbf{x}_o = (2, -1, 0)^\intercal$ is the particular solution of (1.23) that corresponds to $r = 0$.

5. We do not have to use $\mathbf{x}_o = (2, -1, 0)^\intercal$. Any particular solution of the *non-homogenous system* (1.23) will do. For example, if we take $r = -2$ in (1.24) we have a particular solution $\mathbf{x}_1 = (-8, -7, -2)^\intercal$ and we can write the general solution (1.24) in the form

$$\mathbf{S}: \qquad \mathbf{x} = \begin{pmatrix} -8 \\ -7 \\ -2 \end{pmatrix} + t\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}, \qquad -\infty < t < \infty \tag{1.28}$$

The general solution to the non-homogenous system (NH)
(NH) $\qquad x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$
is
$\mathbf{x} = \mathbf{p} + \mathbf{h}$
where
$\mathbf{h}$ is the general solution to the associated homogenous system
(H) $\qquad x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$
and $\mathbf{p}$ is any particular solution of (NH)

1.5 Exercises

1. In each of the following systems:

   (a) Find the solution set of the system in parametric form and in vector parametric form.

   (b) Determine the free variables and the basic variables.

(c) What is the dimension of the solution set?

(d) Describe the solution set geometrically.

$$(a)\quad \begin{cases} 2x_1 + x_2 - 2x_3 &= 2 \\ x_1 - 3x_2 - 5x_3 &= 3 \end{cases}, \qquad (b)\quad \begin{cases} -3x_1 + 2x_2 - 6x_3 &= -1 \\ x_1 + x_2 + 2x_3 &= 2 \end{cases}$$

$$(c)\quad \begin{cases} -2x_1 + 4x_2 - 6x_3 + x_4 &= -1 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 2 \end{cases} \qquad (d)\quad \begin{cases} x_1 \; +2x_2 \; +4x_3 &= 9 \\ -2x_1 \; +x_2 \; +x_3 &= -5 \\ 2x_1 \; -x_3 &= 4 \\ 3x_1 \; +x_2 \; +4x_3 &= 2 \end{cases}$$

2. Find the values of a for which the system has
   (a) a unique solution (b) infinitely many solutions (c) inconsistent.

$$\begin{cases} a\,x_1 \quad + x_2 &= 3 \\ 2\,x_1 \quad +(a-2)\,x_2 &= 4 \end{cases}$$

3. Solve each of the following linear systems and determine the number of solutions in each case:

$$(a)\begin{cases} 2x_1 + 5x_2 &= 4 \\ 4_2 - 2x_3 &= 2 \end{cases}, \qquad (b)\begin{cases} 2x_1 + x_2 &= 4 \\ 4_2 + 2x_3 &= 2 \end{cases}, \qquad (c)\begin{cases} 2x_1 + x_2 &= 4 \\ 4_2 + 2x_3 &= 8 \end{cases}$$

4. Determine the values of h such that the system has (i) a unique solution, (ii) infinitely many solutions, (iii) no solutions.

$$(a)\begin{cases} x_1 + hx_2 &= 4 \\ 3x_1 + 6x_2 &= 8 \end{cases}, \qquad (b)\begin{cases} x_1 + hx_2 &= -3 \\ -2x_1 + 4x_2 &= 6 \end{cases}, \qquad (c)\begin{cases} x_1 + 3x_2 &= -2 \\ -4x_1 + hx_2 &= 8 \end{cases}$$

$$(d)\begin{cases} x_1 - 3x_2 &= -2 \\ 5x_1 + hx_2 &= -7 \end{cases}, \qquad (e)\begin{cases} 2x_1 + 3x_2 &= h \\ -6x_1 + 9x_2 &= 5 \end{cases}, \qquad (f)\begin{cases} 2x_1 + 3x_2 &= h \\ 4x_1 + 6x_2 &= 7 \end{cases}$$

5. In each of the following find the equation of the parabola that passes through the given three points.

   (a) $(1, 4), (-1, 0)$ and $(-2, 7)$.

   (b) $(1, -2), (2, 4)$ and $(-1, -2)$.

   Hint: The general equation of a parabola takes the form $y = ax^2 + bx + c$. Write the 3 equations that a, b and c must satisfy if the parabola is to pass through the given points.

6. Find the values of h for which the following system has (a) no solution (b) infinitely many solutions (c) a unique solution.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4 \\ 3x_1 - x_2 \; + 5x_3 &= 2 \\ 4x_1 + x_2 \; + (h^2 - 14)x_3 &= h + 2 \end{aligned}$$

7. Find α, β and γ

$$\begin{aligned} \frac{1}{\alpha} + \frac{2}{\beta} - \frac{4}{\gamma} &= 1 \\ \frac{2}{\alpha} + \frac{3}{\beta} + \frac{8}{\gamma} &= 0 \\ -\frac{1}{\alpha} + \frac{9}{\beta} + \frac{10}{\gamma} &= 5 \end{aligned}$$

8. What conditions must a, b and c satisfy in order for the system to have a solution?

$$\begin{cases} x_1 & -3x_2 & +4x_3 & = a \\ x_1 & -x_2 & -2x_3 & = b \\ 2x_1 & -5x_2 & +5x_3 & = c \end{cases}$$

9. Find a, b and c so that the following system has the solution $x_1 = 3, x_2 = -1$ and $x_3 = 2$.

$$\begin{aligned} x_1 + ax_2 + cx_3 &= 0 \\ bx_1 + cx_2 - 3x_3 &= 1 \\ ax_1 + 2x_2 + bx_3 &= 5 \end{aligned}$$

10. Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d?

$$\begin{aligned} x_1 + 3x_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

11. Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients a and b?

$$\begin{aligned} ax_1 + bx_2 &= f \\ -x_1 + 2x_2 &= g \end{aligned}$$

12. Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients a, b, c and d?

$$\begin{aligned} ax_1 + bx_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

13. Solve the following nonlinear system

$$\begin{cases} \tan x & -3\cos y & +4\sin z & = 5 \\ \tan & -\cos y & -2\sin z & = 3 \\ 2\tan & -5\cos y & +5\sin z & = 9 \end{cases} \tag{1.29}$$

14. Find the value(s) of h for which the system is consistent.

$$\begin{cases} x_1 & -2x_2 & = 4 \\ 4x_1 & -3x_2 & = 1 \\ -2x_1 & +7x_2 & = h \end{cases}$$

15. What condition(s) must the vector **b** satisfy in order for the linear system to be consistent.

$$\begin{cases} 2x_1 & -7x_2 & = b_1 \\ x_1 & -5x_2 & = b_2 \\ -3x_1 & +3x_2 & = b_3 \end{cases}$$

16. In each of the following, determine the value of h that makes the system inconsisten:

(a) $\begin{cases} x & +y & = & -2 \\ 2x & +hy & = & 3 \end{cases}$

(b) $\begin{cases} 2x & -y & = & 4 \\ hx & +3y & = & 2 \end{cases}$

(c) $\begin{cases} x & -y & = & 2 \\ 4x & -4y & = & h \end{cases}$

(d) $\begin{cases} 2x & -y & = & h \\ 10x & -5y & = & h \end{cases}$

2 Linear system in Matrix notation

2.1 Gauss-Jordan elimination

2.1.1. Example Use row reduction to solve the system of equations

$$\begin{aligned} 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - 3x_3 &= 7 \\ x_1 - 3x_2 + 4x_3 &= -4 \end{aligned} \tag{2.1}$$

The augmented matrix of the system is

$$B = \left(\begin{array}{rrr|r} 3 & -7 & 7 & -8 \\ -4 & 6 & -3 & 7 \\ 1 & -3 & 4 & -4 \end{array}\right)$$

We start by moving the third row to the top.

$$B \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ -4 & 6 & -3 & 7 \\ 3 & -7 & 7 & -8 \end{array}\right)$$

Then we eliminate the second and third entries of the first column and replace them by zeros:

$$\xrightarrow{(4R_1+R_2)\mapsto R_2} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & -6 & 13 & -9 \\ 3 & -7 & 7 & -8 \end{array}\right) \xrightarrow{(-3R_1+R_3)\mapsto R_3} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & -6 & 13 & -9 \\ 0 & 2 & -5 & 4 \end{array}\right)$$

Then we eliminate the third entry of the second column and replace it by zero:

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 13 & -9 \end{array}\right) \xrightarrow{(3R_2+R_3)\mapsto R_3} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & -2 & 3 \end{array}\right) = B_{ech} \tag{2.2}$$

Next we want each **leading term (LT)** (first non-zero entry of a row) to be the only non-zero entry in its column. The first one is already so. We need to work on the second and the third ones and eliminate the entries above them.
We also need to make each diagonal leading term to be 1:

$$\xrightarrow{R_2/2} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & 1 & -5/2 & 2 \\ 0 & 0 & -2 & 3 \end{array}\right)$$

$$\xrightarrow{(3R_2+R_1)\mapsto R_1} \left(\begin{array}{rrr|r} 1 & 0 & -7/2 & 2 \\ 0 & 1 & -5/2 & 2 \\ 0 & 0 & -2 & 3 \end{array}\right)$$

$$\xrightarrow{(-R_3/2} \left(\begin{array}{rrr|r} 1 & 0 & -7/2 & 2 \\ 0 & 1 & -5/2 & 2 \\ 0 & 0 & 1 & -3/2 \end{array}\right)$$

$$\xrightarrow{(5R_3/2+R_2)\mapsto R_2} \left(\begin{array}{ccc|c} 1 & 0 & -7/2 & 2 \\ 0 & 1 & 0 & -7/4 \\ 0 & 0 & 1 & -3/2 \end{array}\right)$$
$$\xrightarrow{(7R_3/2+R_1)\mapsto R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -13/4 \\ 0 & 1 & 0 & -7/4 \\ 0 & 0 & 1 & -3/2 \end{array}\right) = B_{rech}$$
$$x_1 = -13/4$$
$$x_2 = -7/4$$
$$x_3 = -3/2$$

In this case the solution set has only one point

$$\mathbf{S} : \mathbf{x} = \begin{pmatrix} -13/4 \\ -7/4 \\ -3/2 \end{pmatrix}$$

2.1.2. Terminology

1. The matrices
$$\begin{pmatrix} 3 & 5 \\ 4 & -8 \end{pmatrix}, \qquad \begin{pmatrix} 3 & -7 & 7 \\ -4 & 6 & -3 \\ 1 & -3 & 4 \end{pmatrix}$$
are called **coefficient matrices**.

2. The matrices
$$\left(\begin{array}{cc|c} 3 & 5 & 4 \\ 4 & -8 & 12 \end{array}\right), \qquad \left(\begin{array}{ccc|c} 3 & -7 & 7 & -8 \\ -4 & 6 & -3 & 7 \\ 1 & -3 & 4 & -4 \end{array}\right)$$
are called **augmented matrices**.

3. **Zero-rows** and non zero-rows.

4. **The leading term (LT)** in a row is the first non-zero entry in that row.

5. **Pivot columns** are the columns in the original matrix that correspond to **LT**'s.

6. **Row echelon form:** AKA **echelon form:**
$$\begin{pmatrix} 3 & -7 & 7 & -8 \\ 0 & 0 & -3 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
   (a) All zero-rows are at the bottom.
   (b) Leading terms go from top left to bottom right.
   (c) All entries **below** a leading entry are zeros.

7. **Reduced row echelon form** AKA **reduced echelon form:**

$$\begin{pmatrix} 1 & -7 & 0 & -8 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) All zero-rows are at the bottom.
   (b) Leading terms go from top left to bottom right.
   (c) All leading entries are 1's.
   (d) All entries **below and above** a leading entry are all zeros.
   In other words, a leading entry is 1 and is the only non-zero entry in its column.

$$A_{ech} = \left(\begin{array}{cccccc|c} 2 & * & * & * & * & * & * \\ 0 & 0 & 3 & * & * & * & * \\ 0 & 0 & 0 & 0 & 7 & * & * \\ 0 & 0 & 0 & 0 & 0 & -2 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This matrix A_{ech} is in *echelon form* form. An astrix "$*$" means the entry may take any value possibly zero.

To reduce this matrix to the *reduced echelon form* we need to divide each non-zero row by the leading term and then eliminate entries above the "ones" and obtain:

$$\left(\begin{array}{ccccccc|c} 1 & * & 0 & * & * & 0 & 0 & * \\ 0 & 0 & 1 & * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

8. **Theorem 1**
   The reduced row echelon form is unique.
   But an echelon form is not unique.

2.1.3. Coefficient matrix and augmented matrix of a linear system.

$$3x_1 - 7x_2 + 7x_3 = -8 \tag{2.3}$$

$$
\begin{aligned}
-4x_1 + 6x_2 - 3x_3 &= 7 \\
x_1 - 3x_2 + 4x_3 &= -4 \\
2x_1 - 5x_2 + 3x_3 &= 1
\end{aligned}
$$

The coefficient matrix of the system is

$$
A = \begin{pmatrix} 3 & -7 & 7 \\ -4 & 6 & -3 \\ 1 & -3 & 4 \\ 2 & -5 & 3 \end{pmatrix}
$$

The augmented matrix of the system is

$$
B = \left(\begin{array}{ccc|c} 3 & -7 & 7 & -8 \\ -4 & 6 & -3 & 7 \\ 1 & -3 & 4 & -4 \\ 2 & -5 & 3 & 1 \end{array}\right)
$$

2.1.4. Row-equivalent matrices.

Two matrices A and B are said to be
row-equivalent
iff
one of them can be obtained from the other
(can be reduced to the other)
using elementary row operations.

If A is equivalent to B and B is equivalent to C
then
A is equivalent to C.

Elementary row operations are reversible.

If two matrices A and B can be reduced to the sam matrix R
then
they are row equivalent.

2.1.5 Theorem.

Row operations
do not change the span of the rows of a matrix.

That is

$$\text{span}\{\text{rows of } A\} = \text{span}\,\{\text{rows of } A_{ech}\}$$

2.1.6. Multiplying a vector by a matrix.
Write a matrix a with n columns $\mathbf{a}_1, \mathbf{a}_2, \cdots$ and $\mathbf{a}_n$ as

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

As before we write a vector with n components as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The multiplication $A\mathbf{x}$ (with A is to the left of $\mathbf{x}$) is defined as

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

2.1.7. Nonsingular (square) matrices

A square matrix A is **nonsingular**
iff
it is the coefficient matrix
of a homogenous system
(H) $\quad A\mathbf{x} = \mathbf{0}$
has a unique solution.
That is
(H) has only the trivial (zero) solution.

2.1.8. Example: Inconsistent system This is an example of system with no solution. Solve the system of equations

$$\begin{aligned} 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - x_3 &= 7 \\ x_1 - 3x_2 + 4x_3 &= -4 \end{aligned} \tag{2.4}$$

The coefficient matrix for the system is

$$A = \begin{pmatrix} 3 & -7 & 7 \\ -4 & 6 & -1 \\ 1 & -3 & 4 \end{pmatrix}$$

The augmented matrix for the system is

$$B = \left(\begin{array}{ccc|c} 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \\ 1 & -3 & 4 & -4 \end{array}\right)$$

The first goal is to eliminate the second and third entries of the first column and replace them by zeros:

$$\begin{aligned} &\xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ -4 & 6 & -1 & 7 \\ 3 & -7 & 7 & -8 \end{array}\right) \\ &\xrightarrow{(4R_1+R_2) \mapsto R_2} \left(\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & -6 & 15 & -9 \\ 3 & -7 & 7 & -8 \end{array}\right) \\ &\xrightarrow{(-3R_1+R_3) \mapsto R_3} \left(\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & -6 & 15 & -9 \\ 0 & 2 & -5 & 4 \end{array}\right) \end{aligned}$$

The second step is to eliminate the entries below the LT in the 2^{nd} row, that is the third entry of the second column and replace it by zero:

$$\begin{aligned} &\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{array}\right) \\ &\xrightarrow{3R_2+R_3 \mapsto R_3} \left(\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right) \end{aligned}$$

In this case the last equation reads

$$0x_1 + 0x_2 + 0x_3 = 3$$

That is

$$0 = 3$$

In this case the system has **no solution** and is called **inconsistent**.

Notice that in this example we didn't need to reduce the augmented matrix matrix. We stop as soon as we obtain a row of the form $\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & * \end{array}\right)$, where $0 \neq *$. But then that row leads to the equation

$$0 = * \neq 0$$

which is false. And the system is inconsistent. I.e. , has no solution.
The coefficient matrix A has the echelon form

$$A_{ech} = \begin{pmatrix} 1 & -3 & 4 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

2.1.9 Theorem.

Rule 1
A linear system
$A\mathbf{x} = \mathbf{b}$
with augmented matrix
$B = \left(\begin{array}{c|c} A & \mathbf{b} \end{array}\right)$
is consistent
$\Updownarrow$
B_{ech}
doesn't have a row of the form
$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & c \end{array}\right], \qquad c \neq 0$

Rule 2
$A\mathbf{x} = \mathbf{y}$
is consistent for any $\mathbf{y}$
$\Updownarrow$
each row of A_{ech}
has a LT

2.1.10. Example: System with infinitely many solutions Solve the system of equations

$$\begin{aligned} 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - x_3 &= 4 \\ x_1 - 3x_2 + 4x_3 &= -4 \end{aligned} \tag{2.5}$$

The augmented matrix of the system is

$$B = \left(\begin{array}{rrr|r} 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 4 \\ 1 & -3 & 4 & -4 \end{array}\right)$$

The first goal is to eliminate the second and third entries of the first column and replace them by zeros:

$$B \stackrel{R_1 \leftrightarrow R_3}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ -4 & 6 & -1 & 4 \\ 3 & -7 & 7 & -8 \end{array}\right)$$
$$\stackrel{(4R_1+R_2)\mapsto R_2}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & -6 & 15 & -12 \\ 3 & -7 & 7 & -8 \end{array}\right)$$
$$\stackrel{(-3R_1+R_3)\mapsto R_3}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & -6 & 15 & -12 \\ 0 & 2 & -5 & 4 \end{array}\right)$$

The second goal is to eliminate the entries below the LT in the 2^{nd} row, that is the third entry of the second column and replace it by zero:

$$\stackrel{R_2 \leftrightarrow R_3}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -12 \end{array}\right) \tag{2.6}$$
$$\stackrel{R_2/2}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & 1 & -5/2 & 2 \\ 0 & -6 & 15 & -12 \end{array}\right)$$
$$\stackrel{(6R_2+R_3)\mapsto R_3}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & 1 & -5/2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

In this case the last equation reads

$$0 = 0$$

which is correct but not useful. Then we proceed. Next we want each **leading term** (first non-zero entry of a row) to be the only non-zero entry in its column. The first one is already so. We need to work on the second and the third ones and eliminate the entries above them.
We also need to make each diagonal leading term to be 1:

$$\left(\begin{array}{rrr|r} 1 & -3 & 4 & -4 \\ 0 & 1 & -5/2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\stackrel{3R_2+R_1\mapsto R_1}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & 0 & -7/2 & 2 \\ 0 & 1 & -5/2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{2.7}$$

Now we have two equations

$$x_1 - (7/2)x_3 = 2$$
$$x_2 - (5/2)x_3 = 2$$

This can be rewritten as

$$\begin{aligned} x_1 &= (7/2)x_3 + 2 \\ x_2 &= (5/2)x_3 + 2 \\ x_3 &= \text{ free variable} \\ &\quad \text{can take any value } t, -\infty < t < \infty \end{aligned}$$

Once we decide on a value for x_3, say $x_3 = t$, the other two variable are completely and uniquely determined.
The solution set in parametric form is

$$\mathbf{S}: \quad \begin{aligned} x_1 &= (7/2)t + 2 \\ x_2 &= (5/2)t + 2 \\ x_3 &= t \\ -\infty &< t < \infty \end{aligned} \tag{2.8}$$

The solution set in parametric vector form is

$$\mathbf{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7/2 \\ 5/2 \\ 1 \end{pmatrix} \tag{2.9}$$
$$-\infty < t < \infty$$

This is a straight line through the point $(2, 2, 0)^{\intercal}$ along the vector $(7/2, 5/2, 1)^{\intercal}$.

Now we can see that we have infinitely many solutions given by (2.8).

<u>Basic variables</u>: x_1 and x_2.
<u>Free variables</u>: x_3.

Rule 3
<u>Assume $A\mathbf{x} = \mathbf{b}$ is consistent</u>.
<u>Then</u>
it has a unique solution
$\Updownarrow$
each <u>column</u> of A
is a pivot column
$\Updownarrow$
each <u>column</u> of A_{ech}
has a LT
$\Updownarrow$
There is no free variables

Rule 4
Assume $A\mathbf{x} = \mathbf{b}$ is consistent.
Then
The system has infinitely many solutions.
$\Updownarrow$
The system has at least one free variables.
$\Updownarrow$
Some of the columns of A are not pivot columns.
$\Updownarrow$
Some of the columns of A_{ech} have no LT's.

1. **Basic variables** are the variables that correspond to the **leading terms, (LT's)**, equivalently, correspond to **pivot columns**.
2. **Free variables** are the rest of the variables.
3. We **write the basic variables in terms of the free variable**. In order to solve the linear system (2.5) we used (2.7) to write the *basic variables* x_1 and x_2 in terms of the *free variables* x_3.
4. **Parametric description of solutions.** The description of the solution in the form (2.8) or (2.9) are called *Parametric descriptions* of the solution of the linear system (2.5).
5. **Parametric vector description of solutions.** The description of the solution in the form (2.9) is called *Parametric descriptions* of the solution of the linear system (2.5).
6. The solution set (2.9) is an equation of a straight line in a 3-dimensional space. This line passes by the point $(2, 2, 0)^\intercal$ and points in the direction of the vector $< 7/2, 5/2, 1 >^\intercal$.

2.1.11. Important observation

#(basic variables) + #(free variables)
= #(columns)
= #(all variables)

dim(solution set) = #(free variables)

2.1.12. More terminology

1. **Gauss elimination** is the process of reducing a matrix A to an *echelon form*.

2. **Gauss-Jordan elimination** is the process of reducing a matrix A to its *reduced echelon form.*

3. **Pivot position** in a matrix A is the location of a leading term in an *echelon form* of the matrix.

4. **Pivot column** in a matrix A is the column of a *leading term* .

5. **Exercise:** Row-reduce the matrix to a *reduced echelon form* .

$$\begin{pmatrix} 2 & 6 & 10 & 14 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix}$$

2.2 Exercises

1. Solve problems in Exercises 1.5 using matrices.

2. The augmented matrix of a linear system has the form

$$\left(\begin{array}{cc|c} a & 1 & 1 \\ 2 & a-1 & 1 \end{array}\right)$$

Find the values of a for which the system has

(a) a unique solution (b) infinitely many solutions (c) inconsistent.

3. In each of the following determine the values of a and b for which the system is consistent:

$$(i) \quad \begin{pmatrix} 1 & 2 \\ a & 0 \end{pmatrix}\begin{pmatrix} 3 & b \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 6 \\ 12 & 16 \end{pmatrix} \qquad (ii) \quad \begin{pmatrix} 1 & 2 \\ a & 0 \end{pmatrix}\begin{pmatrix} 3 & b \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 12 & 16 \end{pmatrix}$$

3 Inverse of a matrix

$$3x = 7 \to 3^{-1}3x = 3^{-1}7$$
$$\to 1.x = \frac{3}{7} \to x = \frac{3}{7}$$

"3^{-1}" is called "the multiplicative inverse of 3".

$$(3^{-1})(3) = (3)(3^{-1}) = 1$$

Can we do the same with matrices?

$$A\mathbf{x} = \mathbf{b} \to A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

$$I\mathbf{x} = A^{-1}\mathbf{b} \to \mathbf{x} = A^{-1}\mathbf{b}$$

Answer: Not always.
The 2×2 case:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
$$\boxed{A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$
$$\boxed{\det A = ad - bc}$$

Example 1

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right)$$
$$= [I \mid S]$$

$$S = \begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{pmatrix} = \left[\ \mathbf{s}_1 \quad \mathbf{s}_2 \quad \mathbf{s}_3\ \right]$$

What did we actually do?
We solved three equations simultaneously

$$\boxed{A\mathbf{x} = \mathbf{e}_1, \quad A\mathbf{x} = \mathbf{e}_2, \quad A\mathbf{x} = \mathbf{e}_3}$$

And we obtained

$$\boxed{A\mathbf{s}_1 = \mathbf{e}_1, \quad A\mathbf{s}_2 = \mathbf{e}_2, \quad A\mathbf{s}_3 = \mathbf{e}_3}$$

Thus

$$\boxed{A^{-1}\mathbf{e}_1 = \mathbf{s}_1, \quad A^{-1}\mathbf{e}_2 = \mathbf{s}_2, \quad A^{-1}\mathbf{e}_2 = \mathbf{s}_2}$$

Which gives us the columns of A^{-1}

$$\begin{aligned} A^{-1} &= \left[\ A^{-1}\mathbf{e}_1 \quad A^{-1}\mathbf{e}_2 \quad A^{-1}\mathbf{e}_3\ \right] \\ &= \left[\ \mathbf{s}_1 \quad \mathbf{s}_2 \quad \mathbf{s}_3\ \right] \\ &= S \end{aligned}$$

Example 2

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right)$$

This means that A does not have an inverse.
Example 3

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{pmatrix}$$

$$\left(\begin{array}{rrr|rrr} 3 & 1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 5 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 5 & 4 & 1 & 2 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 5 & 9 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 7 & 6 & 1 & 3 & 0 \\ 0 & -25 & -21 & -5 & -10 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 1 & 6/7 & 1/7 & 3/7 & 0 \\ 0 & -25 & -21 & -5 & -10 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 1 & 6/7 & 1/7 & 3/7 & 0 \\ 0 & 0 & 3/7 & -10/7 & 5/7 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 1 & 6/7 & 1/7 & 3/7 & 0 \\ 0 & 0 & 1 & -10/3 & 5/3 & 7/3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 5 & 0 & 43/3 & -14/3 & -28/3 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -10/3 & 5/3 & 7/3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -2/3 & 1/3 & 2/3 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -10/3 & 5/3 & 7/3 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -2/3 & 1/3 & 2/3 \\ 3 & -1 & -2 \\ -10/3 & 5/3 & 7/3 \end{pmatrix}$$

<u>Example 4</u>

$$B = \begin{pmatrix} 3 & 3 & 6 \\ 0 & 1 & 2 \\ -2 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \cdots \rightarrow \left(\begin{array}{rrr|rrr} 1 & 3 & 6 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -6 & 3 \end{array}\right)$$

Thus, B does not have an inverse.

3.0.1 Example. This one happens to start with a row swap.

$$\left(\begin{array}{ccc|ccc} 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \quad \rho_1 \leftrightarrow \rho_2 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 \end{array}\right) \quad \cdots \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/4 & 3/4 \\ 0 & 1 & 0 & 1/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & 3/4 & -3/4 \end{array}\right)$$

3.1 Exercises

1. Determine the value of c for which the matrix does not have an inverse:

$$\begin{pmatrix} 2 & 0 & 3 \\ -3 & 3 & 4 \\ 5 & 0 & c \end{pmatrix}$$

2.

3. Determine the value of c for which the matrix does not have an inverse:

$$\begin{pmatrix} c & -1 & 0 \\ -1 & c & -1 \\ 0 & -1 & c \end{pmatrix}$$

4. Find the inverse, if it exists, by using the Gauss-Jordan method.

   (a) $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 2 & 1/2 \\ 3 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 1 & 0 \end{pmatrix}$

   (e) $\begin{pmatrix} 0 & 1 & 5 \\ 0 & -2 & 4 \\ 2 & 3 & -2 \end{pmatrix}$

   (f) $\begin{pmatrix} 2 & 2 & 3 \\ 1 & -2 & -3 \\ 4 & -2 & -3 \end{pmatrix}$

4 Determinants

Determinants are computed only for square matrices.

4.1 Determinants of 2×2 matrices

There is a simple method for calculating the determinant of a 2×2 matrix.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Interchanging 2 rows changes the sign of the det

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Multiplying a row by $k \neq 0$, Factorizing

$$\begin{aligned} \begin{vmatrix} ka & kb \\ c & d \end{vmatrix} &= kad - kbc = k(ad - bc) \\ &= k \begin{vmatrix} a & b \\ c & d \end{vmatrix} \end{aligned}$$

Adding a multiple of a row to another

$$\begin{aligned} \begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} &= a(d + kb) - b(c + ka) \\ &= ad - bc + akb - bka \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \end{aligned}$$

Example

$$A = \begin{pmatrix} 3 & -2 \\ 5 & 4 \end{pmatrix}, \qquad \det(A) = 22$$

$$\begin{vmatrix} -9 & 6 \\ 5 & 4 \end{vmatrix} = -3 \begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix} = -3(22)$$

$$\begin{vmatrix} 5 & 4 \\ 3 & -2 \end{vmatrix} = - \begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix} = -22$$

4.1.1. The geometric meaning of a 2×2 determinant.
Question: What is the geometric meaning of the numbers 22, -66 and -22?

4.2 Determinants of 3×3 using cofactor expansion:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

We can use the same method with any row or column.

We need the sign convention

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Short cut only for 3×3 determinants

$$\left|\begin{array}{ccc|cc} & & + & + & + \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \\ - & - & - & & \end{array}\right|$$

$$\begin{aligned} \det = {} & a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 \\ & - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3 \end{aligned}$$

Triangular matrices are nice:

$$\left|\begin{array}{ccc|cc} a_1 & a_2 & a_3 & a_1 & a_2 \\ 0 & b_2 & b_3 & 0 & b_2 \\ 0 & 0 & c_3 & 0 & 0 \end{array}\right| = a_1 b_2 c_3$$

$$\left|\begin{array}{ccc|cc} a_1 & 0 & 0 & a_1 & 0 \\ b_1 & b_2 & 0 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}\right| = a_1 b_2 c_3$$

We know how to change a matrix to an upper-triangular matrix using row operations.

What effect do row operations have on determinants?

4.3 Properties of Determinants

(1) Transpose, inverse and multiples

$$(1) \qquad \det A^{\mathsf{T}} = \det A \qquad \det A^{-1} = \frac{1}{\det A} \qquad \det(AB) = \det(A)\det(B)$$

Hoever

$$\det(A+B) \neq \det A + \det B$$

(2) Interchanging two rows

$$(2) \qquad \begin{vmatrix} \vdots \\ R_j \\ \vdots \\ R_k \\ \vdots \end{vmatrix} = - \begin{vmatrix} \vdots \\ R_k \\ \vdots \\ R_j \\ \vdots \end{vmatrix}$$

(3) Factorizing one row

$$(3) \qquad \begin{vmatrix} \vdots \\ cR_j \\ \vdots \\ R_k \\ \vdots \end{vmatrix} = c \begin{vmatrix} \vdots \\ R_j \\ \vdots \\ R_k \\ \vdots \end{vmatrix}$$

(4) Two equal rows

$$(4) \qquad \begin{vmatrix} \vdots \\ R_j \\ \vdots \\ R_k = R_j \\ \vdots \end{vmatrix} = 0$$

(5) Adding a multiple of one row to another

$$(5) \qquad \begin{vmatrix} \vdots \\ R_j \\ \vdots \\ R_k + 5R_j \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ R_j \\ \vdots \\ R_k \\ \vdots \end{vmatrix}$$

(6) Breaking a row into two parts

$$(6) \qquad \begin{vmatrix} \vdots \\ R_k + S_j \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ R_j \\ \vdots \end{vmatrix} + \begin{vmatrix} \vdots \\ S_j \\ \vdots \end{vmatrix}$$

Example

$$\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 5 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 4 & 1 \\ 0 & -3 & -3 & 0 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 0 & -3 & -3 \\ 6 & -2 & -4 \\ -6 & 7 & 7 \end{vmatrix} = (-1)\begin{vmatrix} 0 & -3 & -3 \\ \underline{6} & -2 & -4 \\ 0 & 5 & 3 \end{vmatrix}$$

$$= (-1)[(-1)\underline{6}]\begin{vmatrix} -3 & -3 \\ 5 & 3 \end{vmatrix}$$

$$= (-1)[(-1)3][-9+15] = 36$$

Example

$$\begin{vmatrix} 2 & -2 & -6 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -3 & -5 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -3 & -5 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (2)(1)(1)(-3)(-1) = 6$$

4.3.1 Theorem. *Let A be an $n \times n$ matrix.*

A is invertible
$\Updownarrow$
$\det A \neq 0$
$\Updownarrow$
$\text{rank}A = n$
$\Updownarrow$
A is row-equivalent to I_n
$\Updownarrow$
The homogenous system $A\mathbf{x} = 0$ has only the trivial solution.
$\Updownarrow$
The system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

The columns of A
are linearly independent iff
$\det A \neq 0$

4.4 Exercises

1. Find the determinant of each of the following:

   (a) $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 1/2 \\ 3 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 5 \\ 0 & -2 & 4 \\ 2 & 3 & -2 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 & 3 \\ 1 & -2 & -3 \\ 4 & -2 & -3 \end{pmatrix}$

2. Combine the method of row reduction and cofactor expansion to compute the determinant

$$\begin{vmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{vmatrix}, \quad \begin{vmatrix} 5 & -2 & 2 & 7 \\ 1 & 0 & 0 & 3 \\ -3 & 1 & 5 & 0 \\ 3 & -1 & -9 & 4 \end{vmatrix}, \quad \begin{vmatrix} 2 & -2 & 0 & 3 & 4 \\ 4 & -1 & 0 & 1 & -1 \\ 0 & 5 & 0 & 0 & -1 \\ 3 & 2 & -3 & 4 & 3 \\ 7 & -2 & 0 & 9 & -5 \end{vmatrix}$$

3. Find the determinant of A without multiplying through.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -3 & -3 & 0 \\ 4 & -1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 5 & -3 & 4 \\ 2 & 0 & -4 \end{pmatrix}$$

4. Assume that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$$

Find the following determinants and explain your answer:

$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = ---\qquad \begin{vmatrix} a+3g & b+3h & c+3i \\ d & e & f \\ g & h & i \end{vmatrix} = ---$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ a+3g & b+3h & c+3i \end{vmatrix} = ---\qquad \begin{vmatrix} a & b & c \\ -2d & -2e & -2f \\ g & h & i \end{vmatrix} = ---$$

$$\begin{vmatrix} a+3g & b+3h & c+3i \\ d & e & f \\ a+3g & b+3h & c+3i \end{vmatrix} = ---\qquad \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = ---$$

5. Compute the following determinants of the following matrices by row reduction to echelon form:

$$B = \begin{pmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & -7 \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{pmatrix}$$

6. Combine the method of row reduction and cofactor expansion to compute the determinant

$$\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix}$$

7. If $\det B = -5$, then $\det(B^{-1}B^{\intercal}) = -----$.

8. Find the determinant of A without multiplying through.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -3 & -3 & 0 \\ 4 & -1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 5 & -3 & 4 \\ 2 & 0 & -4 \end{pmatrix}$$

9.

$$\text{If} \qquad C = \begin{pmatrix} 2 & -2 & 13 & 3 & 4 \\ 4 & -1 & 29 & 1 & -1 \\ 0 & 5 & 7 & 0 & -1 \\ 3 & 2 & -3 & 4 & 3 \\ 7 & -2 & 0 & 9 & -5 \end{pmatrix}, \qquad \text{then} \qquad \det(C^{-1}C^{\intercal}) = -----$$

4.5 Cramer's Rule for $n \times n$ systems

Solving 2 equations in 2 unknowns

$$ax + by = r \, , \qquad cx + dy = s \tag{4.1}$$

$$x = \frac{rd - bs}{ad - bc} \, , \qquad y = \frac{as - cr}{ad - bc} \tag{4.2}$$

Using determinants,

$$x = \frac{\begin{vmatrix} r & b \\ s & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \, , \qquad y = \frac{\begin{vmatrix} a & r \\ c & s \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \tag{4.3}$$

Writing the solution of (4.1)
in the form (4.3)
is called
Cramer's rule

It is obvious that the system of equation (4.1) has a unique solution iff $det(A) \neq 0$, where A is the matrix of coefficients.

Exercise: Show that $det(A) = 0$ iff the second row of A is a multiple of the first one. That is,

$$A = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix}$$

Exercise: Show that $det(A) = 0$ iff the second column of A is a multiple of the first one. That is,

$$A = \begin{pmatrix} a & rb \\ c & rc \end{pmatrix}$$

Cramer's Rule for $n \times n$ linear system

$$A\mathbf{x} = \mathbf{b}$$

$$A_i(\mathbf{b}) = \left[\; a_1, \cdots, a_{i-1}, \mathbf{b}, a_{i+1}, \cdots, a_n \; \right]$$
$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}$$

4.5.1. The Inverse A^{-1} using the cofactor method: Notice that

$$\boxed{\text{col}_k(A^{-1}) = A^{-1}\mathbf{e}_k}$$

which is the solution to

$$\boxed{A\mathbf{x} = \mathbf{e}_k}$$

Let A_{ki} be the $(n-1)\times(n-1)$ matrix obtained from A by removing the k^{th} row and the i^{th} column. It follows that

$$\det A_{ki} = (-1)^{k+i}\det A_i(\mathbf{e}_k)$$

The $(-1)^{k+i}$ factor is nothing more then the sign convention.

$$\begin{aligned}(A^{-1})_{ki} &= \frac{\det A_i(\mathbf{e}_k)}{\det A}\\ &= (-1)^{k+i}\frac{\det A_{ki}}{\det A}\\ &= \frac{C_{ki}}{\det A}\\ C_{ki} &= (-1)^{k+i}\det A_{ki}\end{aligned}$$

thus

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n}\\ C_{21} & C_{22} & \cdots & C_{2n}\\ \vdots & \vdots & & \vdots\\ C_{n1} & C_{n2} & \cdots & C_{nn}\end{pmatrix}$$

$$\boxed{C_{ki} \text{ is called the cofactor of } a_{ki}}$$

Area of parallelogram determined by $\begin{pmatrix} a\\ c\end{pmatrix}$ and $\begin{pmatrix} b\\ d\end{pmatrix}$ is

$$\left|\det\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right| = ad - bc$$

Exercise: Completet the following sentence:
Volume of the parallelepiped determined by .... is ....
Theorem. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. Let S be a parallelogram in $\mathbb{R}^2$. Then

$$\boxed{\text{area of } T(S) = |\det A|\ (\text{area of } S)}$$

Theorem. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. Let S be a parallelepiped in $\mathbb{R}^3$. Then

$$\boxed{\text{area of } T(S) = |\det A|\ (\text{area of } S)}$$

Theorem. Let A be a square matrix of size n. Then

The system $A\mathbf{x} = \mathbf{b}$ is consistent
for all $\mathbf{b} \in \mathbb{R}^n$
iff $\det A \neq 0$

This is because

A square matrix A has an inverse
iff
$\det A \neq 0$

4.6 Exercise.

In each of the following use Cramer's rule to solve the linear system if possible. If Cramer's rule does not work, use Gauss-Jordan elimination to find why and if possible solve the system.

$$\begin{cases} x_1 & -3x_2 & +4x_3 & = 8 \\ x_1 & -x_2 & -2x_3 & = 2 \\ 2x_1 & -5x_2 & +3x_3 & = 15 \end{cases}, \qquad \begin{cases} 3\,x_1 & -7\,x_2 & +7\,x_3 & = & -8 \\ -4\,x_1 & +6\,x_2 & -\,x_3 & = & 7 \\ x_1 & -3\,x_2 & +4\,x_3 & = & -4 \end{cases}$$

$$\begin{cases} & 2\,x_2 & -4\,x_3 & = & -2 \\ x_1 & -3\,x_2 & +10\,x_3 & = & 5 \\ x_1 & -\,x_2 & +6\,x_3 & = & 3 \end{cases}$$

5 Vector Spaces

5.1 The geometry of $\mathbb{R}^n$

5.1.1. Example

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -3 \\ -1 \\ -5 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}, \mathbf{b} \begin{pmatrix} 8 \\ 2 \\ 15 \end{pmatrix} \tag{5.1}$$

Find c_1, c_2, and c_3 such that we can write

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

5.1.2. Vectors in $\mathbb{R}^n$. In all the operations listed below that involve two or more vectors $v, w, \cdots$, the vectors $v, w, \cdots$ must lie in the same space $\mathbb{R}^n$.
That is $v, w, \cdots$ must have the same number of components.

- **Addition of two vectors.** We can add two vectors iff they lie in the same space $\mathbb{R}^n$. That is, they have the same number of components.
- **The parallelogram rule.**
- **Rescaling a vector.** That is multiplying a vector by a scalar (real number).
- **The vector connecting two points** $P, Q \in \mathbb{R}^k$. $\overrightarrow{PQ} = Q - P$.
- **An equations for the line through two points** P **and** Q.

$$\mathbf{x} = P + t\,\overrightarrow{PQ}, \qquad t \in \mathbb{R}$$

- **A parametric equation for the plane through three points** P, Q **and** R.

$$\mathbf{x} = P + r\,\overrightarrow{PQ} + t\,\overrightarrow{PR}, \qquad r, t \in \mathbb{R}$$

5.1.3. Length of a vector.

$$\boxed{\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}}$$

5.1.4. Distance between two points P **and** Q**.**

$$d(P, Q) = \|\overrightarrow{PQ}\| = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}$$

5.1.5. Unit vectors If $\|w\| = 1$, it is said to be a unit vector.

5.1.6. The mid point between P **and** Q**.**

$$M = (\frac{x_1 + y_1}{2}, \cdots, \frac{x_n + y_n}{2})$$

5.1.7. The dot product of two vector v and w. The dot product of two vectors **v** and **w** that lie in the same space $\mathbb{R}^n$ is

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

5.1.8. The angle between two vector. The low of cosines (..) leads to the identity

$$\boxed{\mathbf{v} \cdot \mathbf{w} = |v|\ |w|\ \cos\theta}$$

where $\theta \in (-\pi, \pi]$ is the angle between the two vectors v and w.

Recall that $\cos\theta$ is an even function. (What does this mean?)

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$$

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\right)$$

5.1.9. Orthogonal vectors. Two nonzero vectors v and w are said to be orthogonal iff the angle between them is $\theta = \pm\pi/2$.

$$\boxed{\mathbf{v} \perp \mathbf{w} \quad \Leftrightarrow \quad \theta = \pm\pi/2 \quad \Leftrightarrow \quad \mathbf{v} \cdot \mathbf{w} = 0}$$

5.1.10. Note. It does not matter whether we measure the angle θ from v to w or from w to v (and have a difference of a minus sign) because $\cos\theta$ is an even function. That is $\cos(-\theta) = \cos(\cdots)$.

5.1.11. The cross product of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^3$.

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Question: Why is $\mathbf{u} \times \mathbf{v}$ orthogonal to both **u** and **v**?

$$\boxed{\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2}$$

5.1.12. Notice

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

2. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**. That is

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0, \qquad \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$$

5.2 Exercises

Homework: Do problems 1-12.

1. Find the point that lie in the middle of the line segment that connects the two points where $P = (3, 2, -1, 4)^\intercal$ and $Q = (3, -2, 4, , 7)^\intercal$.

2. Find the length of the vector $\overrightarrow{PQ}$ where $P = (2, -1, 3, 4)^\intercal$ and $Q = (4, 2, -3, 1)^\intercal$.

3. Find the length of the line segment that connects the two points $P = (2, -1, 3, 4)^\intercal$ and $Q = (4, 2, -3, 1)^\intercal$.

4. Find the dot-products $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$: $\mathbf{v} = (3, -2, 4, , 7)^\intercal$ and $\mathbf{u} = (1, 2, 3, -1)^\intercal$.

   What do you notice?

5. Find the angles between the two vector $(3, -2, 4, , 7, 2)^\intercal$ and $(1, 2, 3, -15)^\intercal$.

6. Find the angles between the two vector $(2, -1, 3, 4)^\intercal$ and $(4, 2, -3, 1)^\intercal$.

7. Find an equation for the hyperplane that contains the origin and perpendicular to the vector $\vec{n} = (-3, 2, -1, 5, 7)^\intercal$.

8. Find an equation for the hyperplane that contains the point $(2, -1, 3, 4)$ and perpendicular to the vector $\vec{n} = (2, 4, -5, 7)^\intercal$.

9. Find an equation for the plane that contains the point (x_1, x_2, x_3, x_4) and perpendicular to the vector $\vec{n} = (a, b, c, d)^\intercal$.

10. **** Find a unit vector that starts at the point $(2, -1, 2, -4)^\intercal$ and points in the direction of the point $(-1, 3, 5, 2)^\intercal$.

11. **** Find a vector of length 5 that starts at the point $(-3, -2, 1, -4, 3)^\intercal$ and points in the direction of the point $(3, 2, -1, 2, 4)^\intercal$.

12. Find an equation for the plane that contains the three points

$$(1, -1, 3, 4)^\intercal, \quad (2, 3, -1, 5)^\intercal, \quad (3, -2, 6, 4)^\intercal$$

13. Find an equation for the plane that contains the point $(2, -1, 4)^\intercal$ and the line

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + r \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \mid r \in \mathbb{R} \right\}$$

14. Show that

$$\boxed{\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \sin\theta, \qquad 0 \le \theta \le \pi}$$

Notice that we measure the angle θ counterclockwise.

15. Show that

$$\boxed{\|\mathbf{u}\times\mathbf{v}\| = \text{ area of the parallelogram with sides } \mathbf{u} \text{ and } \mathbf{v}.}$$

16. Find the cross-products $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$: $\mathbf{v}=(2,5,3)^\intercal$ and $\mathbf{u}=(4,-2,1)^\intercal$.

    What do you notice?

17. Find the area of the parallelogram with sides $(2,1,-3)$ and $(4,1,2)$.

18. Find the intersection of each pair if possible:

$$(a)\quad \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}+r\begin{pmatrix}0\\3\\0\end{pmatrix}+t\begin{pmatrix}2\\0\\4\end{pmatrix}\middle|\, r,t\in\mathbb{R}\right\},\qquad \left\{r\begin{pmatrix}1\\1\\1\end{pmatrix}+t\begin{pmatrix}0\\1\\3\end{pmatrix}\middle|\, r,t\in\mathbb{R}\right\}$$

$$(b)\quad \left\{\begin{pmatrix}2\\0\\1\end{pmatrix}+r\begin{pmatrix}1\\1\\-1\end{pmatrix}\middle|\, r\in\mathbb{R}\right\},\qquad \left\{r\begin{pmatrix}0\\1\\2\end{pmatrix}+t\begin{pmatrix}1\\1\\4\end{pmatrix}\middle|\, r,t\in\mathbb{R}\right\}$$

$$(c)\quad \left\{\begin{pmatrix}1\\1\\2\end{pmatrix}+r\begin{pmatrix}0\\1\\1\end{pmatrix}\middle|\, r\in\mathbb{R}\right\},\qquad \left\{\begin{pmatrix}1\\3\\-2\end{pmatrix}+t\begin{pmatrix}0\\1\\2\end{pmatrix}\middle|\, t\in\mathbb{R}\right\}$$

5.3 Definition and Examples of Vector Spaces and subspaces

5.3.1 Definition (Vector spaces). A set $\mathbf{V}$ is called a vector space iff for any $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in $\mathbf{V}$ and α and β in $\mathbb{R}$ the following holds

1. $\mathbf{u} + \mathbf{v}$ is also in $\mathbf{V}$.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $\mathbf{u} + 0 = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = 0$.
5. $\alpha\mathbf{u}$ is in $\mathbf{V}$.
6. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
7. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
8. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{v}$.
9. $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$

5.3.2 Definition (Subspaces). A nonempty subset $\mathbf{W}$ of a vector space $\mathbf{V}$ is itself a vector space (and called a vector subspace) iff whenever $\mathbf{w}_1 \in \mathbf{W}$ and $\mathbf{w}_2 \in \mathbf{W}$ and $\alpha \in \mathbb{R}$ then the following holds:

1. $\mathbf{w}_1 + \mathbf{w}_2$ is in $\mathbf{W}$.
2. $\alpha\mathbf{w}_1 \in \mathbf{W}$.

   What about $\alpha\mathbf{w}_2$, is it also in $\mathbf{W}$?

More generaly

> A subspace of a vector space $\mathbf{V}$
> is any nonempty subset $\mathbf{W} \subseteq V$ which is
> <u>closed under linear combinations</u>
> <u>of any number of vectors</u>.
>
> That is,
>
> given any collection of k vectors
> $\mathbf{w}_1, \cdots, \mathbf{w}_k$ in $\mathbf{W}$
> and any k real numbers $a_1, \cdots, a_k$,
> then the linear combination
> $a_1\mathbf{w}_1 + \cdots + a_k\mathbf{w}_k$
> is also in $\mathbf{W}$.

Question. Why are these two definitions equivalent?

5.4 Exercises

1. For each of the given sets answer the following questions:

   - Is it a vector space under the standard addition and scalar multiplication?
   - If it is not, determine all the properties that fail.
   - If it is, find its dimension.
   - Sketch when possible.

   (a) The set $F(\mathbb{R}, \mathbb{R})$ consisting of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$.

   (b) The set $\mathcal{M}_{t\times 4}$ consisting of all matrices of size 3×4.

   (c) $\mathbf{V} = \{(x, y) \in \mathbb{R} \mid x \geq 0, y \geq 0\}$.

   (d) $\mathbf{V} = \{(x, y) \in \mathbb{R} \mid x > 0, y > 0\}$.

   (e) $\mathbf{V} = \{(x, y) \in \mathbb{R} \mid xy > 0\}$.

   (f) $\mathbf{V} = \{(x, y) \in \mathbb{R} \mid xy \geq 0\}$.

   (g) $\mathcal{P}_3 =$ the set of all polynomials $p(t)$ of degree 3.

2. For each of the following, determine the ambient space and then determine whether the given set is a subspace. If it is a vector space, find its dimension.

   (a) $\mathbf{V} = \left\{ \begin{pmatrix} a & 0 & c \\ d & e & 0 \end{pmatrix} \mid a, b, c, d, e \text{ are real numbers} \right\}$.

   (b) $\mathbf{V} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \text{ are real numbers and } a + d = 0 \right\}$

   (c) $\mathbf{V} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \in \mathbb{R} \right\}$

   (d)
   $$\mathbf{V}_1 = \left\{ t \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + r \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \mid -\infty < t, r < \infty \right\}$$
   $$\mathbf{V}_2 = \left\{ \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + r \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \mid -\infty < t, r < \infty \right\}$$

   (e) i. $\mathbf{V}_1 =$ all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that takes the form $f(t) = a\cos 5t + b\sin 5t$, where a and b are real numbers.

   ii. $\mathbf{V}_2 =$ all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that takes the form $f(t) = ae^{3t} + be^{-2t}$, where a and b are real numbers.

   (f) i. $\mathcal{B}_1 =$ the set of all polynomials $p(t)$ of degree 3 that satisfies $p(0) = 5$.

   ii. $\mathcal{B}_2 =$ the set of all polynomials $p(t)$ of degree 3 that satisfies $p(0) = 0$.

(g) i. $\mathcal{B}_1 =$ the set of all polynomials $p(t)$ of degree 4 that satisfies $d^2p/dt^2(0) = 0$.
ii. $\mathcal{B}_2 =$ the set of all polynomials $p(t)$ of degree 4 that satisfies $d^2p/dt^2(0) = 7$.

(h) $\mathbf{W} =$ the solution set of the homogenous system

$$\begin{cases} x_1 & -3\,x_2 & +\,x_3 & +\,x_4 & = & 0 \\ 2\,x_1 & -6\,x_2 & -\,x_3 & +\,x_4 & = & 0 \\ -3\,x_1 & +9\,x_2 & +3\,x_3 & -\,x_4 & = & 0 \\ 4\,x_1 & -12\,x_2 & +4\,x_3 & +\,4x_4 & = & 0 \end{cases}$$

(i) $\mathbf{W} =$ the solution set of the non-homogenous system

$$\begin{cases} x_1 & -3\,x_2 & +\,x_3 & +\,x_4 & = & 3 \\ 2\,x_1 & -6\,x_2 & -\,x_3 & +\,x_4 & = & -6 \\ -3\,x_1 & +9\,x_2 & +3\,x_3 & -\,x_4 & = & 15 \\ 4\,x_1 & -12\,x_2 & +4\,x_3 & +\,4x_4 & = & 12 \end{cases}$$

(j) $\mathbf{W} =$ the solution set of the homogenous system

$$\begin{cases} x_1 & -3x_2 & +x_3 & = 0 \\ 2x_1 & -6x_2 & -x_3 & = 0 \\ 3x_1 & -9x_2 & & = 0 \\ 4x_1 & -12x_2 & +4x_3 & = 0 \end{cases} \qquad \text{(NH)}$$

(k) $\mathbf{W} =$ the solution set of the non-homogenous system

$$\begin{cases} x_1 & -3x_2 & +x_3 & = 1 \\ 2x_1 & -6x_2 & -x_3 & = 1 \\ 3x_1 & -9x_2 & & = 2 \\ 4x_1 & -12x_2 & +4x_3 & = 4 \end{cases} \qquad \text{(NH)}$$

5.5 Projections and orthogonal projections.

Let $\mathbf{F}$ be a force that pulls an object along the line determined by $\mathbf{v}$ (possibly in opposite direction). And let the angle between $\mathbf{F}$ and $\mathbf{v}$ be θ. For example if we pull a box along a surface using a robe that makes an angle of $\pi/6$ with the surface and applying a force of magnitude $\|F\|$.
Question. How much of the force is actually used and how much is lost?

- The part that is used is called **the component of F along v**. It could be negative if we pull the object in the direction opposite to $\mathbf{v}$. In this case $\pi/2 \le \theta \le \pi$ and $-1 \le \cos\theta \le 0$.

The **component of F along v**

$$comp_{\mathbf{v}}\mathbf{F} = \|F\|\cos\theta = \frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|}$$

- We can make this into a vector by multiplying it by a unit vector in the direction of $\mathbf{v}$.

The **vector projection of F along v**

$$proj_{\mathbf{v}}\mathbf{F} = \|F\|\cos\theta\,\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\,\mathbf{v}$$

- Recall that $\|\mathbf{F}\|^2 = \|\mathbf{F}\|^2(\cos^2\theta + \sin^2\theta)$. Thus the lost part of the force $\|\mathbf{F}\|$ is

The **component of F orthogonal to v**

$$comp^{\perp}_{\mathbf{v}}\mathbf{F} = \|F\|\sin\theta = \frac{\|\mathbf{F}\times\mathbf{v}\|}{\|\mathbf{v}\|}$$

- Now we make this lost quantity into a vector

The **orthogonal projection** of $\mathbf{F} \perp \mathbf{v}$

$$proj^{\perp}_{\mathbf{v}}\mathbf{F} = \mathbf{F} - \frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$$

Remarks. Note that

- $comp_{\mathbf{v}}\mathbf{F}$ and $comp^{\perp}_{\mathbf{v}}\mathbf{F}$ are scalars (real numbers)
- while $proj_{\mathbf{v}}\mathbf{F}$ and $proj^{\perp}_{\mathbf{v}}\mathbf{F}$ are vectors.
- $proj_{\mathbf{v}}\mathbf{F}$ can point in opposite direction of $\mathbf{v}$. When?

5.6 Exercises

For each of the following, if possible, find the component, projection and orthogonal projection of the first vector relative to the second. Then find the same objects for the second relative to the first.

1. $(2, 3, -1, 4)^\intercal, \quad (1, -2, 3, 1)^\intercal.$
2. $(3, 0, 2, -1, 1)^\intercal, \quad (3, 1, -1, 4, 5)^\intercal.$
3. $(2, 3, -1, 4)^\intercal, \quad (1, -2, 3, 1, -3)^\intercal.$

5.7 Linear Combinations of Vectors

Let $\mathbf{v}_1, \cdots, \mathbf{v}_4$ be given (known) 4 vector in the 7-dimensional space $\mathbb{R}^7$(i.e. each $\mathbf{v}$ has 7 components).

1. Let $c_1, \cdots, c_4$ be any 4 real numbers.

   The vector $\mathbf{b}$ given by
   $\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_4\mathbf{v}_4$
   is called a **linear combination of the vectors** $\mathbf{v}_1, \cdots, \mathbf{v}_4$.

   The vector $\mathbf{b}$ is (lies, lives) in the $\cdots$-dimensional space $\mathbb{R}^{\cdots}$(i.e. $\mathbf{b}$ has $\cdots$ components).

2. Nothing special about the numbers 4 and 7. We can replace 4 by any positive integer n and 7 by any positive integer$k$: Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be given (known) n vector in the k-dimensional space $\mathbb{R}^k$(i.e. each $\mathbf{v}$ has k components). Let $c_1, \cdots, c_n$ be any n real numbers.

3. In Example 5.7.4 the vector $\mathbf{b}$ can be written as a linear combination of the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_4$ in infinitely many ways given by (5.8). Any choice of r and t gives us a way of writing $\mathbf{b}$ as a linear combination of the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_4$.

5.7.1 Definition.

The vector $\mathbf{b}$ given by
$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$
is called a **linear combination of the vectors** $\mathbf{v}_1, \cdots, \mathbf{v}_n$.

In matrix notation

$$\mathbf{b} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \mathbf{c}$$

The vector $\mathbf{b}$ is (lies, lives) in the $\cdots$-dimensional space $\mathbb{R}^{\cdots}$(i.e. $\mathbf{b}$ has $\cdots$ components).

5.7.2 Example.

$$3\mathbf{v}_1 - 5\mathbf{v}_2 = 3\begin{pmatrix} 11 \\ -3 \\ 13 \end{pmatrix} - 5\begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix} = \begin{pmatrix} 8 \\ -19 \\ 79 \end{pmatrix}$$

5.7.3 Example. In each of the following Determine whether $\mathbf{b}$ can be written as a linear combination of (lies in the span of; lies in the plane spanned by) $\mathbf{v}_1$ and $\mathbf{v}_2$:

1. $\mathbf{v}_1 = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 10 \\ -6 \\ 12 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 11 \\ -6 \\ 9 \end{pmatrix}$

2. $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$

3. $\mathbf{v}_1 = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 10 \\ -6 \\ 12 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 11 \\ -6 \\ 9 \end{pmatrix}$

Solution.

(a) Try to solve the system

$$\begin{aligned} 3x_1 + 10x_2 &= 11 \\ -2x_1 - 6x_2 &= -6 \\ 5x_1 + 12x_2 &= 9 \end{aligned}$$

In matrix notation

$$\begin{pmatrix} 3 & 10 \\ -2 & -6 \\ 5 & 12 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 11 \\ -6 \\ 9 \end{pmatrix}$$

If we can solve this system, then the answer is: Yes **b** can be written as .... If we cannot then the answer is: No, **b** cannot be written as ....
(b)& (c) Can be solved similarly.

5.7.4 Example. Find, if possible, c_1, c_2, c_3 and c_4 such that

$$\begin{pmatrix} 3 \\ -6 \\ 15 \\ 12 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ -3 \\ 4 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 4 \end{pmatrix} \tag{5.2}$$

We can ask the same question in a different way:
If possible write the vector **b** as a linear combination of the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_4$.

$$\mathbf{b} = \begin{pmatrix} 3 \\ -6 \\ -3 \\ 12 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 4 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} \tag{5.3}$$

Solution.

In either case we need to solve the system

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_1 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{b} \tag{5.4}$$

That is: Find c_1, c_2, c_3 and c_4 such that

$$\begin{pmatrix} 3 \\ -6 \\ -15 \\ 12 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ -3 \\ 4 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 4 \end{pmatrix} \tag{5.5}$$

thus we need to solve the system

$$\begin{cases} x_1 & -3x_2 & +x_3 & +x_4 & = 3 \\ 2x_1 & -6x_2 & -x_3 & +x_4 & = -6 \\ 3x_1 & +9x_2 & +3x_3 & -x_4 & = 15 \\ 4x_1 & -12x_2 & +4x_3 & +4x_4 & = 12 \end{cases} \tag{5.6}$$

Use Eq(1) to eliminate x_1 from the other three equations

$$\begin{cases} (-2eq_1 + eq_2) & \mapsto eq_2 \\ (3eq_1 + eq_3) & \mapsto eq_3 \\ (-4eq_1 + eq_4) & \mapsto eq_4 \end{cases} \Longrightarrow \begin{cases} x_1 & -3x_2 & +x_3 & +x_4 & = 3 \\ & & -3x_3 & -x_4 & = -12 \\ & & 6x_3 & +2x_4 & = 24 \\ & & & 0 & = 0 \end{cases}$$

Notice that the fourth equation became irrelevant.
Use Eq(....) to eliminate $\cdots$ from Eq(3) and Eq(...); and then divide the second by -3 and we have:

$$\xrightarrow[(......+.......)\mapsto eq....]{(......+.......)\mapsto eq....} \begin{cases} x_1 & -3x_2 & & +(2/3)x_4 & = -1 \\ & & -3x_3 & -x_4 & = -12 \\ & & & 0 & = 0 \\ & & & 0 & = 0 \end{cases}$$

$$\xrightarrow{eq_2/-3} \begin{cases} x_1 & -3x_2 & & +(2/3)x_4 & = -1 \\ & & +x_3 & +(1/3)x_4 & = 4 \\ & & & 0 & = 0 \\ & & & 0 & = 0 \end{cases}$$

Thus x_1 and x_2 are basic variables and x_2 and x_4 are free variables. Take $x_2 = r$ and $x_4 = t$ as parameters and obtain

$$\begin{cases} x_1 & & & & = -1 + 3r - (2/3)t \\ & x_2 & & & = r \\ & & x_3 & & = 4 - (1/3)t \\ & & & x_4 & = t \end{cases}$$

The solution set to the system (5.6) is

$$\mathbf{S}: \quad \mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 4 \\ 0 \end{pmatrix} + r \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2/3 \\ 0 \\ -1/3 \\ 1 \end{pmatrix}, \qquad -\infty < r, t < \infty \tag{5.7}$$

Description of $\mathbf{S}$: It is the plane passing through the point $(-1, 0, 4, 0)$ and generated by the two vectors $(3, 1, 0, 0)$ and $(2/3, 0, -1/3, 1)$.
Dimension of $\mathbf{S}$ = 2.
Answer to the question we are asked: The set $\mathbf{S}$ (5.7) is the solution set to the linear system (1.12) but it is not the answer to the question we are asked. We are asked to write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_4$. The answer to this question is that we can write $\mathbf{b}$ as a linear

combination of $\mathbf{v}_1, \cdots, \mathbf{v}_4$ in infinitely many ways, namely, for any $r \in \mathbb{R}$ and any $t \in \mathbb{R}$ we can write $\mathbf{b}$ as the linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_4$ given by

$$\begin{pmatrix} 3 \\ -6 \\ -15 \\ 12 \end{pmatrix} = [-1 + 3r - (2/3)t] \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + r \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \end{pmatrix} + [4 - (1/3)t] \begin{pmatrix} 1 \\ -1 \\ -3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 4 \end{pmatrix} \tag{5.8}$$

For example if we take $r = 3$ and $t = -2$ we have $\mathbf{b}$ as the linear combination

$$\begin{pmatrix} 3 \\ -6 \\ -15 \\ 12 \end{pmatrix} = [\cdots] \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + 3 \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \end{pmatrix} + [\cdots] \begin{pmatrix} 1 \\ -1 \\ -3 \\ 4 \end{pmatrix} + [\cdots] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 4 \end{pmatrix}$$

5.8 Exercises.

1. Determine whether the vector $\mathbf{a}$ is a linear combination of the others. If it is find all possible ways of doing that.

   (a) $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{a}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

   (b) $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{a}_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$

   (c) $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{a}_1 = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{a}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

   (d) $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$, $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$, $\mathbf{a}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

   (e) $\mathbf{a} = \begin{pmatrix} 3 \\ -17 \\ 17 \\ 7 \end{pmatrix}$, $\mathbf{a}_1 = \begin{pmatrix} 2 \\ -3 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 6 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{a}_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \\ 3 \end{pmatrix}$

2. Find all possible ways that the vector **b** can be written as a linear combination of the others.

$$\mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

3. Determine if the matrix A is a linear combination of the others:

   (a) $A = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -2 & 3 \\ 1 & 4 \end{pmatrix}, \quad M_3 = \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 3 & -1 \\ 2 & -2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}$

4. Write the product $A\mathbf{x}$ as a linear combination of the columns of A:

$$A = \begin{pmatrix} 1 & 2 & -8 \\ 2 & 3 & 7 \\ -3 & -1 & 1 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$$

5. Write each column of AB as a linear combination of the columns of A.

$$A = \begin{pmatrix} 1 & 2 & -8 \\ 2 & 3 & 7 \\ -3 & -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 5 & 4 \\ -1 & -2 & 3 \\ 3 & -7 & 6 \end{pmatrix}$$

6. If possible, write the given polynomial as a linear combination of $p_1(x) = 1 + x$ and $p_2(x) = x^2$.

   (a) $p(x) = 2x^2 - 3x - 1$.

   (b) $p(x) = -x^2 + 3x31$.

7. If possible, write the given polynomial as a linear combiation of

$$p_1(x) = 1 + x, \quad p_2(x) = -x, \quad p_3(x) = x^2 + 1, \quad p_4(x) = 2x^3 - x + 1$$

   (a) $p(x) = x^3 - 2x + 1$.

   (b) $p(x) = -4x^3$.

8. Describe all 2×2 matrices that can be written as a linear combination of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

5.9 The span of Vectors

Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be given (known) n vector in the k-dimensional space $\mathbb{R}^k$(i.e. each $\mathbf{v}$ has k components).

5.9.1 Definition.

Span$\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$
is the collection of all possible linear combinations
of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots$ and $\mathbf{v}_n$.

$$\Updownarrow$$

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n | c_1, c_2, \cdots, c_n \in \mathbb{R}\}$$

Thus

span$\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$
is the collection of all $\mathbf{b}$ for which the system
$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$
is consistent (i.e. has a solution).

℧℧ This is not the solution set of this system.

5.9.2 Theorem.

$\mathbf{b}$ *is in span*$\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$

$$\Updownarrow$$

$\mathbf{b}$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$

$$\Updownarrow$$

the linear system $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$
is consistent.

$$\Updownarrow$$

In matrix notation
$A\mathbf{x} = \mathbf{b}$
is consistent
where $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$.

Thus

To test whether a vector $\mathbf{b}$ *is in span*$\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ *we try to solve the system*
$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$.
If we can, the answer is "yes it is".

5.9.3 Example. Find the value(s) of h for which $\mathbf{b}$ lies in the plane spanned by (a linear combination of) the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ h \end{pmatrix}$$

Solution.

Find the value(s) of h for which the system

$$x_1 \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ h \end{pmatrix}$$

is consistent (i.e. has a solution). As usual we start by witting the system explicitly.

$$\left\{ \begin{array}{lll} x_1 & -2x_2 & = 4 \\ 4x_1 & -3x_2 & = 1 \\ -2x_1 & +7x_2 & = h \end{array} \right. \longrightarrow \left\{ \begin{array}{lll} x_1 & -2x_2 & = 4 \\ & 5x_2 & = -15 \\ & 3x_2 & = h+8 \end{array} \right. \longrightarrow \left\{ \begin{array}{lll} x_1 & -2x_2 & = 4 \\ & 5x_2 & = -15 \\ & 0 & = h+17 \end{array} \right.$$

In order for the system to be consistent we shouldn't have an equation of the form $0 = c$ with $c \neq 0$.

Answer.

Thus $\mathbf{b}$ lies in the plane spanned by (a linear combination of) the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ iff (if and only if)

$$h = -17$$

5.9.4 Example. 1. Find the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$.

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

2. Determine which of the following vectors lie in the space generated (spanned) by $\{\mathbf{v}_1, \mathbf{v}_2\}$.

$$\mathbf{a}_1 = \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 5 \\ -4 \\ 7 \end{pmatrix}$$

Answer: We need to find all possible vectors $\mathbf{y}$ that can be written as a linear combination of $\mathbf{v}_1 \& \mathbf{v}_2$:

$$\mathbf{y} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$$

$$\Leftrightarrow \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{y}$$

$$\left(\begin{array}{cc|c} 2 & 3 & y_1 \\ 1 & -1 & y_2 \\ -3 & 2 & y_3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & y_2 \\ 2 & -3 & y_1 \\ -3 & 2 & y_3 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & -1 & y_2 \\ 0 & -1 & y_1 - 2y_2 \\ 0 & -1 & y_3 + 3y_2 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & -1 & y_2 \\ 0 & -1 & y_1 - 2y_2 \\ 0 & 0 & y_3 - y_1 + 5y_2 \end{array}\right)$$

$$\boxed{\begin{array}{c}\text{This system is consistent} \\ \Updownarrow \\ y_3 - y_1 + 5y_2 = 0\end{array}}$$

1. Thus

$$\boxed{\begin{array}{c}\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \\ \{\mathbf{y} \in \mathbb{R}^3 \mid -y_1 + 5y_2 + y_3 = 0\}\end{array}}$$

2. The vector $\mathbf{a}_1$ is in span$\{\mathbf{v}_1, \mathbf{v}_2\}$ because

$$-4 + 10 - 6 = 0$$

The vector $\mathbf{a}_2$ is not in span$\{\mathbf{v}_1, \mathbf{v}_2\}$ because

$$-5 - 20 + 7 = -18 \neq 0$$

5.9.5 Example. The following four questions are equivalent. That is have the same answer.

1. Find the value(s) of b_1, b_2 and b_3 for which $\mathbf{b}$ lie in the plane spanned by (a linear combination of) the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.
2. Find the set of all $\mathbf{b}$'s that are a linear combination of the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.
3. Find and describe the span of $\mathbf{a}_1$ and $\mathbf{a}_2$.
4. Find the set of all $\mathbf{b}$ the linear system $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ is consistent.

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$$

Solution.

As we mentioned, these four questions have the same answer which is the answer to part (d). Thus we need to find the set of all vectors **b** for which the following system s consistent.

$$x_1 \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Notice that we switched $\mathbf{a}_1$ and $\mathbf{a}_2$. This is allowed since we can label vectors any which way we like as long as we do not change the labeling during the course of the work.

As usual we start by witting the system explicitly.

$$\left\{\begin{array}{lll} -5x_1 & +3x_2 & = b_1 \\ x_1 & -2x_2 & = b_2 \\ 3x_1 & +4x_2 & = b_3 \end{array}\right. \longrightarrow \left\{\begin{array}{lll} x_1 & -2x_2 & = b_2 \\ -5x_1 & +3x_2 & = b_1 \\ 3x_1 & +4x_2 & = b_3 \end{array}\right. \longrightarrow \left\{\begin{array}{lll} x_1 & -2x_2 & = b_2 \\ & -7x_2 & = b_1 + 5b_2 \\ & 10x_2 & = b_3 - 3b_1 \end{array}\right.$$

$$\left\{\begin{array}{lll} x_1 & -2x_2 & = b_2 \\ & -70x_2 & = 10b_1 + 50b_2 \\ & 70x_2 & = 7b_3 - 21b_1 \end{array}\right. \longrightarrow \left\{\begin{array}{lll} x_1 & -2x_2 & = b_2 \\ & -70x_2 & = 10b_1 + 50b_2 \\ & 0 & = 10b_1 + 29b_2 + 7b_3 \end{array}\right.$$

In order for the system to be consistent we shouldn't have an equation of the form $0 = c$ with $c \neq 0$.

Answer.

Thus **b** lies in the plane spanned by (a linear combination of) the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ iff (if and only if)

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2\} = \{\mathbf{b} \in \mathbb{R}^3 | 10b_1 + 29b_2 + 7b_3 = 0\}$$

5.9.6 Definition. Let A be an $r \times c$ matrix

$$A = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_c \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_r \end{pmatrix}$$

1. The column space of a matrix A = the range of a matrix A

$$\begin{array}{c} \text{col } A = \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_c\} = \text{rng } A \\ = \text{the set of all } \mathbf{y} \text{ for which } A\mathbf{x} = \mathbf{y} \text{ is consistent.} \end{array}$$

2. The row space of a matrix A

$$\begin{array}{c} \text{row } A = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \vdots, \mathbf{r}_r\} = \text{rng } A \\ = \text{the set of all } \mathbf{y} \text{ for which } A\mathbf{x} = \mathbf{y} \text{ is consistent.} \end{array}$$

3. The <u>nul space of a matrix</u> A

$$\boxed{\begin{array}{c}\text{nul } A = \text{the set of all solutions of } A\mathbf{x} = \mathbf{0} \\ = \{\mathbf{x} \mid A\mathbf{x} = 0\}\end{array}}$$

5.9.7 Example. For each of the vectors $\mathbf{a}_1, \cdots, \mathbf{a}_4$:

1. Determine whether it lies in the span of $\mathbf{v}_1, \cdots, \mathbf{v}_3$.
2. Determine whether it is a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_3$.
3. Determine whether the linear system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{a}$ is consistent.
4. Find and describe the span of $\mathbf{v}_1, \cdots, \mathbf{v}_3$.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 4 \end{pmatrix},$$
$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 5 \\ 1 \\ 1 \\ -3 \end{pmatrix}, \quad \mathbf{a}_4 = \begin{pmatrix} 4 \\ 1 \\ 5 \\ 3 \end{pmatrix},$$

<u>**Solution**</u>.

Notice that the answers to the first three questions (1-3) are the same. for the three of them we need to find out whether the linear system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{a}$ is consistent. That is, to answer question 3.

$$\begin{cases} x_1 & -3x_2 & +x_3 & = b_1 \\ 2x_1 & -6x_2 & -x_3 & = b_2 \\ -3x_1 & -9x_2 & & = b_2 \\ 4x_1 & -12x_2 & +4x_3 & = b_4 \end{cases}$$

$$\begin{cases} x_1 & -3x_2 & +x_3 & = b_1 \\ & & -3x_3 & = b_2 - 2b_1 \\ & & -3x_3 & = b_3 - 3b_1 \\ & & 0 & = b_4 - 4b_1 \end{cases} \longrightarrow \begin{cases} x_1 & -3x_2 & +x_3 & = b_1 \\ & & x_3 & = -(b_2 - 2b_1)/3 \\ & & 0 & = b_3 - 3b_1 - (b_2 - 2b_1) \\ & & 0 & = b_4 - 4b_1 \end{cases}$$

Thus, in order for the linear system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$ is consistent, the vector $\mathbf{b}$ must satisfy both of the two conditions

$$\begin{cases} b_3 - b_1 - b_2 & = 0 \\ b_4 - 4b_1 & = 0 \end{cases} \tag{*}$$

Check which of the 4 vector satisfies these two conditions:

- $\mathbf{a}_1$: $4-4=0, \quad -1-1+2=0$.
- $\mathbf{a}_2$: $0-8\neq 0$. In this case we do not need to check the second condition.
- $\mathbf{a}_3$: $-3-20\neq 0$.
- $\mathbf{a}_3$: $3-16\neq 0$.

Answer 1-3:

1. Only $\mathbf{a}_1$ lies in the span of of $\mathbf{v}_1, \cdots, \mathbf{v}_3$.
2. Only $\mathbf{a}_1$ is a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_3$.
3. Only $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{a}$ is consistent.

To answer part (4) we reduce the system to the reduced echelon form:

$$\begin{cases} x_1 \quad -3x_2 \qquad\quad = (b_1+b_2)/3 \\ \qquad\qquad\quad x_3 = -(b_2-2b_1)/3 \\ \qquad\qquad\quad 0 = b_3-3b_1-(b_2-2b_1) \\ \qquad\qquad\quad 0 = b_4-4b_1 \end{cases}$$

Thus, if the system is consistent, (i.e. $\mathbf{b}$ satisfies the 2 conditions in (*)) the system has infinitely many solutions with x_2 as a free variable:

$$\begin{aligned} x_1 &= (b_1+b_2)/3+3r \\ x_2 &= r, \qquad\qquad -\infty < r < \infty \\ x_3 &= (2b_1-b_2)/3 \end{aligned}$$

When $\mathbf{b}=\mathbf{a}_1$, the solution is

$$\begin{aligned} x_1 &= 2/3+3r \\ x_2 &= r, \qquad\qquad -\infty < r < \infty \\ x_3 &= 1/3 \end{aligned}$$

In vector parametric form,

$$\mathbf{x} = \begin{pmatrix} 2/3 \\ 0 \\ 1/3 \end{pmatrix} + r\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \qquad -\infty < r < \infty$$

5.9.8 Example. Consider the vectors

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} -7 \\ -5 \\ 3 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$$

1. Find and describe the span of $\mathbf{a}_1$ and $\mathbf{a}_2$.
2. For each $\mathbf{v}$, determine whether it lies in $\text{span}\{\mathbf{a}_1, \mathbf{a}_2\}$.
3. For each $\mathbf{v}$, determine whether it can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.
4. For each $\mathbf{v}$, determine whether the linear system of $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{v}$ is consistent. If it is, find $\mathbf{x}$.

Solution.

Notice that parts (2-4) have the same answer: For each $\mathbf{v}$ the answer is either "yes" on all of them or "no" for all of them. This is because to answer any of them we need to decide whether $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{v}$ has a solution.

1. There are two methods to answer question (1). First we describe each. Then we give the advantage of each, or when to use each.

   (a) **First method:** Observe that the span of the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ is a plane through the origin. (Why?) Therefore we find a vector normal to both $\mathbf{a}_1$ and $\mathbf{a}_2$:

   $$\mathbf{a}_1 \times \mathbf{a}_2 = \begin{pmatrix} 12 \\ 15 \\ -3 \end{pmatrix}, \qquad \mathbf{n} = (1/3)\mathbf{a}_1 \times \mathbf{a}_2 = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \tag{5.9}$$

   We divide by the common factor 3 only to use smaller numbers.

   Now the equation of the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$ is

   $$\mathbf{n} \cdot \mathbf{x} = 0, \qquad \text{that is ,} \qquad 4x_1 + 5x_2 - x_3 = 0$$

   Thus the span of $\{\mathbf{a}_1, \mathbf{a}_2\}$ is the plane

   $$\text{span}\{\mathbf{a}_1, \mathbf{a}_2\} = \{\mathbf{b} \in \mathbb{R}^3 | 4x_1 + 5x_2 - x_3 = 0\}$$

   This answers question (1) using the first method.

   (b) **Second method:** Recall that Theorem 5.9.2 tells us that

   $$\boxed{\begin{array}{c} \mathbf{b} \text{ is in span}\{\mathbf{a}_1, \mathbf{a}_2\} \\ \Updownarrow \\ \mathbf{b} \text{ is a linear combination of } \mathbf{a}_1 \text{ and } \mathbf{a}_2 \\ \Updownarrow \\ \text{the linear system } x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \text{ is consistent.} \end{array}}$$

   Thus, to answer any of the four questions we need to find $\mathbf{b} = (b_1, b_2, b_3)^\intercal$ for which the following system is consistent (i.e. has a solution):

$$x_1 \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + x_2 \begin{pmatrix} -7 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

As usual we start by witting the system explicitly.

$$\begin{cases} 2x_1 & -7x_2 & = b_1 \\ x_1 & -5x_2 & = b_2 \\ -3x_1 & +3x_2 & = b_3 \end{cases} \longrightarrow \begin{cases} x_1 & -5x_2 & = b_2 \\ 2x_1 & -7x_2 & = b_1 \\ -3x_1 & +3x_2 & = b_3 \end{cases} \longrightarrow \begin{cases} x_1 & -5x_2 & = b_2 \\ & 3x_2 & = b_1 - 2b_2 \\ & -12x_2 & = b_3 + 3b_2 \end{cases}$$

$$\begin{cases} x_1 & -5x_2 & = b_2 \\ & 3x_2 & = b_1 - 2b_2 \\ & 0 & = b_3 + 3b_2 + 4(b_1 - 2b_2) \end{cases} \longrightarrow \begin{cases} x_1 & -5x_2 & = b_2 \\ & 3x_2 & = b_1 - 2b_2 \\ & 0 & = 4b_1 - 5b_2 + b_3 \end{cases}$$

In order for the system to be consistent we shouldn't have an equation of the form $0 = c$ with $c \neq 0$.

Thus the span of $\{\mathbf{a}_1, \mathbf{a}_2\}$ is the plane

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2\} = \{\mathbf{b} \in \mathbb{R}^3 | 4x_1 + 5x_2 - x_3 = 0\}$$

This answers question (1).

Which method to use?: Before answering (2-4) Let us find out which method to use:

- The first method is simpler but it works only in $\mathbb{R}^3$ not in any higher dimension.
- The second method works in any dimension.

2. A vector $\mathbf{v}$ lies in the plane spanned by (a linear combination of) the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ iff (if and only if)

$$4b_1 - 5b_2 + b_3 = 0 \qquad (*)$$

- $\mathbf{v}_1 : -8 - 15 + 5 \neq 0 :$ No, $\mathbf{v}_1$ does not lie in span$\{\mathbf{a}_1, \mathbf{a}_2\}$.
- $\mathbf{v}_2 : 4 - 10 + 6 = 0 :$ Yes, $\mathbf{v}_2$ lies in span$\{\mathbf{a}_1, \mathbf{a}_2\}$.
- $\mathbf{v}_3 : 8 - 15 + 7 = 0 :$ Yes, $\mathbf{v}_2$ lies in span$\{\mathbf{a}_1, \mathbf{a}_2\}$.
- $\mathbf{v}_4 : 12 - 10 + 6 \neq 0 :$ No, $\mathbf{v}_1$ does not lie in span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

3. From the definition answer to part (2) is yes iff the answer to part (3) is yes. That is

- No, $\mathbf{v}_1$ cannot be written as a linear combination.
- Yes, $\mathbf{v}_2$ can be written as a linear combination.
- Yes, $\mathbf{v}_3$ can be written as a linear combination.
- No, $\mathbf{v}_4$ cannot be written as a linear combination.

4. We need to answer this question only for $\mathbf{v}_2$ and $\mathbf{v}_3$. We simplify the system to the reduced echelon form:

$$\begin{cases} x_1 & -5x_2 & = b_2 \\ & 3x_2 & = b_1 - 2b_2 \\ & 0 & = 4b_1 - 5b_2 + b_3 \end{cases} \longrightarrow \begin{cases} x_1 & = (5b_1 - 7b_2)/3 \\ x_2 & = (b_1 - 2b_2)/3 \\ 0 & = 4b_1 - 5b_2 + b_3 \end{cases}$$

- $\mathbf{v}_2 = (1, 2, 6)^\mathsf{T}$

$$\begin{cases} x_1 & = (5(1) - 7(2))/3 = -3 \\ x_2 & = ((1) - 2(2))/3 = -1 \\ 0 & = 0 \end{cases}$$

$$\begin{cases} x_1 & = -3 \\ x_2 & = -1 \end{cases} \qquad \text{or in vector form} \qquad \mathbf{x} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

  Thus (check)

$$-3\mathbf{a}_1 - \mathbf{a}_2 = \mathbf{v}_2$$

- $\mathbf{v}_3 = (2, 3, 7)^\mathsf{T}$: hence,

$$\begin{cases} x_1 & = (5(2) - 7(3))/3 = -11/3 \\ x_2 & = ((2) - 2(3))/3 = -4/3 \\ 0 & = 0 \end{cases}$$

  Thus the solution to $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{v}$ is

$$\begin{cases} x_1 & = -11/3 \\ x_2 & = -4/3 \end{cases} \qquad \text{or in vector form} \qquad \mathbf{x} = \begin{pmatrix} -11/3 \\ -4/3 \end{pmatrix}$$

  Thus (check)

$$-(11/3)\mathbf{a}_1 - (4/3)\mathbf{a}_2 = \mathbf{v}_3$$

5.10 Exercises.

1. Answer all questions in Exercises 5.8 after replacing the phrase "*linear combination of*" by the phrase "*span of*".

2. In each of the following describe the span of B:

   (a) $B = \{(1, 3, -2)^\mathsf{T}, 4, -1, 5)^\mathsf{T}\}$

   (b) $B = \left\{ \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \right\}$

   (c) $B = \{x, (1 + x)^2, x^2 + 3x + 1\}$

   (d) $B = \{x^2 - 4, 2 - x, x^2 + x + 1\}$

3. The columns of a 9×4 matrix A satisfies the following: *Three times the second column plus twice the third column minus the first one equals five times the fourth column minus six times the first column.*

   (a) Find a nonzero solution to the homogenous system $A\mathbf{x} = 0$.

   (b) For some $\mathbf{b} \in \mathbb{R}^9$ the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. How many solution does it have?

5.11 Linear independence.

Linear independence of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$:
Two nonzero vectors $\mathbf{v}_1 \& \mathbf{v}_2$ are **linearly independent** iff any of the following equivalent situations occurs:

1. $\mathbf{v}_1 \& \mathbf{v}_2$ are not parallel.
2. span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is 2-d, i.e. a plane.
3. Neither of them is a multiple of the other. That is $\mathbf{v}_2 \neq c\mathbf{v}_1$.
4. the only solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$ is $c_1 = c_2 = 0$.

Linear independence of 3 vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

> Three nonzero vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
> are **linearly independent**
> iff
> none of them is a linear combination
> of the other two.

Any of the following situations is used to test for linear independence:

1. The only solution to the homogenous equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

   is $c_1 = c_2 = c_3 = 0$, i.e the vector $\mathbf{c} = \mathbf{0}$.

2. The homogenous system

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \mathbf{c} = \mathbf{0}$$

   has only the trivial solution $\mathbf{c} = \mathbf{0}$.

3. None of them is a linear combination of the other. For example, we cannot write

$$\mathbf{v}_3 \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

4. None of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ lies in the plan spanned by the other two.
5. span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is 3-d.

General case:

A set of n nonzero vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$
is called linearly independent
iff
none of them can be written as
a linear combination of the others.
Otherwise
it is called linearly dependent.

5.11.1 Theorem. *Let $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be a set of n nonzero vectors. The following are equivalent:*
iff any of the following equivalent situations occur

1. *The set $\mathcal{V}$ is linearly independent. None of them is a linear combination of the others. For example, we cannot write*
$$\mathbf{v}_n \neq c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1}$$

2. *The only solution of the homogenous system*
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$
is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$

   Notice that condition can be written in matrix notation as follows:

   The system
$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_2 \end{bmatrix} \mathbf{c} = \mathbf{0}$$
has only the trivial solution $\mathbf{c} = \mathbf{0}$.

3. *None of them lies in the space spanned by the others.*

4. *$\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ span an n dimensional space.*

5.11.2. Example. Determine whether the following three 3-vectors are linearly independent or not. If they are not, write $\mathbf{v}_3$ as a linear combination of the other two.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 1 \\ 10 \\ 10 \end{pmatrix}$$

Solution.

The three vectors are linearly independent iff the homogenous $A\mathbf{x} = 0$ has a nontrivial solution.

$$A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 4 & 10 \\ 3 & 1 & 10 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 8 & 8 \\ 0 & 7 & 7 \end{pmatrix}$$

$$\to A_{ech} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 8 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

The homogenous system $A\mathbf{x} = 0$ has a nontrivial solution because A_{ech} has a zero-row. Thus the three vectors are linearly dependent.

In order to write $\mathbf{v}_3$ as a linear combination of the other two we proceed:

$$A_{ech} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 8 & 8 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

thus

$$-3\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = 0$$

and hence

$$\mathbf{v}_3 = 3\mathbf{v}_1 + \mathbf{v}_2$$

In fact if we go back and draw a vertical line between the second and third columns we have:

$$B = \left(\mathbf{v}_1 \quad \mathbf{v}_2 \,\middle|\, \mathbf{v}_3 \right) \longrightarrow \cdots \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & 8 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow B_{ech} \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

and hence

$$\mathbf{v}_3 = 3\mathbf{v}_1 + \mathbf{v}_2$$

as above □

5.11.3. Example. Answer the following questions for each case where $\mathbf{A} = \left[\ \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\ \right]$

1. Find the value(s) of h for which $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent.
2. Find the value(s) of h for which $\mathbf{v}_3$ lies in the Span$\{\mathbf{v}_1, \mathbf{v}_2\}$
3. Find the value(s) of h for which the linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$ has a non-trivial solution where the vector columns of $\mathbf{A}$ are the 3 vectors above.

$$(a) \qquad \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 5 \\ h \end{pmatrix}$$

$$(b) \qquad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}$$

Solution.

First notice that

- Answer(1): $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent iff $A\mathbf{c} = 0$ has only the trivial solution. To determine that we reduce the augmented matrix $B = \left[\begin{array}{ccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{0} \end{array}\right]$ to an echelon form B_{ech}.
  - If each column in B_{ech} has a LT, then we have no free variables and the system $A\mathbf{c} = 0$ has only the zero solution.
  - If one of the columns does not have a LT (in this case B_{ech} has a zero row) then we have at least one FV and the system $A\mathbf{c} = 0$ has infinitely many nonzero solutions.
- Answer(2) = complement of Answer(1). This is because if $\mathbf{v}_3$ lies in the Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent.
- Answer(3) = Answer(2) = complement of Answer(1). This is because $A\mathbf{c} = 0$ has a nontrivial solution iff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent.

(a) To answer any of these questions we need to put the augmented matrix $\left[\begin{array}{ccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{0} \end{array}\right]$ in *echelon form.*

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & h & 0 \end{array}\right) \longrightarrow \cdots \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & h-6 & 0 \end{array}\right)$$

We have two possibilities
(a) $h \neq 6$: In this case the system $\mathbf{Ax} = \mathbf{0}$ has a unique solution, which is the trivial solution. In this case $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent.
(b)$h = 6$: We have one free variable because

$$\#(\text{col}) - \#(\text{LT}) = 3\text{-}2 = 1$$

Hence, there are infinitely many nontrivial solutions for $h = 6$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent.
<u>Conclusion</u>:

1. $h \neq 6$.
2. $h = 6$.
3. $h = 6$.

5.11.4. Example. Answer the following questions for each case where $\mathbf{A} = \left[\begin{array}{ccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{array}\right]$

1. Find the value(s) of h for which $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent.
2. Find the value(s) of h for which $\mathbf{v}_3$ lies in the Span$\{\mathbf{v}_1, \mathbf{v}_2\}$
3. Find the value(s) of h for which the linear system $\mathbf{Ax} = \mathbf{0}$ has a non-trivial solution where the vector columns of $\mathbf{A}$ are the 3 vectors above.

$$(a)\qquad \begin{pmatrix}1\\5\\-3\end{pmatrix},\ \begin{pmatrix}-2\\-9\\6\end{pmatrix},\ \begin{pmatrix}3\\h\\-9\end{pmatrix}$$

$$(b)\qquad \mathbf{v}_1=\begin{pmatrix}1\\5\\-3\end{pmatrix},\mathbf{v}_2=\begin{pmatrix}-2\\-9\\6\end{pmatrix},\mathbf{v}_3=\begin{pmatrix}3\\h\\-9\end{pmatrix}$$

To answer any of these questions we need to put the augmented matrix $\left[\begin{array}{ccc|c}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{0}\end{array}\right]$ in *echelon form.*

$$\left(\begin{array}{ccc|c}1 & -2 & 3 & 0\\0 & 1 & h-15 & 0\\0 & 0 & 0 & 0\end{array}\right)$$

Here the zero-row in A_{ech} does not lead to any inconsistency because we are dealing with a homogenous system $\mathbf{Ax}=\mathbf{0}$. We have one free variable because

#(col) - #(LT) = 3-2 =1

Hence, there are infinitely many solutions *for all h.*
Conclusion:

1. No h.
2. All h.
3. All h.

5.11.5. Example. Answer the following questions for each case where $\mathbf{A}=\left[\begin{array}{ccc}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{array}\right]$

1. Find the value(s) of h for which $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ are linearly independent.
2. Find the value(s) of h for which $\mathbf{v}_3$ lies in the Span$\{\mathbf{v}_1,\mathbf{v}_2\}$
3. Find the value(s) of h for which the linear system $\mathbf{Ax}=\mathbf{0}$ has a non-trivial solution where the vector columns of $\mathbf{A}$ are the 3 vectors above.

$$(a)\qquad \begin{pmatrix}1\\-3\\2\end{pmatrix},\ \begin{pmatrix}-3\\9\\-6\end{pmatrix},\ \begin{pmatrix}5\\-7\\h\end{pmatrix}$$

$$(b)\qquad \mathbf{v}_1=\begin{pmatrix}1\\-3\\2\end{pmatrix},\mathbf{v}_2=\begin{pmatrix}-3\\9\\-6\end{pmatrix},\mathbf{v}_3=\begin{pmatrix}5\\-7\\h\end{pmatrix}$$

To answer any of these questions we need to put the augmented matrix $\left[\begin{array}{ccc|c}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{0}\end{array}\right]$ in *echelon form.*

$$\begin{pmatrix}1 & -3 & 0\\0 & 0 & 1\\0 & 0 & h-10\end{pmatrix}$$

If $h = 8$, the last row is a zero row. If $h \neq 8$, replace R_3 by $R_3 - (h-8)R_2$. We obtain

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the solution to $\mathbf{Ax} = \mathbf{0}$ is

$$x_2 = 0, \quad x_3 = t, \quad x_1 = 3t, \quad -\infty < t < \infty$$

for all h. Conclusion:

1. No h.
2. All h.
3. All h.

5.11.6. Example. The following five 4-vectors cannot be linearly independent. Why?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ -2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 2 \\ -8 \\ 7 \\ 11 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} -9 \\ 2 \\ 1 \\ -8 \end{pmatrix}$$

1. Find the largest set of vectors among $\{\mathbf{v}_1, \cdots, \mathbf{v}_5\}$ that are linearly independent.
2. Let $\mathbf{V} = \text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_5\}$. What is the smallest set among $\{\mathbf{v}_1, \cdots, \mathbf{v}_5\}$ that span $\mathbf{V}$?
3. What is the dimension of $\mathbf{V}$?

Solution.

$$A = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_5 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{pmatrix}$$

$$\to A_{ech} = \begin{pmatrix} 1 & 3 & 3 & 2 & -9 \\ 0 & 1 & 2 & -1 & -4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 1:
The largest set of vectors among $\{\mathbf{v}_1, \cdots, \mathbf{v}_5\}$ that are linearly independent is $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$.

Notice that these are the LT vectors in A not A_{ech}.

Question 2:
The smallest set of vectors
among $\{\mathbf{v}_1, \cdots, \mathbf{v}_5\}$
that span H is **again** $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$.

Again notice that
these are
the LT vectors in A not A_{ech}.

Question 3:
if $H = \text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_5\}$
$\dim H = \#(\text{members of } B) = \#(\text{LT's}) = 3.$

□

5.11.7 Theorem. *The columns of a square matrix A are linearly independent*
iff
$\det A \neq 0$.

Question: Show that Theorem 5.11.7 follows from Theorem 4.3.1.

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5.12 Basis for $V := \text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$.

A basis for $V := \text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a maximal linearly independent subset V.
Equivalently
it is a minimal spanning subset of V.
Practically
we look for a subset of V that (1) spans V and (2) is linearly independent.
It does not have to include any vector from the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$.
In fact we look for the simplest one, that is, with as many zero components as possibly.

5.12.1. Example. Let $\mathbf{v}_1, \cdots, \mathbf{v}_6$ be the columns of the matrix A. Let $H = \{\mathbf{v}_1, \cdots, \mathbf{v}_6\}$.

$$A = \begin{pmatrix} 0 & -3 & -6 & 6 & -3 & 15 \\ -1 & -2 & -1 & 3 & 3 & 1 \\ -2 & -3 & 0 & 3 & 10 & -1 \\ 1 & 4 & 5 & -9 & 5 & -7 \end{pmatrix}$$

1. Find the largest possible subset of linearly independent vectors among $\{\mathbf{v}_1, \cdots, \mathbf{v}_6\}$.
2. Find the smallest possible subset among $\{\mathbf{v}_1, \cdots, \mathbf{v}_6\}$ that you can use to express all vectors in H as linear combinations.
3. Write each of the remaining vectors in $\{\mathbf{v}_1, \cdots, \mathbf{v}_6\}$ as a linear combination of the ones you found in the first part.
4. Find a basis for the subspace H.

$$A \longrightarrow \begin{pmatrix} 1 & 4 & 5 & -9 & 5 & -7 \\ 0 & -3 & -6 & 6 & -3 & 15 \\ 0 & 2 & 4 & -6 & 8 & -6 \\ 0 & 5 & 10 & -15 & 20 & -15 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 4 & 5 & -9 & 5 & -7 \\ 0 & 1 & 2 & -3 & 4 & -3 \\ 0 & -3 & -6 & 6 & -3 & 15 \\ 0 & 2 & 4 & -6 & 8 & -6 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 4 & 5 & -9 & 5 & -7 \\ 0 & 1 & 2 & -3 & 4 & -3 \\ 0 & -3 & -6 & 6 & -3 & 15 \\ 0 & 2 & 4 & -6 & 8 & -6 \end{pmatrix}$$

$$\longrightarrow A_{ech} \begin{pmatrix} [1] & 4 & 5 & -9 & 5 & -7 \\ 0 & [1] & 2 & -3 & 4 & -3 \\ 0 & 0 & 0 & [3] & 9 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, a largest set of linearly independent vectors among the columns of A ar the pivot columna $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$.

To write each of the rest as a linear combination of these ones, we consider them one at a time. Start with $\mathbf{a}_6$. We pretend that we started with $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ and $\mathbf{a}_6$ as $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4|\mathbf{a}_6)$ and done the Gauss elimination. Then we would have arrived at the same A_{ech} with $\mathbf{a}_3^{aug}$ and $\mathbf{a}_5^{aug}$ removed and $\mathbf{a}_6^{aug}$ at the end. That is $(\mathbf{a}_1^{aug}\mathbf{a}_2^{aug}\mathbf{a}_4^{aug}|\mathbf{a}_6^{aug})$,

$$\left(\begin{array}{ccc|c} [1] & 4 & -9 & -7 \\ 0 & [1] & -3 & -3 \\ 0 & 0 & [3] & -6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} [1] & 0 & 0 & 11 \\ 0 & [1] & 0 & -9 \\ 0 & 0 & [1] & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and the solution is $x_1 = 11, x_2 = -9, x_3 = -2$, and hence

$$\mathbf{a}_6 = 11\mathbf{a}_1 - 9\mathbf{a}_2 - 2\mathbf{a}_4$$

Now repeat the same process with $\mathbf{a}_3$ and $_\mathbf{5}$. That is $(\mathbf{a}_1^{aug}\mathbf{a}_2^{aug}\mathbf{a}_4^{aug}|\mathbf{a}_3^{aug})$ and $(\mathbf{a}_1^{aug}\mathbf{a}_2^{aug}\mathbf{a}_4^{aug}|\mathbf{a}_5^{aug})$.

5.12.2. Subspaces of $\mathbb{R}^n$

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\boxed{H = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}}$$

That is, H consists of all vectors of the form

$$\boxed{\mathbf{y} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2, \quad a_1, a_2 \in \mathbb{R}}$$

$$\boxed{H = \{\mathbf{v} \in V \mid \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2, \quad c_1, c_2 \in \mathbb{R}\}}$$

Recall the parallelogram rule.
Notice that

(b) If $\mathbf{u} \in H, c \in \mathbb{R}$, then the entire line that $\mathbf{u}$ determines through the origin lies in H. That is,

$$\text{If } \boxed{\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2, \quad c \in \mathbb{R}}$$

$$\text{Then } \boxed{c\mathbf{u} = (ca_1)\mathbf{u}_1 + (ca_2)\mathbf{u}_2 \in H}$$

(c) If $\mathbf{u}, \mathbf{v} \in H$, then $\mathbf{u} + \mathbf{v} \in H$. That is

$$\boxed{\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2, \quad \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2}$$

$$\boxed{\Rightarrow \mathbf{u} + \mathbf{v} = (a_1 + b_2)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2}$$

$$\boxed{\Rightarrow \mathbf{u} + \mathbf{v} \in H}$$

Notice that the zero vector $\mathbf{0} \in H$ because

$$\boxed{\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 \in H}$$

There is nothing special about these two vectors:

1. The span of any two 3-vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$, will have the same properties.
2. The span of any two n-vectors, $\mathbf{v}_1, \mathbf{v}_2$ in any higher dimension space $\in \mathbb{R}^n$, has the same properties.
3. In fact, the span of any number of n-vectors, $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_k$ in any space $\in \mathbb{R}^n$ has the same properties (a-c).

Definition:

$$\boxed{\text{A set } H \text{ in } \mathbb{R}^n \text{ is called a } \underline{\text{subspace}} \text{ of } \mathbb{R}^n}$$

$$\boxed{\text{if it satisfies (a-b).}}$$

Definition:

$$\boxed{\text{the subspace spanned by } \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k}$$

$$\boxed{= \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}}$$

5.13 Spanning sets.

Review section 5.9 using matrices.

6 Linear Transformations

6.1 Isomorphisms

6.1.1 Definitions

Let V and W be two vector spaces.

Recall that

A map $T : V \longrightarrow W$ is called
a **linear transformation**
or a **homomorphism**
iff

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_1), \qquad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in V$$

$$T(c\mathbf{v}) = cT(\mathbf{v}), \qquad \text{for all } \mathbf{v} \in V, c \in \mathbb{R}$$

A map $T : V \longrightarrow W$ is called
a **linear transformation**
or a **homomorphism**
iff
it can be represented by a matrix.

A map $T : V \longrightarrow W$ is said to be
one-to-one or **injective**
iff

$$T(\mathbf{v}_1) = T(\mathbf{v}_2) \Leftrightarrow \mathbf{v}_1 = \mathbf{v}_2$$

equivalently

$$T(\mathbf{v}) = 0 \Leftrightarrow \mathbf{v} = 0$$

A map $T : V \longrightarrow W$ is said to be
onto or **surjective**
iff
for any $\mathbf{w} \in W$
the equation
$T(\mathbf{x}) = \mathbf{w}$
has at least one solution

Let V and W be two vector spaces.

A map $T : V \longrightarrow W$ is called
a **correspondence**
iff it is
one-to-one and
onto

A map $T : V \longrightarrow W$ is called
an **isomorphism**
iff it is
a correspondence,
and a linear transformation

In short, we have:

A map $T : V \longrightarrow W$ is called
an **isomorphism**
iff it is
a linear transformation
one-to-one and
onto

An isomorphism $T : V \longrightarrow V$ is called
an **automorphism**

6.1.2 Dimension characterizes isomorphisms.

Two vector spaces are isomorphic

iff

they have the same dimension.

6.2 Homomorphisms = Linear maps.

A map $T : V \longrightarrow W$ is called
a **linear transformation**
or a **homomorphism**
iff

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_1), \qquad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in V$$
$$T(c\mathbf{v}) = cT(\mathbf{v}), \qquad \text{for all } \mathbf{v} \in V, c \in \mathbb{R}$$

6.2.1. Exercises.

1. Three matrices A_1, A_2 and A_3 has echelon forms B_1, B_2 and B_3 where "♣" stands for a leading term.

   For which of the three matrices the homomorphism $h : \mathbb{R}^{\cdots} \to \mathbb{R}^{\cdots}$ is (a) onto and for which (b) one-to-one.

$$B_1 = \begin{pmatrix} \clubsuit & * & * \\ 0 & \clubsuit & * \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \clubsuit & * & * \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} \clubsuit & * & * \\ 0 & 0 & \clubsuit \end{pmatrix}$$

2. A map $h : \mathbb{R}^c \to \mathbb{R}^r$ is given by $h(\mathbf{x}) = A\mathbf{x}$ where A is an $\cdots \times \cdots$ matrix. Assume that an echelon form of A has a zero row.

   (a) Is the map h linear?

   (b) Is the map h onto?

   (c) Is the map h one-to-one?

   (d) Does the equation $A\mathbf{x} = y$ has a solution for each $y \in \mathbb{R}^{\cdots}$?

   (e) Does the equation $h(x) = y$ has a solution for each $y \in \mathbb{R}^{\cdots}$?

3. A map $h : \mathbb{R}^c \to \mathbb{R}^r$ is given by $h(\mathbf{x}) = A\mathbf{x}$ where A is an $\cdots \times \cdots$ matrix. Assume that $r > c$.

   (a) Is the map h linear?

   (b) Is the map h onto?

(c) Is the map h one-to-one?

(d) Does the equation $A\mathbf{x} = y$ has a solution for each $y \in \mathbb{R}^{\cdots}$?

(e) Does the equation $h(x) = y$ has a solution for each $y \in \mathbb{R}^{\cdots}$?

4. A homomorphism $h : \mathbb{R}^3 \to \mathbb{R}^2$ satisfies $h(\mathbf{e}_1) = (7, 2)^\intercal, h(\mathbf{e}_2) = (-1, 3)^\intercal$ and $h(\mathbf{e}_3) = (5, 4)^\intercal$.

   (a) Find $h(\mathbf{x})$ explicitly.

   (b) Represent the map h by a matrix.

> A linear map $h : V \to W$ is completely determined by its action on a basis of $\cdots\cdots$.

6.3 The matrix of a linear transformation

In $\mathbb{R}^3$ let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise. Find the matrix of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$, $T : \mathbb{R}^3 \to \mathbb{R}^4$ and

$$T_A(\mathbf{e}_1) = \begin{pmatrix} 2 \\ -4 \\ 6 \\ 5 \end{pmatrix}, \quad T_A(\mathbf{e}_2) = \begin{pmatrix} -3 \\ -2 \\ 9 \\ 6 \end{pmatrix},$$

$$T_A(\mathbf{e}_3) = \begin{pmatrix} 4 \\ 0 \\ 3 \\ -2 \end{pmatrix}$$

Answer:

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) \\ &= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + x_3 T(\mathbf{e}_3) \\ &= x_1 \begin{pmatrix} 2 \\ -4 \\ 6 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ -2 \\ 9 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 0 \\ 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -3 & 4 \\ -4 & -2 & 0 \\ 6 & 9 & 3 \\ 5 & 6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

Therefore

$$A_T = \begin{pmatrix} 2 & -3 & 4 \\ -4 & -2 & 0 \\ 6 & 9 & 3 \\ 5 & 6 & -2 \end{pmatrix}$$

Example. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation

$$T_A(\mathbf{x}) = A\mathbf{x}, \quad \approx \quad \mathbf{x} \mapsto \mathbf{y} = A\mathbf{x},$$

$$A = \begin{pmatrix} 2 & -3 & 11 \\ -3 & 4 & -3 \\ 5 & 2 & 6 \\ 7 & 5 & -1 \end{pmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$$

$$Ae_1 = \begin{pmatrix} 2 & -3 & 11 \\ -3 & 4 & -3 \\ 5 & 2 & 6 \\ 7 & 5 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \\ 7 \end{pmatrix}$$

$$A\mathbf{e}_2 = \mathbf{a}_2, \qquad A\mathbf{e}_3 = \mathbf{a}_3$$

$$\begin{aligned} T_A(\mathbf{x}) &= T_A(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) \\ &= x_1T_A(\mathbf{e}_1) + x_2T_A(\mathbf{e}_2) + x_3T_A(\mathbf{e}_3) \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 \\ T_A(\mathbf{x}) &= x_1 \begin{pmatrix} 2 \\ -3 \\ 5 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 4 \\ 2 \\ 5 \end{pmatrix} \\ &+ x_3 \begin{pmatrix} 11 \\ -3 \\ 6 \\ 1 \end{pmatrix} \end{aligned}$$

6.3.1 Example. Rotation in the plane: Let $R_\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin with by angle ϕ, where the positive direction is counterclockwise.
Answer: Notice that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}$$

Then

$$A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$A_{\pi/6} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

6.3.2 Example. Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\mathbf{e}_1$ into $3\mathbf{e}_1 + 4\mathbf{e}_2$ and maps $\mathbf{e}_2$ into $2\mathbf{e}_1 - 5\mathbf{e}_2$.
Answer: The transformation T maps

$$\mathbf{e}_1 \mapsto \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \qquad \mathbf{e}_2 \mapsto \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

Thus, The matrix of T is

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_1) \end{bmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & -5 \end{pmatrix}$$

6.3.3 Exercise. 1. Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotate each vector by an angle $\pi/6$ then maps $\mathbf{e}_1$ into $-2\mathbf{e}_1 + 3\mathbf{e}_2$ and maps $\mathbf{e}_2$ into $5\mathbf{e}_1 - \mathbf{e}_2$, then rotate each vector by an angle $\pi/6$.

2. Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\mathbf{e}_1$ into $-2\mathbf{e}_1 + 3\mathbf{e}_2$ and maps $\mathbf{e}_2$ into $5\mathbf{e}_1 - \mathbf{e}_2$.

3. Are the two linear systems the same?

6.4 The Null space, Row space and Col space of A.

6.4.1. The Null space of A. Let A be a matrix of size $r \times c$.

The Null space of A Nul A is the solution set of the homogenous system $A\mathbf{x} = \mathbf{0}$
Nul A = The set of all $\mathbf{x} \in \mathbb{R}^c$ that satisfies $A\mathbf{x} = \mathbf{0}$ $\dim(\text{Nul } A) = \#(\text{free variables})$

6.4.2. The column space of A. Let A be a matrix of size $r \times c$

The column space of A denoted by Col$A$ or Range A = The span of the columns of A. $\Updownarrow$ The set of all $y \in \mathbb{R}^r$ which is a linear combination of the columns of A. $\Updownarrow$ The set of all $\mathbf{y} \in \mathbb{R}^r$ for which $A\mathbf{x} = \mathbf{y}$ has a solution.
$\dim(\text{Col } A) = \#(\text{basic variables})$
$\dim(\text{Nul } A) + \dim(\text{Col } A) =$ .... $\dim(\text{Nul } A) =$ ....

6.4.3. The row space of A. Let A be a matrix of size $r \times c$

The row space of A
denoted by Row$A$
=
The span of the rows of A.

6.4.1 Nul$A$ is orthogonal to Row$A$.

If $\mathbf{v} \in \text{Nul}A$, that is $A\mathbf{v} = \mathbf{0}$, then

$$A\mathbf{v} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_4 \end{pmatrix} \mathbf{v} = \begin{pmatrix} \mathbf{r}_1 \cdot \mathbf{v} \\ \mathbf{r}_2 \cdot \mathbf{v} \\ \vdots \\ \mathbf{r}_4 \cdot \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus $\mathbf{v}$ is orthogonal to all the rows of A. Hence

$$\text{Nul}A \perp \text{Row}A$$

6.4.2 The rank and nullity of A.

rank of A := #(basic variables) = #(LT's in A_{ech})
nullity of A := #(free variables)

Thus, we can translate (2.1.11) to

rank of A + nullity of A = #(variables)
=#(columns)