Fundamentals of Linear Algebra

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PREFACE

Linear algebra has evolved as a branch of mathematics with wide range of applications to the natural sciences, to engineering, to computer sciences, to management and social sciences, and more.

This book is addressed primarely to second and third your college students who have already had a course in calculus and analytic geometry. It is the result of lecture notes given by the author at The University of North Texas and the University of Texas at Austin. It has been designed for use either as a supplement of standard textbooks or as a textbook for a formal course in linear algebra.

This book is not a "traditional" book in the sense that it does not include any applications to the material discussed. Its aim is solely to learn the basic theory of linear algebra within a semester period. Instructors may wish to incorporate material from various fields of applications into a course.

I have included as many problems as possible of varying degrees of difficulty. Most of the exercises are computational, others are routine and seek to fix some ideas in the reader's mind; yet others are of theoretical nature and have the intention to enhance the reader's mathematical reasoning. After all doing mathematics is the way to learn mathematics.

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Chapter 1

Exercise 42
Which of the following equations are not linear and why:
(a) $x_1^2 + 3x_2 - 2x_3 = 5.$
(b) $x_1 + x_1x_2 + 2x_3 = 1.$
(c) $x_1 + \frac{2}{x_2} + x_3 = 5.$

Solution.
(a) The given equation is linear by (1.1).
(b) The equation is not linear because of the term x_1x_2.
(c) The equation is nonlinear because x_2 has a negative power ■

Exercise 43
Show that $(2s + 12t + 13, s, -s - 3t - 3, t)$ *is a solution to the system*

$$\left\{\begin{array}{ccccccccc} 2x_1 & + & 5x_2 & + & 9x_3 & + & 3x_4 & = & -1 \\ x_1 & + & 2x_2 & + & 4x_3 & & & = & 1 \end{array}\right.$$

Solution.
Substituting these values for x_1, x_2, x_3, and x_4 in each equation.

$$\begin{array}{ccccc} 2x_1 + 5x_2 + 9x_3 + 3x_4 & = & 2(2s + 12t + 13) + 5s + 9(-s - 3t - 3) + 3t & = & -1 \\ x_1 + 2x_2 + 4x_3 & = & (2s + 12t + 13) + 2s + 4(-s - 3t - 3) & = & 1. \end{array}$$

Since both equations are satisfied, then it is a solution for all s and t ■

Exercise 44
Solve each of the following systems using the method of elimination:
(a)

$$\begin{cases} 4x_1 - 3x_2 = 0 \\ 2x_1 + 3x_2 = 18 \end{cases}$$

(b)

$$\begin{cases} 4x_1 - 6x_2 = 10 \\ 6x_1 - 9x_2 = 15 \end{cases}$$

(c)

$$\begin{cases} 2x_1 + x_2 = 3 \\ 2x_1 + x_2 = 1 \end{cases}$$

Which of the above systems is consistent and which is inconsistent?

Solution.
(a) Adding the two equations to obtain $6x_1 = 18$ or $x_1 = 3$. Substituting this value for x_1 in one of the given equations and then solving for x_2 we find $x_2 = 4$. So system is consistent.

(b) The augmented matrix of the given system is

$$\begin{pmatrix} 4 & -6 & 10 \\ 6 & -9 & 15 \end{pmatrix}$$

Divide the first row by 4 to obtain

$$\begin{pmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ 6 & -9 & 15 \end{pmatrix}$$

Now, add to the second row -6 times the first row to obtain

$$\begin{pmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, $x_2 = s$ is a free variable. Solving for x_1 we find $x_1 = \frac{5+3t}{2}$.The system is consistent.

(c) Note that according to the given equation $1 = 3$ which is impossible. So the given system is inconsistent ■

Exercise 45
Find the general solution of the linear system

$$\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 = -3 \\ 2x_1 - x_2 + 3x_3 - x_4 = 0 \end{cases}$$

Solution.
The augmented matrix of the given system is

$$\begin{pmatrix} 1 & -2 & 3 & 1 & -3 \\ 2 & -1 & 3 & -1 & 0 \end{pmatrix}$$

A corresponding row-echelon matrix is obtained by adding negative two times the first row to the second row.

$$\begin{pmatrix} 1 & -2 & 3 & 1 & -3 \\ 0 & 3 & -3 & -3 & 6 \end{pmatrix}$$

Thus $x_3 = s$ and $x_4 = t$ are free variables. Solving for the leading variables one finds $x_1 = 1 - s + t$ and $x_2 = 2 + s + t$ ■

Exercise 46
Find $a, b,$ and c so that the system

$$\left\{\begin{array}{ccccccc} x_1 & + & ax_2 & + & cx_3 & = & 0 \\ bx_1 & + & cx_2 & - & 3x_3 & = & 1 \\ ax_1 & + & 2x_2 & + & bx_3 & = & 5 \end{array}\right.$$

has the solution $x_1 = 3, x_2 = -1, x_3 = 2$.

Solution.
Simply substitute these values into the given system to obtain

$$\left\{\begin{array}{ccccccc} -a & & & + & 2c & = & -3 \\ & & 3b & - & c & = & 7 \\ 3a & + & 2b & & & = & 7 \end{array}\right.$$

The augmented matrix of the system is

$$\begin{pmatrix} -1 & 0 & 2 & -3 \\ 0 & 3 & -1 & 7 \\ 3 & 2 & 0 & 7 \end{pmatrix}$$

A row-echelon form of this matrix is obtained as follows.
Step 1: $r_1 \leftarrow -r_1$

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 3 & -1 & 7 \\ 3 & 2 & 0 & 7 \end{pmatrix}$$

Step 2: $r_3 \leftarrow r_3 - 3r_1$

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 3 & -1 & 7 \\ 0 & 2 & 6 & -2 \end{pmatrix}$$

Step 3: $r_2 \leftarrow r_2 - r_3$

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -7 & 9 \\ 0 & 2 & 6 & -2 \end{pmatrix}$$

Step 4: $r_3 \leftarrow r_3 - 2r_2$

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -7 & 9 \\ 0 & 0 & 20 & -20 \end{pmatrix}$$

Step 5: $r_3 \leftarrow \frac{1}{20}r_3$

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -7 & 9 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The corresponding system is

$$\left\{\begin{array}{rrrrr} a & & - & 2c & = & 3 \\ & b & - & 7c & = & 9 \\ & & & c & = & -1 \end{array}\right.$$

Using back substitution we find the solution $a = 1, b = 2, c = -1$ ■

Exercise 47

Find a relationship between a,b,c so that the following system is consistent

$$\left\{\begin{array}{rrrrrrr} x_1 & + & x_2 & + & 2x_3 & = & a \\ x_1 & & & + & x_3 & = & b \\ 2x_1 & + & x_2 & + & 3x_3 & = & c \end{array}\right.$$

Solution.

The augmented matrix of the system is

$$\begin{pmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{pmatrix}$$

We reduce this matrix into row-echelon form as follows.

Step 1: $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 - 2r_1$

$$\begin{pmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{pmatrix}$$

Step 2: $r_2 \leftarrow -r_2$

$$\begin{pmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & -1 & -1 & c-2a \end{pmatrix}$$

Step 3: $r_3 \leftarrow r_3 + r_2$

$$\begin{pmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & c-a-b \end{pmatrix}$$

The system is consistent provided that $a + b - c = 0$ ■

Exercise 48
For which values of a will the following system have (a) no solutions? (b) exactly one solution? (c) infinitely many solutions?

$$\left\{\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & 3x_3 & = & 4\\ 3x_1 & - & x_2 & + & 5x_3 & = & 2\\ 4x_1 & + & x_2 & + & (a^2-14)x_3 & = & a+2\end{array}\right.$$

Solution.
The augmented matrix is

$$\left(\begin{array}{cccc} 1 & 2 & -3 & 4\\ 3 & -1 & 5 & 2\\ 4 & 1 & a^2-14 & a+2\end{array}\right)$$

The reduction of this matrix to row-echelon form is outlined below.

Step 1: $r_2 \leftarrow r_2 - 3r_1$ and $r_3 \leftarrow r_3 - 4r_1$

$$\left(\begin{array}{cccc} 1 & 2 & -3 & 4\\ 0 & -7 & 14 & -10\\ 0 & -7 & a^2-2 & a-14\end{array}\right)$$

Step 2: $r_3 \leftarrow r_3 - r_2$

$$\left(\begin{array}{cccc} 1 & 2 & -3 & 4\\ 0 & -7 & 14 & -10\\ 0 & 0 & a^2-16 & a-4\end{array}\right)$$

The corresponding system is

$$\left\{\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & 3x_3 & = & 4\\ & - & 7x_2 & + & 14x_3 & = & -10\\ & & & & (a^2-16)x_3 & = & a-4\end{array}\right.$$

(a) If $a = -4$ then the last equation becomes $0 = -8$ which is impossible. Therefore, the system is inconsistent.
(b) If $a \neq \pm 4$ then the system has exactly one solution, namely, $x_1 = \frac{8a+15}{7(a+4)}, x_2 = \frac{10a+54}{7(a+4)}, x_3 = \frac{1}{a+4}$.
(c) If $a = 4$ then the system has infinitely many solutions. In this case, $x_3 = t$ is the free variable and $x_1 = \frac{8-7t}{7}$ and $x_2 = \frac{10+14t}{7}$ ■

Exercise 49
Find the values of A,B,C in the following partial fraction

$$\frac{x^2-x+3}{(x^2+2)(2x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{2x-1}.$$

Solution.
By multiplying both sides of the equation by $(x^2+2)(2x-1)$ and then equating coefficients of like powers of x we obtain the following system

$$\begin{cases} 2A \quad\quad + C = 1 \\ -A + 2B \quad\quad = -1 \\ \quad - B + 2C = 3 \end{cases}$$

Replace r_3 by $2r_3 + r_2$ to obtain

$$\begin{cases} 2A \quad\quad + C = 1 \\ -A + 2B \quad\quad = -1 \\ -A \quad\quad + 4C = 5 \end{cases}$$

Next, replace r_3 by $2r_3 + r_1$ to obtain

$$\begin{cases} 2A \quad\quad + C = 1 \\ -A + 2B \quad\quad = -1 \\ \quad\quad 9C = 11 \end{cases}$$

Solving backward, we find $A = -\frac{1}{9}, B = -\frac{5}{9}$, and $C = \frac{11}{9}$ ■

Exercise 50
Find a quadratic equation of the form $y = ax^2 + bx + c$ *that goes through the points* $(-2, 20), (1, 5),$ *and* $(3, 25)$.

Solution.
The components of these points satisfy the given quadratic equation. This leads to the following system

$$\begin{cases} 4a - 2b + c = 20 \\ a + b + c = 5 \\ 9a + 3b + c = 25 \end{cases}$$

Apply Gauss algorithm as follows.

Step1. $r_1 \leftrightarrow r_2$

$$\begin{cases} a + b + c = 5 \\ 4a - 2b + c = 20 \\ 9a + 3b + c = 25 \end{cases}$$

Step 2. $r_2 \leftarrow r_2 - 4r_1$ and $r_3 \leftarrow r_3 - 9r_1$

$$\begin{cases} a + b + c = 5 \\ \quad - 6b - 3c = 0 \\ \quad - 6b - 8c = -20 \end{cases}$$

Step 3. $r_3 \leftarrow r_3 - r_2$

$$\left\{\begin{array}{rcrcrcr} a & + & b & + & c & = & 5 \\ & - & 6b & - & 3c & = & 0 \\ & & & - & 5c & = & -20 \end{array}\right.$$

Solving by the method of backward substitution we find $a = 3, b = -2$, and $c = 4$ ■

Exercise 51
For which value(s) of the constant k does the following system have (a) no solutions? (b) exactly one solution? (c) infinitely many solutions?

$$\left\{\begin{array}{rcrcl} x_1 & - & x_2 & = & 3 \\ 2x_1 & - & 2x_2 & = & k \end{array}\right.$$

Solution.
(a) The system has no solutions if $\frac{k}{2} \neq 3$, i.e. $k \neq 6$.
(b) The system has no unique solution for any value of k.
(c) The system has infinitely many solution if $k = 6$. The general solution is given by $x_1 = 3 + t, x_2 = t$ ■

Exercise 52
Find a linear equation in the unknowns x_1 and x_2 that has a general solution $x_1 = 5 + 2t, x_2 = t$.

Solution.
Since $x_2 = t$ then $x_1 = 5 + 2x_2$ that is $x_1 - 2x_2 = 5$ ■

Exercise 53
Consider the linear system

$$\left\{\begin{array}{rcrcrcrcl} 2x_1 & + & 3x_2 & - & 4x_3 & + & x_4 & = & 5 \\ -2x_1 & & & + & x_3 & & & = & 7 \\ 3x_1 & + & 2x_2 & & & - & 4x_3 & = & 3 \end{array}\right.$$

(a) Find the coefficient and augmented matrices of the linear system.
(b) Find the matrix notation.

Solution.
(a) If A is the coefficient matrix and B is the augmented matrix then

$$A = \left(\begin{array}{rrrr} 2 & 3 & -4 & 1 \\ -2 & 0 & 1 & 0 \\ 3 & 2 & 0 & -4 \end{array}\right), B = \left(\begin{array}{rrrrr} 2 & 3 & -4 & 1 & 5 \\ -2 & 0 & 1 & 0 & 7 \\ 3 & 2 & 0 & -4 & 3 \end{array}\right)$$

(b) The given system can be written in matrix form as follows

$$\left(\begin{array}{rrrr} 2 & 3 & -4 & 1 \\ -2 & 0 & 1 & 0 \\ 3 & 2 & 0 & -4 \end{array}\right) \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right) = \left(\begin{array}{r} 0 \\ 8 \\ -9 \end{array}\right) ■$$

Exercise 54
Solve the following system using elementary row operations on the augmented matrix:

$$\left\{\begin{array}{ccccccc} 5x_1 & - & 5x_2 & - & 15x_3 & = & 40 \\ 4x_1 & - & 2x_2 & - & 6x_3 & = & 19 \\ 3x_1 & - & 6x_2 & - & 17x_3 & = & 41 \end{array}\right.$$

Solution.
The augmented matrix of the system is

$$\left(\begin{array}{cccc} 5 & -5 & -15 & 40 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array}\right)$$

The reduction of this matrix to row-echelon form is

Step 1: $r_1 \leftarrow \frac{1}{5}r_1$

$$\left(\begin{array}{cccc} 1 & -1 & -3 & 8 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array}\right)$$

Step 2: $r_2 \leftarrow r_2 - 4r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\left(\begin{array}{cccc} 1 & -1 & -3 & 8 \\ 0 & 2 & 6 & -13 \\ 0 & -3 & -8 & 17 \end{array}\right)$$

Step 3: $r_2 \leftarrow r_2 + r_3$

$$\left(\begin{array}{cccc} 1 & -1 & -3 & 8 \\ 0 & -1 & -2 & 4 \\ 0 & -3 & -8 & 17 \end{array}\right)$$

Step 4: $r_3 \leftarrow r_3 - 3r_2$

$$\left(\begin{array}{cccc} 1 & -1 & -3 & 8 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & -2 & 5 \end{array}\right)$$

Step 5: $r_2 \leftarrow -r_2$ and $r_3 \leftarrow -\frac{1}{2}r_3$

$$\left(\begin{array}{cccc} 1 & -1 & -3 & 8 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right)$$

It follows that $x_3 = -\frac{5}{2}, x_2 = -6$, and $x_1 = -\frac{11}{2}$ ■

Exercise 55
Solve the following system.

$$\left\{\begin{array}{ccccccc} 2x_1 & + & x_2 & + & x_3 & = & -1 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & & & - & 2x_3 & = & 5 \end{array}\right.$$

Solution.
The augmented matrix is given by

$$\left(\begin{array}{cccc} 2 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 3 & 0 & -2 & 5 \end{array}\right)$$

The reduction of the augmented matrix to row-echelon form is as follows.

Step 1: $r_1 \leftrightarrow r_2$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & -1 \\ 3 & 0 & -2 & 5 \end{array}\right)$$

Step 2: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & -6 & -5 & 5 \end{array}\right)$$

Step 3: $r_3 \leftarrow r_3 - 2r_2$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 0 & -3 & 7 \end{array}\right)$$

Step 4: $r_2 \leftarrow -\frac{1}{3}r_2$ and $r_3 \leftarrow -\frac{1}{3}r_3$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{7}{3} \end{array}\right)$$

The solution is given by: $x_1 = \frac{1}{9}, x_2 = \frac{10}{9}, x_3 = -\frac{7}{3}$ ■

Exercise 56
Which of the following matrices are not in reduced row-ehelon from and why?
(a)

$$\left(\begin{array}{cccc} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

(b)

$$\left(\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

(c)

$$\left(\begin{array}{ccc} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

Solution.
(a) No, because the matrix fails condition 1 of the definition. Rows of zeros must be at the bottom of the matrix.
(b) No, because the matrix fails condition 2 of the definition. Leading entry in row 2 must be 1 and not 2.
(c) Yes. The given matrix satisfies conditions 1 - 4 ■

Exercise 57
Use Gaussian elimination to convert the following matrix into a row-echelon matrix.

$$\begin{pmatrix} 1 & -3 & 1 & -1 & 0 & -1 \\ -1 & 3 & 0 & 3 & 1 & 3 \\ 2 & -6 & 3 & 0 & -1 & 2 \\ -1 & 3 & 1 & 5 & 1 & 6 \end{pmatrix}$$

Solution.
We follow the steps in Gauss-Jordan algorithm.

Step 1: $r_2 \leftarrow r_2 + r_1, r_3 \leftarrow r_3 - 2r_1$, and $r_4 \leftarrow r_4 + r_1$

$$\begin{pmatrix} 1 & -3 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 2 & 4 & 1 & 5 \end{pmatrix}$$

Step 2: $r_3 \leftarrow r_3 - r_2$ and $r_4 \leftarrow r_4 = 2r_2$

$$\begin{pmatrix} 1 & -3 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Step 3: $r_3 \leftarrow -\frac{1}{2}r_3$

$$\begin{pmatrix} 1 & -3 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Step 4: $r_4 \leftarrow r_4 + r_3$

$$\begin{pmatrix} 1 & -3 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \blacksquare$$

Exercise 58
Use Gauss-Jordan elimination to convert the following matrix into reduced row-

echelon form.

$$\begin{pmatrix} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \\ -5 & -5 & -5 & -14 \end{pmatrix}$$

Solution.
Using the Gauss-Jordan to bring the given matrix into reduced row-echelon form as follows.

Step 1: $r_1 \leftrightarrow r_3$

$$\begin{pmatrix} 1 & -1 & -1 & -11 \\ 6 & -1 & -2 & -36 \\ -2 & 1 & 1 & 15 \\ -5 & -5 & -5 & -14 \end{pmatrix}$$

Step 2: $r_2 \leftarrow r_2 - 6r_1, r_3 \leftarrow r_3 + 2r_1$, and $r_4 \leftarrow r_4 + 5r_1$

$$\begin{pmatrix} 1 & -1 & -1 & -11 \\ 0 & 5 & 4 & 30 \\ 0 & -1 & -1 & -7 \\ 0 & -10 & -10 & -69 \end{pmatrix}$$

Step 3: $r_2 \leftrightarrow -r_3$

$$\begin{pmatrix} 1 & -1 & -1 & -11 \\ 0 & 1 & 1 & 7 \\ 0 & 5 & 4 & 30 \\ 0 & -10 & -10 & -69 \end{pmatrix}$$

Step 4: $r_3 \leftarrow r_3 - 5r_2$ and $r_4 \leftarrow r_4 + 10r_2$

$$\begin{pmatrix} 1 & -1 & -1 & -11 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & -9 \end{pmatrix}$$

Step 5: $r_3 \leftarrow -r_3$ and $r_4 \leftarrow -\frac{1}{9}r_4$

$$\begin{pmatrix} 1 & -1 & -1 & -11 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 6: $r_1 \leftarrow r_1 + r_2$

$$\begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 7: $r_2 \leftarrow r_2 - r_3$

$$\begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 8: $r_1 \leftarrow r_1 + 5r_4$ and $r_2 \leftarrow r_2 - 6r_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \blacksquare$$

Exercise 59

Solve the following system using Gauss-Jordan elimination.

$$\left\{\begin{array}{rcrcrcrcr} 3x_1 & + & x_2 & + & 7x_3 & + & 2x_4 & = & 13 \\ 2x_1 & - & 4x_2 & + & 14x_3 & - & x_4 & = & -10 \\ 5x_1 & + & 11x_2 & - & 7x_3 & + & 8x_4 & = & 59 \\ 2x_1 & + & 5x_2 & - & 4x_3 & - & 3x_4 & = & 39 \end{array}\right.$$

Solution.

The augmented matrix of the system is

$$\begin{pmatrix} 3 & 1 & 7 & 2 & 13 \\ 2 & -4 & 14 & -1 & -10 \\ 5 & 11 & -7 & 8 & 59 \\ 2 & 5 & -4 & -3 & 39 \end{pmatrix}$$

The reduction of the augmented matrix into reduced row-echelon form is

Step 1: $r_1 \leftarrow r_1 - r_2$

$$\begin{pmatrix} 1 & 5 & -7 & 3 & 23 \\ 2 & -4 & 14 & -1 & -10 \\ 5 & 11 & -7 & 8 & 59 \\ 2 & 5 & -4 & -3 & 39 \end{pmatrix}$$

Step 2: $r_2 \leftarrow r_2 - 2r_1, r_3 \leftarrow r_3 - 5r_1$, and $r_4 \leftarrow r_4 - 2r_1$

$$\begin{pmatrix} 1 & 5 & -7 & 3 & 23 \\ 0 & -14 & 28 & -7 & -56 \\ 0 & -14 & 28 & -7 & -56 \\ 0 & -5 & 10 & -9 & -7 \end{pmatrix}$$

Step 3: $r_3 \leftarrow r_3 - r_2$

$$\begin{pmatrix} 1 & 5 & -7 & 3 & 23 \\ 0 & -14 & 28 & -7 & -56 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 10 & -9 & -7 \end{pmatrix}$$

Step 4: $r_4 \leftrightarrow r_3$ and $r_2 \leftarrow -\frac{1}{14}r_2$

$$\begin{pmatrix} 1 & 5 & -7 & 3 & 23 \\ 0 & 1 & -2 & \frac{1}{2} & 4 \\ 0 & -5 & 10 & -9 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 5: $r_1 \leftarrow r_1 - 5r_2$ and $r_3 \leftarrow r_3 + 5r_2$

$$\begin{pmatrix} 1 & 0 & 3 & .5 & 3 \\ 0 & 1 & -2 & .5 & 4 \\ 0 & 0 & 0 & -6.5 & 13 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 6: $r_3 \leftarrow -\frac{2}{13}r_3$

$$\begin{pmatrix} 1 & 0 & 3 & .5 & 3 \\ 0 & 1 & -2 & .5 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 7: $r_1 \leftarrow r_1 - .5r_3$ and $r_2 \leftarrow r_2 - .5r_r$

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding system is given by

$$\begin{cases} x_1 \quad\quad + 3x_3 \quad\quad = 4 \\ \quad\quad x_2 - 2x_3 \quad\quad = 5 \\ \quad\quad\quad\quad\quad\quad x_4 = -2 \end{cases}$$

The general solution is: $x_1 = 4 - 3t, x_2 = 5 + 2t, x_3 = t, x_4 = -2$ ■

Exercise 60
Find the rank of each of the following matrices.
(a)

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 4 & 0 & -2 & 1 \\ 3 & -1 & 0 & 4 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{pmatrix}$$

Solution.
(a) We reduce the given matrix to row-echelon form.

Step 1: $r_3 \leftarrow r_3 + 4r_1$ and $r_4 \leftarrow r_4 + 3r_1$

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & -4 & -2 & 1 \\ 0 & -4 & 0 & 4 \end{pmatrix}$$

Step 2: $r_4 \leftarrow r_4 - r_3$

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & -4 & -2 & 1 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

Step 3: $r_1 \leftarrow -r_1$ and $r_2 \leftrightarrow r_3$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

Step 4: $r_4 \leftarrow r_3 - r_4$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 5: $r_2 \leftarrow -\frac{1}{4}r_2$ and $r_3 \leftarrow \frac{1}{2}r_3$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & .5 & -.25 \\ 0 & 0 & 1 & 1.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, the rank of the given matrix is 3.
(b) Apply the Gauss algorithm as follows.

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 + r_1$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & -4 & 4 \end{pmatrix}$$

Step 2: $r_3 \leftarrow r_3 + 2r_2$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 3: $r_2 \leftarrow \frac{1}{2}r_2$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, the rank is 2 ■

Exercise 61

Choose h and k such that the following system has (a) no solutions, (b) exactly one solution, and (c) infinitely many solutions.

$$\left\{ \begin{array}{rcrcr} x_1 & - & 3x_2 & = & 1 \\ 2x_1 & - & hx_2 & = & k \end{array} \right.$$

Solution.

The augmented matrix of the system is

$$\begin{pmatrix} 1 & -3 & 1 \\ 2 & -h & k \end{pmatrix}$$

By performing the operation $r_2 \leftarrow r_2 - 2r_1$ we find

$$\begin{pmatrix} 1 & -3 & 1 \\ 0 & 6-h & k-2 \end{pmatrix}$$

(a) The system is inconsistent if $h = 6$ and $k \neq 2$.
(b) The system has exactly one solution if $h \neq 6$ and for any k. The solution is given by the formula:$x_1 = \frac{3k-h}{6-h}, x_2 = \frac{k-2}{6-h}$.
(c) The system has infinitely many solutions if $h = 6$ and $k = 2$. The parametric form of the solution is: $x_1 = 1 + 3s, x_2 = s$ ■

Exercise 62

Solve the linear system whose augmented matrix is reduced to the following reduced row-echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{pmatrix}$$

Solution.

The free variable is $x_4 = s$. Solving by back-substitution one finds $x_1 = 8 + 7s, x_2 = 2 - 3s$, and $x_3 = -5 - s$ ■

Exercise 63

Solve the linear system whose augmented matrix is reduced to the following row-echelon form

$$\begin{pmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution.
Because of the last row the system is inconsistent ■

Exercise 64
Solve the linear system whose augmented matrix is given by

$$\begin{pmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{pmatrix}$$

Solution.
The reduction of the augmented matrix to row-echelon form is

Step 1: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\begin{pmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{pmatrix}$$

Step 2: $r_2 \leftarrow -r_2$

$$\begin{pmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{pmatrix}$$

Step 3: $r_3 \leftarrow r_3 + 10r_2$

$$\begin{pmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{pmatrix}$$

Step 4: $r_3 \leftarrow -\frac{1}{52}r_3$

$$\begin{pmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Using backward substitution we find the solution: $x_1 = 3, x_2 = 1, x_3 = 2$ ■

Exercise 65
Find the value(s) of a for which the following system has a nontrivial solution. Find the general solution.

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_1 + 3x_2 + 6x_3 = 0 \\ 2x_1 + 3x_2 + ax_3 = 0 \end{cases}$$

Solution.
The augmented matrix of the system is

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 6 & 0 \\ 2 & 3 & a & 0 \end{pmatrix}$$

Reducing this matrix to row-echelon form as follows.

Step 1: $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 - 2r_1$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & -1 & a-2 & 0 \end{pmatrix}$$

Step 2: $r_3 \leftarrow r_3 + r_2$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & a+3 & 0 \end{pmatrix}$$

If $a = -3$ then the rank of the coefficient matrix is less than the number of unknowns. Therefore, by Theorem 6 the system has a nontrivial solution and consequently infinitely many solutions. By letting $x_3 = s$ we find $x_1 = 9s$ and $x_2 = -5s$ ■

Exercise 66

Solve the following homogeneous system.

$$\left\{ \begin{array}{ccccccccc} x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 0 \\ 2x_1 & + & 2x_2 & & & - & x_4 & = & 0 \\ 3x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 0 \end{array} \right.$$

Solution.

The augmented matrix of the system is

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 3 & 1 & 2 & 1 & 0 \end{pmatrix}$$

Reducing the augmented matrix to row-echelon form.

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -4 & -3 & 0 \\ 0 & 4 & -4 & -2 & 0 \end{pmatrix}$$

Step 2: $r_3 \leftarrow r_3 - r_2$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -4 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Step 3: $r_2 \leftarrow \frac{1}{4}r_2$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

It follows that the rank of the coefficient matrix is less than the number of unknowns so the system has infinitely many solutions given by the formula $x_1 = -s, x_2 = s, x_3 = s,$ and $x_4 = 0$ ■

Exercise 67
Let A be an $m \times n$ matrix.
(a) Prove that if y and z are solutions to the homogeneous system $Ax = 0$ then $y + z$ and cy are also solutions, where c is a number.
(b) Give a counterexample to show that the above is false for nonhomogeneous systems.

Solution.
(a) Suppose that $Ax = Ay = 0$ and $c \in \mathbb{R}$ then $A(x + y) = Ax = Ay = 0$ and $A(cx) = cAx = 0$.
(b) The ordered pair $(1, 0)$ satisfies the nonhomogeneous system

$$\left\{\begin{array}{cccc} x_1 & + & x_2 & = & 1 \\ x_1 & - & x_2 & = 1 \end{array}\right.$$

However, $2(1, 0)$ is not a solution ■

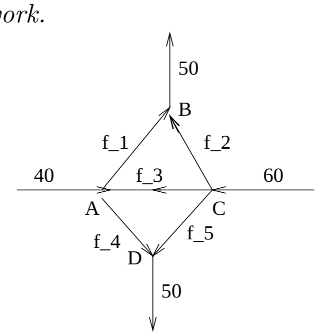
Exercise 68
Show that the converse of Theorem 6 is false. That is, show the existence of a nontrivial solution does not imply that the number of unknowns is greater than the number of equations.

Solution.
Consider the system

$$\left\{\begin{array}{ccccc} x_1 & + & x_2 & = & 0 \\ 2x_1 & + & 2x_2 & = & 0 \\ 3x_1 & + & 3x_2 & = & 0 \end{array}\right.$$

The system consists of three lines that coincide ■

Exercise 69 *(Network Flow)*
The junction rule of a network says that at each junction in the network the total flow into the junction must equal the total flows out. To illustrate the use of this rule, consider the network shown in the accompanying diagram. Find the possible flows in the network.



Solution.
Equating the flow in with the flow out at each intersection, we obtain the following system of four equations in the unknowns f_1, f_2, f_3, f_4, and f_5.

$$\begin{cases} f_1 \quad - f_3 + f_4 = 40 (\text{Intersection}A) \\ f_1 + f_2 = 50 (\text{Intersection}B) \\ f_2 + f_3 + f_5 = 60 (\text{Intersection}C) \\ f_4 + f_5 = 50 (\text{Intersection}D) \end{cases}$$

The augmented matrix of the system is

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 & 1 & 50 \end{pmatrix}$$

The reduction of this system to row-echelon form is carried out as follows.

Step 1: $r_2 \leftarrow r_2 - r_1$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 40 \\ 0 & 1 & 1 & -1 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 & 1 & 50 \end{pmatrix}$$

Step 2: $r_3 \leftarrow r_3 - r_2$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 40 \\ 0 & 1 & 1 & -1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 1 & 1 & 50 \end{pmatrix}$$

Step 3: $r_4 \leftarrow r_4 - r_3$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 40 \\ 0 & 1 & 1 & -1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This shows that f_3 and f_5 play the role of parameters. Now, solving for the remaining variables we find the general solution of the system

$$\begin{cases} f_1 = f_3 + f_5 - 10 \\ f_2 = -f_3 - f_5 + 60 \\ f_4 = 50 - f_5 \end{cases}$$

which gives all the possible flows ■

Chapter 2

Exercise 108
Compute the matrix

$$3\left(\begin{array}{cc} 2 & 1 \\ -1 & 0 \end{array}\right)^T - 2\left(\begin{array}{cc} 1 & -1 \\ 2 & 3 \end{array}\right)$$

Solution.

$$3\left(\begin{array}{cc} 2 & 1 \\ -1 & 0 \end{array}\right)^T - 2\left(\begin{array}{cc} 1 & -1 \\ 2 & 3 \end{array}\right) = \left(\begin{array}{cc} 4 & -1 \\ -1 & -6 \end{array}\right) \blacksquare$$

Exercise 109
Find $w, x, y,$ and z.

$$\left(\begin{array}{ccc} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{array}\right) = \left(\begin{array}{ccc} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{array}\right)$$

Solution.
Equating the corresponding entries we find $w = -1, x = -3, y = 0$, and $z = 5$ $\blacksquare$

Exercise 110
Determine two numbers s and t such that the following matrix is symmetric.

$$A = \left(\begin{array}{ccc} 2 & s & t \\ 2s & 0 & s+t \\ 3 & 3 & t \end{array}\right)$$

Solution.
Since A is symmetric then $A^T = A$; that is,

$$A^T = \left(\begin{array}{ccc} 2 & 2s & 3 \\ s & 0 & 3 \\ t & s+t & t \end{array}\right) = \left(\begin{array}{ccc} 2 & s & t \\ 2s & 0 & s+t \\ 3 & 3 & t \end{array}\right)$$

Equating corresponding entries we find that $s = 0$ and $t = 3$ $\blacksquare$

Exercise 111
Let A be a 2×2 matrix. Show that

$$A = a\left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) + b\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right) + c\left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right) + d\left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right)$$

Solution.
Let A be the matrix

$$A = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$$

Then

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =$$
$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} =$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \blacksquare$$

Exercise 112
Let $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$. *If* $rA + sB + tC = \mathbf{0}$ *show that* $s = r = t = 0$.

Solution.
A simple arithmetic yields the matrix

$$rA + sB + tC = \begin{bmatrix} r + 3t & r + s & -r + 2s + t \end{bmatrix}$$

The condition $rA + sB + tC = \mathbf{0}$ yields the system

$$\begin{cases} r & & & + & 3t & = & 0 \\ r & + & s & & & = & 0 \\ -r & + & 2s & + & t & = & 0 \end{cases}$$

The augmented matrix is

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

Step 1: $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 + r_1$

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & 4 & 0 \end{pmatrix}$$

Step 2: $r_3 \leftarrow r_3 - 2r_2$

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{pmatrix}$$

Solving the corresponding system we find $r = s = t = 0$ $\blacksquare$

Exercise 113
Show that the product of two diagonal matrices is again a diagonal matrix.

Solution.
Let $D = (d_{ij})$ be a diagonal matrix of size $m \times n$ and $D' = (d'_{ij})$ be an $n \times p$ diagonal matrix. Let $DD' = (a_{ij})$. Then $a_{ij} = \sum_{k=1}^n d_{ik}d'_{kj}$. If $i = j$ then $a_{ii} = d_{ii}d'_{ii}$ since for $i \neq k$ we have $d_{ik} = d'_{ki} = 0$. Now, if $i \neq j$ then $d_{ik}d'_{kj} = 0$ for all possible values of k. Hence, DD' is diagonal $\blacksquare$

Exercise 114
Let A be an arbitrary matrix. Under what conditions is the product AA^T defined?

Solution.
Suppose A is an $m \times n$ matrix. Then A^T is an $n \times m$ matrix. Since the number of columns of A is equal to the number of rows of A^T then AA^T is always defined ■

Exercise 115
(a) Show that $AB = BA$ if and only if $(A-B)(A+B) = A^2 - B^2$.
(b) Show that $AB = BA$ if and only if $(A+B)^2 = A^2 + 2AB + B^2$.

Solution.
(a) Suppose that $AB = BA$ then $(A-B)(A+B) = A^2+AB-BA-B^2 = A^2-B^2$ since $AB - BA = \mathbf{0}$. Conversely, suppose that $(A-B)(A+B) = A^2 - B^2$ then by expanding the left-hand side we obtain $A^2 - AB + BA - B^2 = A^2 - B^2$. This implies that $AB = BA$.
(b) Suppose that $AB = BA$ then $(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$. Conversely, suppose that $(A+B)^2 = A^2 + 2AB + B^2$. By expanding the left-hand side we obtain $A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$ and this leads to $AB = BA$ ■

Exercise 116
Let A be a matrix of size $m \times n$. Denote the columns of A by $C_1, C_2, \cdots, C_n$. Let x be the $n \times 1$ matrix with entries $x_1, x_2, \cdots, x_n$. Show that $Ax = x_1C_1 + x_2C_2 + \cdots + x_nC_n$.

Solution.
Suppose that $A = (a_{ij})$. Then

$$\begin{aligned}
Ax &= \left(\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{mn} & \cdots & a_{mn} \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right) = \left(\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{array}\right) \\
&= x_1\left(\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{array}\right) + x_2\left(\begin{array}{c} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{array}\right) + \cdots + x_n\left(\begin{array}{c} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{array}\right) \\
&= x_1C_1 + x_2C_2 + \cdots + x_nC_n
\end{aligned}$$ ■

Exercise 117
Let A be an $m \times n$ matrix. Show that if $yA = \mathbf{0}$ for all $y \in \mathbb{R}^m$ then $A = \mathbf{0}$.

Solution.
Suppose the contrary. Then there exist indices i and j such that $a_{ij} \neq 0$. Let y be the $1 \times m$ matrix such that the jth column is 1 and 0 elsewhere. Then the $(1, j)-$th entry of yA is $a_{ij} \neq 0$, a contradiction ■

Exercise 118
An $n \times n$ matrix A is said to be **idempotent** *if $A^2 = A$.*
(a) Show that the matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is idempotent
(b) Show that if A is idempotent then the matrix $(I_n - A)$ is also idempotent.

Solution.
(a) Easy calculation shows that $A^2 = A$.
(b) Suppose that $A^2 = A$ then $(I_n - A)^2 = I_n - 2A + A^2 = I_n - 2A + A = I_n - A$ ■

Exercise 119
The purpose of this exercise is to show that the rule $(ab)^n = a^n b^n$ does not hold with matrix multiplication. Consider the matrices

$$A = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$$

Show that $(AB)^2 \neq A^2B^2$.

Solution.
We have

$$(AB)^2 = \begin{pmatrix} 100 & -432 \\ 0 & 289 \end{pmatrix}$$

and

$$A^2B^2 = \begin{pmatrix} 160 & -460 \\ -5 & 195 \end{pmatrix} ■$$

Exercise 120
Show that $AB = BA$ if and only if $A^TB^T = B^TA^T$.

Solution.
$AB = BA$ if and only if $(AB)^T = (BA)^T$ if and only if $B^TA^T = A^TB^T$ ■

Exercise 121
Suppose $AB = BA$ and n is a non-negative integer.
(a) Use induction to show that $AB^n = B^nA$.
(b) Use induction and (a) to show that $(AB)^n = A^nB^n$.

Solution.
(a) The equality holds for $n = 0$ and $n = 1$. Suppose that $AB^n = B^nA$. Then $AB^{n+1} = (AB)B^n = (BA)B^n = B(AB^n) = B(B^nA) = B^{n+1}A$.
(b) The equality holds for $n = 1$. Suppose that $(AB)^n = A^nB^n$. Then $(AB)^{n+1} = (AB)^n(AB) = (A^nB^n)(AB) = (A^nB^n)(BA) = A^n(B^nB)A = A^n(B^{n+1}A) = A^n(AB^{n+1}) = A^{n+1}B^{n+1}$ ■

Exercise 122
Let A and B be symmetric matrices. Show that AB is symmetric if and only if $AB = BA$.

Solution.
AB is symmetric if and only if $(AB)^T = AB$ if and only if $B^TA^T = AB$ if and only if $AB = BA$ ■

Exercise 123
Show that $tr(AA^T)$ is the sum of the squares of all the entries of A.

Solution.
If $A = (a_{ij})$ then $A^T = (a_{ji})$. Hence, $AA^T = (\sum_{k=1}^n a_{ik}a_{jk})$. Using the definition of the trace we have

$$tr(AA^T) = \sum_{i=1}^n (\sum_{k=1}^n a_{ik}a_{ik}) = \sum_{i=1}^n \sum_{k=1}^n a_{ik}^2 \blacksquare$$

Exercise 124
(a) Find two 2×2 singular matrices whose sum in nonsingular.
(b) Find two 2×2 nonsingular matrices whose sum is singular.

Solution.
(a)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \blacksquare$$

Exercise 125
Show that the matrix

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{pmatrix}$$

is singular.

Solution.
If B is a 3×3 matrix such that $BA = I_3$ then

$$b_{31}(0) + b_{32}(0) + b_{33}(0) = 0$$

But this is equal to the $(3, 3)$ entry of I_3 which is 1. A contradiction ■

Exercise 126
Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

Find A^{-3}.

Solution.

$$A^{-3} = (A^{-1})^3 = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 41 & -30 \\ -15 & 11 \end{pmatrix} \blacksquare$$

Exercise 127
Let

$$A^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$$

Find A.

Solution.
Using Exercise 94, we find

$$A = (A^{-1})^{-1} = \begin{pmatrix} \frac{5}{13} & \frac{1}{13} \\ -\frac{3}{13} & \frac{2}{13} \end{pmatrix} \blacksquare$$

Exercise 128
Let A *and* B *be square matrices such that* $AB = \mathbf{0}$. *Show that if* A *is invertible then* B *is the zero matrix.*

Solution.
If A is invertible then $B = I_nB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}\mathbf{0} = \mathbf{0}$ $\blacksquare$

Exercise 129
Find the inverse of the matrix

$$A = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$$

Solution.
Using Exercise 94 we find

$$A^{-1} = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \blacksquare$$

Exercise 130
Which of the following are elementary matrices?
(a)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution.
(a) No. This matrix is obtained by performing two operations: $r_2 \leftrightarrow r_3$ and $r_1 \leftarrow r_1 + r_3$.
(b) Yes: $r_2 \leftarrow r_2 - 5r_1$.
(c) Yes: $r_2 \leftarrow r_2 + 9r_3$.
(d) No: $r_1 \leftarrow 2r_1$ and $r_1 \leftarrow r_1 + 2r_4$ ■

Exercise 131
Let A be a 4×3 matrix. Find the elementary matrix E, which as a premultiplier of A, that is, as EA, performs the following elementary row operations on A :
(a) Multiplies the second row of A by -2.
(b) Adds 3 times the third row of A to the fourth row of A.
(c) Interchanges the first and third rows of A.

Solution.
(a)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ■$$

Exercise 132
For each of the following elementary matrices, describe the corresponding elementary row operation and write the inverse.
(a)

$$E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(b)

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Solution.
(a) $r_1 \leftrightarrow r_3$, $E^{-1} = E$.
(b) $r_2 \leftarrow r_2 - 2r_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) $r_3 \leftarrow 5r_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix} \blacksquare$$

Exercise 133
Consider the matrices

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{pmatrix}, B = \begin{pmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{pmatrix}, C = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{pmatrix}$$

Find elementary matrices $E_1, E_2, E_3,$ and E_4 such that
$(a) E_1 A = B, (b) E_2 B = A, (c) E_3 A = C, (d) E_4 C = A.$

Solution.
(a)

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(b) $E_2 = E_1$.
(c)

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

(d)

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \blacksquare$$

Exercise 134
Determine if the following matrix is invertible.

$$\begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$$

Solution.
Consider the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right)$$

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 + r_1$

$$\left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right)$$

Step 2: $r_3 \leftarrow r_3 + r_2$

$$\left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right)$$

It follows that the given matrix is row equivalent to a matrix with a row of zeros. By Theorem 255 this matrix is singular $\blacksquare$

Exercise 135
For what values of a does the following homogeneous system have a nontrivial solution?

$$\begin{cases} (a-1)x_1 + 2x_2 = 0 \\ 2x_1 + (a-1)x_2 = 0 \end{cases}$$

Solution.
Let A be the coefficient matrix of the given system, i.e.

$$A = \begin{pmatrix} a-1 & 2 \\ 2 & a-1 \end{pmatrix}$$

By Exercise 94, A is invertible if and only if $(a-1)^2-4\neq 0$,i.e. $a\neq 3$ and $a\neq -1$. In this case the trivial solution is the only solution. Hence, in order to have nontrivial solution we must have $a=-1$ or $a=3$ ■

Exercise 136
Find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 1\\ 0 & 2 & 3\\ 5 & 5 & 1\end{pmatrix}$$

Solution.
We first construct the matrix

$$\left(\begin{array}{ccccccc} 1 & 1 & 1 & | & 1 & 0 & 0\\ 0 & 2 & 3 & | & 0 & 1 & 0\\ 5 & 5 & 1 & | & 0 & 0 & 1\end{array}\right)$$

Step 1: $r_3\leftarrow r_3-5r_1$

$$\left(\begin{array}{ccccccc} 1 & 1 & 1 & | & 1 & 0 & 0\\ 0 & 2 & 3 & | & 0 & 1 & 0\\ 0 & 0 & -4 & | & -5 & 0 & 1\end{array}\right)$$

Step 2: $r_2\leftarrow \frac{1}{2}r_2$ and $r_3\leftarrow -\frac{1}{4}r_3$

$$\left(\begin{array}{ccccccc} 1 & 1 & 1 & | & 1 & 0 & 0\\ 0 & 1 & \frac{3}{2} & | & 0 & \frac{1}{2} & 0\\ 0 & 0 & 1 & | & \frac{5}{4} & 0 & -\frac{1}{4}\end{array}\right)$$

Step 3: $r_1\leftarrow r_1-r_2$

$$\left(\begin{array}{ccccccc} 1 & 0 & -\frac{1}{2} & | & 1 & -\frac{1}{2} & 0\\ 0 & 1 & \frac{3}{2} & | & 0 & \frac{1}{2} & 0\\ 0 & 0 & 1 & | & \frac{5}{4} & 0 & -\frac{1}{4}\end{array}\right)$$

Step 4: $r_2\leftarrow r_2-\frac{3}{2}r_3$ and $r_1\leftarrow r_1+\frac{1}{2}r_3$

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & | & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8}\\ 0 & 1 & 0 & | & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8}\\ 0 & 0 & 1 & | & \frac{5}{4} & 0 & -\frac{1}{4}\end{array}\right)$$

It follows that

$$A^{-1}=\begin{pmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8}\\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8}\\ \frac{5}{4} & 0 & -\frac{1}{4}\end{pmatrix}■$$

Exercise 137
What conditions must b_1,b_2,b_3 satisfy in order for the following system to be consistent?

$$\left\{\begin{array}{ccccccc} x_1 & + & x_2 & + & 2x_3 & = & b_1\\ x_1 & & & + & x_3 & = & b_2\\ 2x_1 & + & x_2 & + & 3x_3 & = & b_3\end{array}\right.$$

Solution.
The augmented matrix of the system is

$$\left(\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{array}\right)$$

Step 1: $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 - 2r_1$

$$\left(\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{array}\right)$$

Step 2: $r_3 \leftarrow r_3 - r_2$

$$\left(\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right)$$

The system is consistent if and only if $b_3 - b_2 - b_1 = 0$ ■

Exercise 138
Prove that if A is symmetric and nonsingular than A^{-1} is symmetric.

Solution.
Let A be an invertible and symmetric $n \times n$ matrix. Then $(A^{-1})^T = (A^T)^{-1} = A^{-1}$. That is, A^{-1} is symmetric ■

Exercise 139
If

$$D = \left(\begin{array}{ccc} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{array}\right)$$

find D^{-1}.

Solution.
According to Exercise 100 we have

$$D^{-1} = \left(\begin{array}{ccc} \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{array}\right) ■$$

Exercise 140
Prove that a square matrix A is nonsingular if and only if A is a product of elementary matrices.

Solution.
Suppose first that A is nonsingular. Then by Theorem 19, A is row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that $I_n =$

$E_kE_{k-1}\cdots E_1A$. Then $A = E_1^{-1}E_2^{-1}\cdots E_k^{-1}$. But each E_i^{-1} is an elementary matrix by Theorem 18.
Conversely, suppose that $A = E_1E_2\cdots E_k$. Then $(E_1E_2\cdots E_k)^{-1}A = I_n$. That is, A is nonsingular ■

Exercise 141
Prove that two $m \times n$ matrices A and B are row equivalent if and only if there exists a nonsingular matrix P such that $B = PA$.

Solution.
Suppose that $A \sim B$. Then there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that $B = E_kE_{k-1}\cdots E_1A$. Let $P = E_kE_{k-1}\cdots E_1$. Then by Theorem 18 and Theorem 15 (a), P is nonsingular.
Conversely, suppose that $B = PA$, for some nonsingular matrix P. By Theorem 19, $P \sim I_n$. That is, $I_n = E_kE_{k-1}\cdots E_1P$. Thus, $B = E_1^{-1}E_2^{-1}\cdots E_k^{-1}A$ and this implies that $A \sim B$ ■

Exercise 142
Let A and B be two $n \times n$ matrices. Suppose A is row equivalent to B. Prove that A is nonsingular if and only if B is nonsingular.

Solution.
Suppose that $A \sim B$. Then by the previous exercise, $B = PA$, with P nonsingular. If A is nonsingular then by Theorem 15 (a), B is nonsingular. Conversely, if B is nonsingular then $A = P^{-1}B$ is nonsingular ■

Exercise 143
Show that the product of two lower (resp. upper) triangular matrices is again lower (resp. upper) triangular.

Solution.
We prove by induction on n that the product of tow lower triangular matrices is lower triangular. For $n = 2$ we have

$$\begin{aligned} AB &= \left(\begin{array}{cc} a_{11} & 0 \\ a_{21} & a_{22} \end{array}\right)\left(\begin{array}{cc} b_{11} & 0 \\ b_{21} & b_{22} \end{array}\right) \\ &= \left(\begin{array}{cc} a_{11}b_{11} & 0 \\ a_{21}b_{11} + a_{22}b_{21} & a_{22}b_{22} \end{array}\right) \end{aligned}$$

Thus, AB is lower triangular. So suppose that the result is true for all lower triangular matrices of sise $\leq n-1$. Let A and B be two lower triangular matrices of size $n \times n$. Then

$$\begin{aligned} AB &= \left(\begin{array}{cc} a & 0 \\ X & A_1 \end{array}\right)\left(\begin{array}{cc} b & 0 \\ Y & B_2 \end{array}\right) \\ &= \left(\begin{array}{cc} ab & 0 \\ bX + A_1Y & A_1B_1 \end{array}\right) \end{aligned}$$

where X and Y are $(n-1)\times 1$ matrices and A_1, B_1 are $(n-1)\times(n-1)$ lower triangular matrices. By the induction hypothesis, A_1B_1 is lower triangular so that AB is lower triangular.
The fact that the product of two upper triangular matrices is upper triangular follows by transposition. ■

Exercise 144
Show that a 2×2 lower triangular matrix is invertible if and only if $a_{11}a_{22}\neq 0$ and in this case the inverse is also lower triangular.

Solution.
By Exercise 94, the lower triangular matrix

$$A = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$$

is invertible if and only in $a_{11}a_{22}\neq 0$ and in this case

$$A^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & 0 \\ -\frac{a_{21}}{a_{11}a_{22}} & \frac{1}{a_{22}} \end{pmatrix} ■$$

Exercise 145
Let A be an $n\times n$ matrix and suppose that the system $Ax = \mathbf{0}$ has only the trivial solution. Show that $A^kx = \mathbf{0}$ has only the trivial solution for any positive integer k.

Solution.
Since $Ax = \mathbf{0}$ has only the trivial solution then A is invertible. By indiction on k and Theorem 15(a), A^k is invertible and consequently the system $A^kx = \mathbf{0}$ has only the trivial solution ■

Exercise 146
Show that if A and B are two $n\times n$ matrices then $A \sim B$.

Solution.
Since A is invertible then $A \sim I_n$. That is, there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that $I_n = E_kE_{k-1}\cdots E_1A$. Similarly, there exist elementary matrices $F_1, F_2, \cdots, F_l$ such that $I_n = F_lF_{l-1}\cdots F_1B$. Hence, $A = E_1^{-1}E_2^{-1}\cdots E_k^{-1}F_lF_{l-1}\cdots F_1B$. That is, $A \sim B$ ■

Exercise 147
Show that an $n\times n$ matrix invertible matrix A satisfies the property (P): If $AB = AC$ then $B = C$.

Solution.
Suppose that A is invertible. Let B abd C be two $n\times n$ matrices such that $AB = AC$. Then $A(B-C) = \mathbf{0}$. Multiplying both sides by A^{-1} to obtain $B - C = \mathbf{0}$ or $B = C$ ■

Exercise 148
Show that an $n \times n$ matrix A is invertible if and only if the equation $yA = \mathbf{0}$ has only the trivial solution.

Solution.
Suppose that A is invertible and let y be a $1 \times n$ matrix such that $yA = \mathbf{0}$. Multiplying from the left by A^{-1} we find $y = \mathbf{0}$.
Conversely, suppose that the equation $yA = \mathbf{0}$ has only the trivial solution. Then $A^T x = \mathbf{0}$ has only the trivial solution. By Theorem 19, A^T is invertible and consequently A is invertible ■

Exercise 149
Let A be an $n \times n$ matrix such that $A^n = \mathbf{0}$. Show that $I_n - A$ is invertible and find its inverse.

Solution.
We have $(I_n - A)(I_n + A + A^2 + \cdots + A^{n-1} = I_n + A + \cdots + A^{n-1} - A - A^2 - \cdots - A^{n-1} - A^n = I_n$. Thus, $I_n - A$ is invertible and $(I_n - A)^{-1} = I_n + A + \cdots + A^{n-1}$ ■

Exercise 150
Let $A = (a_{ij}(t))$ be an $m \times n$ matrix whose entries are differentiable functions of the variable t. We define $\frac{dA}{dt} = (\frac{da_{ij}}{dt})$. Show that if the entries in A and B are differentiable functions of t and the sizes of the matrices are such that the stated operations can be performed, then
(a) $\frac{d}{dt}(kA) = k\frac{dA}{dt}$.
(b) $\frac{d}{dt}(A + B) = \frac{dA}{dt} + \frac{dB}{dt}$.
(c) $\frac{d}{dt}(AB) = \frac{dA}{dt}B + A\frac{dB}{dt}$.

Solution.
(a) $\frac{d}{dt}(kA) = (\frac{d}{dt}(ka_{ij})) = (k\frac{da_{ij}}{dt}) = k(\frac{da_{ij}}{dt}) = k\frac{dA}{dt}$.
(b) $\frac{d}{dt}(A + B) = (\frac{d}{dt}(a_{ij} + b_{ij})) = (\frac{da_{ij}}{dt} + \frac{db_{ij}}{dt}) = \frac{dA}{dt} + \frac{dB}{dt}$.
(c) Use the product rule of differentiation ■

Exercise 151
Let A be an $n \times n$ invertible matrix with entries being differentiable functions of t. Find a formula for $\frac{dA^{-1}}{dt}$.

Solution.
Since A is invertible then $AA^{-1} = I_n$. Taking derivative of both sides and using part (c) of the previous exercise we find $\frac{dA}{dt}A^{-1} + A\frac{dA^{-1}}{dt} = 0$. Thus, $A\frac{dA^{-1}}{dt} = -\frac{dA}{dt}A^{-1}$. Now premultiply by A^{-1} to obtain $\frac{dA^{-1}}{dt} = -A^{-1}\frac{dA}{dt}A^{-1}$ ■

Exercise 152 *(Sherman-Morrison formula)*
Let A be an $n \times n$ invertible matrix. Let u, v be $n \times 1$ matrices such that $v^T A^{-1} u + 1 \neq 0$. Show that
(a) $(A + uv^T)(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}) = I_n$.
(b) Deduce from (a) that A is invertible.

Solution.
(a) Using the properties of matrix multiplication we find

$(A+uv^T)(A^{-1}-\frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}) = I_n - \frac{uv^TA^{-1}}{1+v^TA^{-1}u} + uv^TA^{-1} - \frac{uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} =$
$I_n + \frac{uv^TA^{-1}v^TA^{-1}u - uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} = I_n + \frac{v^TA^{-1}u(uv^TA^{-1}-uv^TA^{-1})}{1+v^TA^{-1}u} = I_n.$
(b) It follows from Theorem 20 and (a)that $A+uv^T$ is invertible and

$$(A+uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \blacksquare$$

Exercise 153
Let $x=(x_1,x_2,\cdots x_m)^T$ be an $m\times 1$ matrix and $y=(y_1,y_2,\cdots,y_n)^T$ be an $n\times n$ matrix. Construct the $m\times n$ matrix xy^T.

Solution.
Using matrix multiplication we have

$$\begin{aligned} xy^T &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}(y_1,y_2,\cdots,y_n) \\ &= \begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{pmatrix} \blacksquare \end{aligned}$$

Exercise 154
Show that a triangular matrix A with the property $A^T = A^{-1}$ is always diagonal.

Solution.
Since A is invertible and upper triangular then A^{-1} is also upper triangular. Since A is upper triangular then A^T is lower triangular. But $A^T = A^{-1}$. Thus, A^T is both upper and lower triangular so it must be diagonal. Hence, A is also diagonal $\blacksquare$

Chapter 3

Exercise 186
(a) Find the number of inversions in the permutation (41352).
(b) Is this permutation even or odd?

Solution.
(a) There are five inversions: $(41),(43),(42),(32),(52)$.
(b) The given permutation is odd $\blacksquare$

Exercise 187
Evaluate the determinant of each of the following matrices
(a)

$$A = \begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{pmatrix}$$

Solution.
(a) $|A| = (3)(4) - (-2)(5) = 22.$
(b) $|A| = -2(-14-6) - 7(20+6) + 6(40-3) = 0$ ■

Exercise 188
Find all values of t for which the determinant of the following matrix is zero.

$$A = \begin{pmatrix} t-4 & 0 & 0 \\ 0 & t & 0 \\ 0 & 3 & t-1 \end{pmatrix}$$

Solution.
$|A| = t(t-1)(t-4) = 0$ implies $t = 0, t = 1$, or $t = 4$ ■

Exercise 189
Solve for x

$$\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}$$

Solution.
Evaluating the determinants on both sides we find

$$x(1-x) + 3 = x(x-5) + 18 - 3(6-x)$$

Simplifying to obtain the quadratic equation $2x^2 - 3x - 3 = 0$ whose roots are $x_1 = \frac{3-\sqrt{33}}{4}$ and $x_2 = \frac{3+\sqrt{33}}{4}$ ■

Exercise 190
Evaluate the determinant of the following matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution.
Since the given matrix has a row consisting of 0 then $|A| = 0$ ■

Exercise 191
Evaluate the determinant of the following matrix.

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}$$

Solution.
The given matrix is upper triangular so that the determinant is the product of entries on the main diagonal, i.e. equals to -1296 ■

Exercise 192
Use the row reduction technique to find the determinant of the following matrix.

$$A = \begin{pmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -6 & 4 & 3 \end{pmatrix}$$

Solution.
Step 1: $r_1 \leftarrow r_1 + r_4$

$$B_1 = \begin{pmatrix} 1 & -1 & 1 & 1 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -6 & 4 & 3 \end{pmatrix}$$

and $|B_1| = |A|$.
Step 2: $r_2 \leftarrow r_2 + 2r_1$ and $r_3 \leftarrow r_3 + r_4$

$$B_2 = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & -5 & 4 & -3 \\ 0 & -3 & 2 & 5 \\ -1 & -6 & 4 & 3 \end{pmatrix}$$

and $|B_2| = |A|$.
Step 3: $r_4 \leftarrow r_4 + r_1$

$$B_3 = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & -5 & 4 & -3 \\ 0 & -3 & 2 & 5 \\ 0 & -7 & 5 & 4 \end{pmatrix}$$

and $|B_3| = |A|$.
Step 4: $r_2 \leftarrow r_2 - 2r_3$

$$B_4 = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -13 \\ 0 & -3 & 2 & 5 \\ 0 & -7 & 5 & 4 \end{pmatrix}$$

and $|B_4| = |A|$.
Step 5: $r_3 \leftarrow r_3 + 3r_2$ and $r_4 \leftarrow r_4 + 7r_2$

$$B_5 = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 2 & -34 \\ 0 & 0 & 5 & -87 \end{pmatrix}$$

and $|B_5| = |A|$.
Step 6: $r_3 \leftarrow \frac{1}{2}r_3$

$$B_6 = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -17 \\ 0 & 0 & 5 & -87 \end{pmatrix}$$

and $|B_6| = \frac{1}{2}|A|$.
Step 7: $r_4 \leftarrow r_4 - 5r_3$

$$B_7 = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -17 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

and $|B_7| = \frac{1}{2}|A|$. Thus, $|A| = 2|B_7| = -4$ ■

Exercise 193
Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6,$$

find
(a)

$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix},$$

(b)

$$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$$

(c)

$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$$

(d)

$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

Solution.
(a) The process involves row interchanges twice so

$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -6$$

(b) We have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \frac{1}{3}\begin{vmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= \frac{1}{12}\begin{vmatrix} 3a & 3b & 3c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix} = -\frac{1}{12}\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$$

Thus,

$$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = 72$$

(c)

$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

(d)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\frac{1}{3}\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g & h & i \end{vmatrix} = -\frac{1}{3}\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

Thus,

$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} = 18 \blacksquare$$

Exercise 194
Determine by inspection the determinant of the following matrix.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix}$$

Solution.
The determinant is 0 since the first and the fifth rows are proportional ■

Exercise 195
Find the determinant of the 1×1 *matrix* $A = (3)$.

Solution.
By definition the determinant of a 1×1 matrix is just the entry of that matrix. Thus, $|A| = 3$ ■

Exercise 196
Let A *be a* 3×3 *matrix such that* $|2A| = 6$. *Find* $|A|$.

Solution.
We have, $6 = |2A| = 2^3|A| = 8|A|$. Thus, $|A| = \frac{3}{4}$ ■

Exercise 197
Show that if n *is any positive integer then* $|A^n| = |A|^n$.

Solution.
The proof is by induction on $n \geq 1$. The equality is valid for $n = 1$. Suppose that it is valid up to n. Then $|A^{n+1}| = |A^n A| = |A^n||A| = |A|^n|A| = |A|^{n+1}$ ■

Exercise 198
Show that if A *is an* $n \times n$ *skew-symmetric and* n *is odd then* $|A| = 0$.

Solution.
Since A is skew-symmetric then $A^T = -A$. Taking the determinant of both sides we find $|A^T| = |-A| = (-1)^n|A| = -|A|$ since n is odd. Thus, $2|A| = 0$ and therefore $|A| = 0$ ■

Exercise 199
Show that if A *is* **orthogonal**, *i.e.* $A^TA = AA^T = I_n$ *then* $|A| = \pm 1$. *Note that* $A^{-1} = A^T$.

Solution.
Taking to determinant of both sides of the equality $A^TA = I_n$ to obtain $|A^T||A| = 1$ or $|A|^2 = 1$ since $|A^T| = |A|$. It follows that $|A| = \pm 1$ ■

Exercise 200
If A *is a nonsingular matrix such that* $A^2 = A$, *what is* $|A|$?

Solution.
Taking the determinant of both sides to obtain $|A^2| = |A|$ or $|A|(|A| - 1) = 0$. Hence, either A is singular or $|A| = 1$ ■

Exercise 201
True or false: If

$$A = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right)$$

then $rank(A) = 3$. *Justify your answer.*

Solution.
Since A has a column of zero then $|A| = |A^T| = 0$. In this case, $rank(A) < 3$ ■

Exercise 202
Find out, without solving the system, whether the following system has a nontrivial solution

$$\left\{\begin{array}{ccccccc} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 0 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 0 \end{array}\right.$$

Solution.
The coefficient matrix

$$A = \left(\begin{array}{ccc} 1 & -2 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}\right)$$

has determinant $|A| = 0$. By Theorem 32, the system has a nontrivial solution ■

Exercise 203
For which values of c does the matrix

$$A = \left(\begin{array}{ccc} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{array}\right)$$

have an inverse.

Solution.
Finding the determinant we get $|A| = 2(c+2)(c-3)$. The determinant is 0 if $c = -2$ or $c = 3$ ■

Exercise 204
If $|A| = 2$ and $|B| = 5$, calculate $|A^3B^{-1}A^TB^2|$.

Solution.
$|A^3B^{-1}A^TB^2| = |A|^3|B|^{-1}|A||B|^2 = |A|^4|B| = 80$ ■

Exercise 205
Show that $|AB| = |BA|$.

Solution.
We have $|AB| = |A||B| = |B||A| = |BA|$ ■

Exercise 206
Show that $|A + B^T| = |A^T + B|$ for any $n \times n$ matrices A and B.

Solution.
We have $|A + B^T| = |(A + B^T)^T| = |A^T + B|$ ■

Exercise 207
Let $A = (a_{ij})$ be a triangular matrix. Show that $|A| \neq 0$ if and only if $a_{ii} \neq 0$, for $1 \leq i \leq n$.

Solution.
Let $A = (a_{ij})$ be a triangular matrix. By Theorem 32, A is nonsingular if and only if $|A| \neq 0$ and this is equivalent to $a_{11}a_{22}\cdots a_{nn} \neq 0$ ■

Exercise 208
Express

$$\left|\begin{array}{cc} a_1+b_1 & c_1+d_1 \\ a_2+b_2 & c_2+d_2 \end{array}\right|$$

as a sum of four determinants whose entries contain no sums.

Solution.

$$\left|\begin{array}{cc} a_1+b_1 & c_1+d_1 \\ a_2+b_2 & c_2+d_2 \end{array}\right| = \left|\begin{array}{cc} a_1 & c_1 \\ a_2 & c_2 \end{array}\right| = \left|\begin{array}{cc} a_1 & d_1 \\ a_2 & d_2 \end{array}\right| = \left|\begin{array}{cc} b_1 & c_1 \\ b_2 & c_2 \end{array}\right| = \left|\begin{array}{cc} b_1 & d_1 \\ b_2 & d_2 \end{array}\right| ■$$

Exercise 209
Let

$$A = \left(\begin{array}{ccc} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{array}\right)$$

(a) Find adj(A).
(b) Compute $|A|$.

Solution.
(a) The matrix of cofactors is

$$\left(\begin{array}{ccc} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{array}\right) = \left(\begin{array}{ccc} -18 & 17 & -6 \\ -6 & -10 & -2 \\ -10 & -1 & 28 \end{array}\right)$$

The adjoint of A is the transpose of the cofactors matrix.

$$adj(A) = \left(\begin{array}{ccc} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{array}\right)$$

(b) $|A| = -18(-280+2) + 6(476-6) - 10(-34-60) = -94$ ■

Exercise 210
Find the determinant of the matrix

$$A = \left(\begin{array}{cccc} 3 & 0 & 0 & 0 \\ 5 & 1 & 2 & 0 \\ 2 & 6 & 0 & -1 \\ -6 & 3 & 1 & 0 \end{array}\right)$$

Solution.
Expanding along the first row

$$|A| = 3C_{11} + 0C_{12} + 0C_{13} + 0C_{14} = -15\blacksquare$$

Exercise 211
Find the determinant of the following **Vandermonde** *matrix.*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$$

Solution.
We have

$$\begin{aligned}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ a & b-c & c \\ a^2 & b^2-c^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ a & b-c & c-a \\ a^2 & b^2-c^2 & c^2-a^2 \end{vmatrix} \\ &= \begin{vmatrix} b-c & c-a \\ b^2-c^2 & c^2-a^2 \end{vmatrix} \\ &= (b-c)(c^2-a^2)-(b^2-c^2)(c-a) \\ &= (b-c)(c-a)(a-b)\end{aligned}$$

Exercise 212
Let A be an $n \times n$ matrix. Show that $|adj(A)| = |A|^{n-1}$.

Solution.
Suppose first that A is invertible. Then $adj(A) = A^{-1}|A|$ so that $|adj(A)| = ||A|A^{-1}| = |A|^n|A^{-1}| = \frac{|A|^n}{|A|} = |A|^{n-1}$. If A is singular then $adj(A)$ is singular. To see this, suppose there exists a square matrix B such that $Badj(A) = adj(A)B = I_n$. Then $A = AI_n = A(adj(A)B) = (Aadj(A))B = 0$ and this leads to $adj(A) = 0$ a contradiction to the fact that $adj(A)$ is nonsingular. Thus, $adj(A)$ is singular and consequently $|adj(A)| = 0 = |A|^{n-1}$ $\blacksquare$

Exercise 213
If

$$A^{-1} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{pmatrix}$$

find $adj(A)$.

Solution.
The determinant of A^{-1} is $|A^{-1}| = -21$ and this implies that $|A| = -\frac{1}{21}$. Thus,

$$adj(A) = |A|A^{-1} = \begin{pmatrix} -\frac{1}{7} & 0 & -\frac{1}{21} \\ 0 & -\frac{2}{21} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{21} & \frac{1}{21} \end{pmatrix}\blacksquare$$

Exercise 214
If $|A| = 2$, *find* $|A^{-1} + adj(A)|$.

Solution.
We have $|A^{-1} + adj(A)| = |A^{-1}(I_n + |A|I_n)| = |A^{-1}|(1 + |A|)^n = \frac{3^n}{|A|} = \frac{3^n}{2}$ ■

Exercise 215
Show that $adj(\alpha A) = \alpha^{n-1}adj(A)$.

Solution.
The equality is valid for $\alpha = 0$. So suppose that $\alpha \neq 0$. Then $adj(\alpha A) = |\alpha A|(\alpha A)^{-1} = (\alpha)^n|A|\frac{1}{\alpha}A^{-1} = (\alpha)^{n-1}|A|A^{-1} = (\alpha)^{n-1}adj(A)$ ■

Exercise 216
Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$$

(a) Find $|A|$.
(b) Find $adj(A)$.
(c) Find A^{-1}.

Solution.
(a) $|A| = 1(21 - 20) - 2(14 - 4) + 3(10 - 3) = 2$.
(b) The matrix of cofactors of A is

$$\begin{pmatrix} 1 & -10 & 7 \\ 1 & 4 & -3 \\ -1 & 2 & -1 \end{pmatrix}$$

The adjoint is the transpose of this cofactors matrix

$$adj(A) \begin{pmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{pmatrix}$$

(c)

$$A^{-1} = \frac{adj(A)}{|A|} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -5 & 2 & 1 \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} ■$$

Exercise 217
Prove that if A *is symmetric then* $adj(A)$ *is also symmetric.*

Solution.
Suppose that $A^T = A$. Then $(adj(A))^T = (|A|A^{-1})^T = |A|(A^{-1})^T = |A|(A^T)^{-1} = |A|A^{-1} = adj(A)$ ■

Exercise 218
Prove that if A is a nonsingular triangular matrix then A^{-1} is also triangular.

Solution.
Suppose that $A = (a_{ij})$ is a lower triangular invertible matrix. Then $a_{ij} = 0$ if $i < j$. Thus, $C_{ij} = 0$ if $i > j$ since in this case C_{ij} is the determinant of a lower triangular matrix with at least one zero on the diagonal. Hence, $adj(A)$ is lower triangular and $A^{-1} = \frac{adj(A)}{|A|}$ is also lower triangular ■

Exercise 219
Let A be an $n \times n$ matrix.
(a) Show that if A has integer entries and $|A| = 1$ then A^{-1} has integer entries as well.
(b) Let $Ax = b$. Show that if the entries of A and b are integers and $|A| = 1$ then the entries of x are also integers.

Solution.
(a) If A has integer entries then $adj(A)$ has integer entries. If $|A| = 1$ then $A^{-1} = adj(A)$ has integer entries.
(b) Since $|A| = 1$ then A is invertible and $x = A^{-1}b$. By (a), A^{-1} has integer entries. Since b has integer entries then $A^{-1}b$ has integer entries ■

Exercise 220
Show that if $A^k = \mathbf{0}$ for some positive integer k then A is singular.

Solution.
Suppose the contrary. Then we can multiply the equation $A^k = \mathbf{0}$ by A^{-1} $k-1$ times to obtain $A = \mathbf{0}$ which is a contradiction ■

Exercise 221
Use Cramer's Rule to solve

$$\left\{\begin{array}{rcrcrcr} x_1 & & & + & 2x_3 & = & 6 \\ -3x_1 & + & 4x_2 & + & 6x_3 & = & 30 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 8 \end{array}\right.$$

Solution.

$$A = \left(\begin{array}{ccc} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{array}\right), |A| = 44.$$

$$A_1 = \left(\begin{array}{ccc} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{array}\right), |A_1| = -40.$$

$$A_2 = \left(\begin{array}{ccc} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{array}\right), |A_2| = 72.$$

$$A_3 = \begin{pmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{pmatrix}, |A| = 152.$$

Thus, $x_1 = \frac{|A_1|}{|A|} = -\frac{10}{11}, x_2 = \frac{|A_2|}{|A|} = \frac{18}{11}, x_3 = \frac{|A_3|}{|A|} = \frac{38}{11}$ ■

Exercise 222
Use Cramer's Rule to solve

$$\left\{ \begin{array}{ccccccc} 5x_1 & + & x_2 & - & x_3 & = & 4 \\ 9x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & - & x_2 & + & 5x_3 & = & 2 \end{array} \right.$$

Solution.

$$A = \begin{pmatrix} 5 & 1 & -1 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{pmatrix}, |A| = -16.$$

$$A_1 = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{pmatrix}, |A_1| = 12.$$

$$A_2 = \begin{pmatrix} 5 & 4 & -1 \\ 9 & 1 & -1 \\ 1 & 2 & 5 \end{pmatrix}, |A_2| = -166.$$

$$A_3 = \begin{pmatrix} 5 & 1 & 4 \\ 9 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}, |A_3| = -42.$$

Thus, $x_1 = \frac{|A_1|}{|A|} = -\frac{3}{4}, x_2 = \frac{|A_2|}{|A|} = \frac{83}{8}, x_3 = \frac{|A_3|}{|A|} = \frac{21}{8}$ ■

Chapter 4

Exercise 283
Show that the midpoint of $P(x_1, y_1, z_1)$ and $Q(x_1, y_2, z_2)$ is the point

$$M(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}).$$

Solution.
Let $M = (x, y, z)$ be the midpoint of the line segment PQ. Then

$$\begin{array}{ccc} \overrightarrow{OM} & = & \overrightarrow{OP} + \overrightarrow{PM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ} \\ & = & (x_1, y_1, z_1) + \frac{1}{2}(x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ & = & (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}) \end{array}$$

■

Exercise 284
(a) Let $\vec{n} = (a, b, c)$ be a vector orthogonal to a plane P. Suppose $P_0(x_0, y_0, z_0)$ is a point in the plane. Write the equation of the plane.
(b) Find an equation of the plane passing through the point $(3, -1, 7)$ and perpendicular to the vector $\vec{n} = (4, 2, -5)$.

Solution.
(a) Let $P = (x, y, z)$ be an arbitrary point in the plane. Then the vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is contained in the plane. Since $\vec{n}$ is orthogonal to the plane then $< \vec{n}, \overrightarrow{P_0P} >= 0$ and this leads to the equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.
(b) Here we have $a = 4, b = 2, c = -5, x_0 = 3, y_0 = -1$, and $z_0 = 7$. Substituting in the equation obtained in (a) to find $4x + 2y - 5z + 25 = 0$ ■

Exercise 285
Find the equation of the plane through the points $P_1(1, 2, -1), P_2(2, 3, 1)$ and $P_3(3, -1, 2)$.

Solution.
First we find the normal vector as follows

$$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 9\vec{i} + \vec{j} - 5\vec{k}$$

Using Exercise282 (a) we find $9(x-1)+(y-2)-5(z-1) = 0$ or $9x+y-5z = 16$■

Exercise 286
Find the parametric equations of the line passing through a point $P_0(x_0, y_0, z_0)$ and parallel to a vector $\vec{v} = (a, b, c)$.

Solution.
Let $P = (x, y, z)$ be an arbitrary point on the line. Then $\overrightarrow{PP_0} = t\vec{v}$; that is, $(x - x_0, y - y_0, z - z_0) = (ta, tb, tc)$. Hence, the parametric equations of the line are $x = x_0 + ta, y = y_0 + bt, z = z_0 + tc$ ■

Exercise 287
Compute $< \vec{u}, \vec{v} >$ when $\vec{u} = (2, -1, 3)$ and $\vec{v} = (1, 4, -1)$.

Solution.
$< \vec{u}, \vec{v} >= (2)(1) + (-1)(4) + (3)(-1) = -5$ ■

Exercise 288
Compute the angle between the vectors $\vec{u} = (-1, 1, 2)$ and $\vec{v} = (2, 1, -1)$.

Solution.
If θ denotes the angle between the vectors then

$$\cos\theta = \frac{< \vec{u}, \vec{v} >}{||\vec{u}||||\vec{v}||} = -\frac{1}{2}.$$

Since $0 \leq \theta \leq \pi$ then $\theta = \frac{2\pi}{3}$ ■

Exercise 289
For any vectors $\vec{u}$ and $\vec{v}$ we have

$$||\vec{u} \times \vec{v}||^2 = ||\vec{u}||^2||\vec{v}||^2 - < \vec{u}, \vec{v} >^2 .$$

Solution.
Suppose that $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$. Then $\vec{u} \times \vec{v} = (y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1)$. Thus,

$$||\vec{u} \times \vec{v}||^2 = (y_1z_2 - y_2z_1)^2 + (x_2z_1 - x_1z_2)^2 + (x_1y_2 - x_2y_1)^2$$

On the other hand,

$$||\vec{u}||^2||\vec{v}||^2 - < \vec{u}, \vec{v} >^2 = (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2.$$

Expanding one finds that the given equality holds ■

Exercise 290
Show that $||\vec{u} \times \vec{v}||$ is the area of the parallelogram with sides $\vec{u}$ and $\vec{v}$.

Solution.
Let θ be the angle between the vectors $\vec{u}$ and $\vec{v}$. Constructing the parallelogram with sides $\vec{u}$ and $\vec{v}$ one finds that its area is $||\vec{u}||||\vec{v}|| \sin\theta$. By the previous exercise, we have

$$\begin{array}{rcl} ||\vec{u} \times \vec{v}||^2 & = & ||\vec{u}||^2||\vec{v}||^2 - < \vec{u}, \vec{v} >^2 \\ & = & ||\vec{u}||^2||\vec{v}||^2 - ||\vec{u}||^2||\vec{v}||^2 \sin^2\theta \\ & = & ||\vec{u}||^2||\vec{v}||^2(1 - \sin^2\theta \\ & = & ||\vec{u}||^2||\vec{v}||^2 \cos^2\theta \end{array}$$

Since $\theta \in [0, \pi]$ then $\sin\theta \geq 0$ and therefore $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$ ■

Exercise 291
Let $\mathbf{P}$ be the collection of polynomials in the indeterminate x. Let $p(x) = a_0 + a_x + a_2x^2 + \cdots$ and $q(x) = b_0 + b_1x + b_2x^2 + c \ldots$ be two polynomials in $\mathbf{P}$. Define the operations:
(a) Addition: $p(x) + q(x) = a_0 + b_0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$
(b) Multiplication by a scalar: $\alpha p(x) = \alpha a_0 + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots$.
Show that $\mathbf{P}$ is a vector space.

Solution.
Since $\mathbf{P}$ is a subset of the vector space of all functions defined on $\mathbb{R}$ then it suffices to show that $\mathbf{P}$ is a subspace. Indeed, the sum of two polynomilas is again a plynomial and the scalar multiplication by a polynomial is also a plynomial ■

Exercise 292
Define on $\mathbb{R}^2$ the following operations:
(i) $(x, y) + (x', y') = (x + x', y + y,)$;
(ii) $\alpha(x, y) = (\alpha y, \alpha x)$.
Show that $\mathbb{R}^2$ with the above operations is not a vector space.

Solution.
Let $x \neq y$. Then $\alpha(\beta(x,y)) = \alpha(\beta y, \beta x) = (\alpha\beta x, \alpha\beta y) \neq (\alpha\beta)(x,y)$ then $\mathbb{R}^2$ with the above operations is not a vector space ■

Exercise 293
Let $U = \{p(x) \in \mathbf{P} : p(3) = 0\}$. Show that U is a subspace of **P.**

Solution.
Let $p, q \in U$ and $\alpha \in \mathbb{R}$. Then $\alpha p + q$ is a polynomial such that $(\alpha p + q)(3) = \alpha p(3) + q(3) = 0$. That is, $\alpha p + q \in U$. This says that U is a subspace of **P** ■

Exercise 294
Let P_n denote the collection of all polynomials of degree n. Show that P_n is a subspace of **P**.

Solution.
Let $p(x) = a_0 + a_1x + \cdots + a_nx^n, q(x) = b_0 + b_1x + \cdots + b_nx^n$, and $\alpha \in R$. Then $(\alpha p + q)(x)(\alpha a_0 b_0) + (\alpha a_1 b_1)x + \cdots + (\alpha a_n b_n)x^n \in P_n$. Thus, P_n is a subspace of **P** ■

Exercise 295
Show that $< f, g >= \int_a^b f(x)g(x)dx$ is an inner product on the space $C([a,b])$ of continuous functions.

Solution.
(a) $< f, f >= \int_a^b f^2(x)dx \geq 0$. If $f \equiv 0$ then $< f, f >= 0$. Conversely, if $< f, f >= 0$ then by the continuity of f, we must have $f \equiv 0$.
(b) $\alpha < f, g >= \alpha \int_a^b f(x)g(x)dx = \int_a^b (\alpha f)(x)g(x)dx =< \alpha f, g >$.
(c) $< f, g >=< g, f >$.
(d) $< f, g + h >=< f, g > + < f, h >$ ■

Exercise 296
Show that if u and v are two orthogonal vectors of an inner product space, i.e. $< u, v >= 0$, then

$$||u + v||^2 = ||u||^2 + ||v||^2.$$

Solution.
Indeed,
$||u + v||^2 =< u + v, u + v >=< u, u > -2 < u, v > + < v, v >= ||u||^2 + ||v||^2$
since $< u, v >= 0$ by orthogonality ■

Exercise 297
(a) Prove that a line through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ under the standard operations.
(b) Prove that a line not through the origin in $\mathbb{R}^3$ in not a subspace of $\mathbb{R}^3$.

Solution.
(a) Let (L) be a line through the origin with direction vector (a, b, c). Then the parametric equations of (L) are given by

$$x(t) = ta, y(t) = tb, z(t) = tc.$$

Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in (L)$ and $\alpha \in \mathbb{R}$. Then

$$\alpha x_1 + x_2 = (\alpha t_1 + t_2)a = t_3 a$$

where $t_3 = \alpha t_1 + t_2$. Similarly, $\alpha y_1 + y_2 = t_3 b$ and $\alpha z_1 + z_2 = t_3 c$. This shows that $\alpha(x_1, y_1, z_1) + (x_2, y_2, z_2) \in (L)$.
(b) Let (L) be the line through the point $(-1, 4, 2)$ and with direction vector $(1, 2, 3)$. Then the parametric equations of (L) are given by $x(t) = -1+t, y(t) = 4+2t, z(t) = 2+3t$. It is clear that $(-1, 4, 2) \in (L), (0, 6, 5) \in (L)$ but $(-1, 4, 2)+(0, 6, 5) = (-1, 11, 7) \notin (L)$ ■

Exercise 298
Show that the set $S = \{(x, y) : x \leq 0\}$ *is not a vector space of* $\mathbb{R}^2$ *under the usual operations of* $\mathbb{R}^2$.

Solution.
$(-1, 0) \in S$ but $-2(-1, 0) = (2, 0) \notin S$ so S is not a vector space ■

Exercise 299
Show that the collection $C([a, b])$ *of all continuous functions on* $[a, b]$ *with the operations:*

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x) \end{aligned}$$

is a vector space.

Solution.
Since for any continuous functions f and g and any scalar α the function $\alpha f + g$ is continuous then $C([a, b])$ is a subspace of $F([a, b])$ and hence a vector space ■

Exercise 300
Let $S = \{(a, b, a+b) : a, b \in \mathbb{R}\}$. *Show that* S *is a subspace of* $\mathbb{R}^3$ *under the usual operations.*

Solution.
Indeed, $\alpha(a, b, a+b) + (a', b', a'+b') = (\alpha(a+a'), \alpha(b+b'), \alpha(a+b+a'+b')) \in S$
■

Exercise 301
Let V *be a vector space. Show that if* $u, v, w \in V$ *are such that* $u + v = u + w$ *then* $v = w$.

Solution.
Using the properties of vector spaces we have $v = v + 0 = v + (u + (-u)) = (v + u) + (-u) = (w + u) + (-u) = w + (u + (-u)) = w + 0 = w$ ■

Exercise 302
Let H and K be subspaces of a vector space V.
(a) The **intersection** *of H and K, denoted by $H \cap K$, is the subset of V that consists of elements that belong to both H and K. Show that $H \cap V$ is a subspace of V.*
(b) The **union** *of H and K, denoted by $H \cup K$, is the susbet of V that consists of all elements that belong to either H or K. Give, an example of two subspaces of V such that $H \cup K$ is not a subspace.*
(c) Show that if $H \subset K$ or $K \subset H$ then $H \cup K$ is a subspace of V.

Solution.
(a) Let $u, v \in H \cap K$ and $\alpha \in R$. Then $u, v \in H$ and $u, v \in K$. Since H and K are subspaces then $\alpha u + v \in H$ and $\alpha u + v \in K$ that is $\alpha u + v \in H \cap K$. This shows that $H \cap K$ is a subspace.
(b) One can easily check that $H = \{(x, 0) : x \in \mathbb{R}\}$ and $K = \{(0, y) : y \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^2$. The vector $(1, 0)$ belongs to H and the vector $(0, 1)$ belongs to K. But $(1, 0)+(0, 1) = (1, 1) \notin H \cup K$. It follows that $H \cup K$ is not a subspace of $\mathbb{R}^2$.
(c) If $H \subset K$ then $H \cup K = K$, a subspace of V. Similarly, if $K \subset H$ then $H \cup K = H$, again a subspace of V ■

Exercise 303
Let u and v be vectors in an inner product vector space. Show that

$$||u + v||^2 + ||u - v||^2 = 2(||u||^2 + ||v||^2).$$

Solution.
Recall that $||u|| = \sqrt{< u, u >}$. Then

$$\begin{aligned} ||u + v||^2 + ||u - v||^2 &= < u + v, u + v > + < u - v, u - v > \\ &= < u, u > +2 < u, v > + < v, v > \\ &+ < u, u > -2 < u, v > + < v, v > \\ &= 2(< u, u > + < v, v >) = 2(||u||^2 + ||v||^2) \blacksquare \end{aligned}$$

Exercise 304
Let u and v be vectors in an inner product vector space. Show that

$$||u + v||^2 - ||u - v||^2 = 4 < u, v > .$$

Solution.
Indeed,

$$\begin{aligned} ||u + v||^2 - ||u - v||^2 &= < u + v, u + v > - < u - v, u - v > \\ &= < u, u > +2 < u, v > + < v, v > \\ &- < u, u > +2 < u, v > - < v, v > \\ &= 4 < u, v > \blacksquare \end{aligned}$$

Exercise 305

(a) Use Cauchy -Schwarz's inequality to show that if $u = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ *and* $v = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ *then*

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \le (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2).$$

(b) Use (a) to show

$$n^2 \le (a_1 + a_2 + \cdots + a_n)(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n})$$

where $a_i > 0$ *for* $1 \le i \le n$.

Solution.

(a) The standard inner product in $\mathbb{R}^n$ is given by $< u, v >= x_1y_1 + x_2y_2 + \cdots + x_ny_n$. By Cauchy - Schwarz inequality we have that $< u, v >^2 \le < u, u >< v, v >$ that is

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \le (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2).$$

(a) Let $x_i = \sqrt{a_i}$ and $y_i = \frac{1}{\sqrt{a_i}}$ in (a) ■

Exercise 306

Let $C([0, 1])$ *be the vector space of continuous functions on* $[0, 1]$. *Define*

$$< f, g >= \int_0^1 f(x)g(x)dx.$$

(a) Show that $< ., . >$ *is an inner product on* $C([0, 1])$.
(b) Show that

$$\left[\int_0^1 f(x)g(x)dx\right]^2 \le \left[\int_0^1 f(x)^2dx\right]\left[\int_0^1 g(x)^2dx\right].$$

This inequality is known as **Holder's inequality**.
(c) Show that

$$\left[\int_0^1 (f(x) + g(x))^2dx\right]^{\frac{1}{2}} \le \left[\int_0^1 f(x)^2dx\right]^{\frac{1}{2}} + \left[\int_0^1 g(x)^2dx\right]^{\frac{1}{2}}.$$

This inequality is known as **Minkowski's inequality**.

Solution.

(a) See Exercise 295
(b) This is just the Cauchy-Schwarz's inequality.
(c) The inequality is definitely valid if $\int_0^1 (f(x) + g(x))^2dx = 0$. So we may assume that this integral is nonzero. In this case, we have

$$\begin{aligned}
\int_0^1 (f(x) + g(x))^2dx &= \int_0^1 (f(x) + g(x))(f(x) + g(x))dx \\
&= \int_0^1 f(x)(f(x) + g(x))dx + \int_0^1 g(x)(f(x) + g(x))dx \\
&\le \left[\int_0^1 f^2(x)dx\right]^{\frac{1}{2}} \left[\int_0^1 (f(x) + g(x))^2dx\right]^{\frac{1}{2}} \\
&+ \left[\int_0^1 g^2(x)dx\right]^{\frac{1}{2}} \left[\int_0^1 (f(x) + g(x))^2dx\right]^{\frac{1}{2}}
\end{aligned}$$

Now divide both sides of this inequality by $\left[\int_0^1 (f(x)+g(x))^2 dx\right]^{\frac{1}{2}}$ to obtain the desired inequality ■

Exercise 307
Let $W = span\{v_1, v_2, \cdots, v_n\}$, where $v_1, v_2, \cdots, v_n$ are vectors in V. Show that any subspace U of V containing the vectors $v_1, v_2, \cdots, v_n$ must contain W, i.e. $W \subset U$. That is, W is the smallest subspace of V containing $v_1, v_2, \cdots, v_n$.

Solution.
Let U be a subspace of V containing the vectors $v_1, v_2, \cdots, v_n$. Let $x \in W$. Then $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ for some scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$. Since U is a subspace then $x \in U$. This gives $x \in U$ and consequently $W \subset U$ ■

Exercise 308
Express the vector $\vec{u} = (-9, -7, -15)$ as a linear combination of the vectors $\vec{v_1} = (2, 1, 4), \vec{v_2} = (1, -1, 3), \vec{v_3} = (3, 2, 5)$.

Solution.
The equation $\vec{u} = \alpha\vec{v_1} + \beta\vec{v_2} + \gamma\vec{v_3}$ gives the system

$$\left\{\begin{array}{ccccccc} 2\alpha & + & \beta & + & 3\gamma & = & -9 \\ \alpha & - & \beta & + & 2\gamma & = & -7 \\ 4\alpha & + & 3\beta & + & 5\gamma & = & -15 \end{array}\right.$$

Solving this system (details omitted) we find $\alpha = -2, \beta = 1$ and $\gamma = -2$ ■

Exercise 309
(a) Show that the vectors $\vec{v_1} = (2, 2, 2), \vec{v_2} = (0, 0, 3)$, and $\vec{v_3} = (0, 1, 1)$ span $\mathbb{R}^3$.
(b) Show that the vectors $\vec{v_1} = (2, -1, 3), \vec{v_2} = (4, 1, 2)$, and $\vec{v_3} = (8, -1, 8)$ do not span $\mathbb{R}^3$.

Solution.
(a) Indeed, this follows because the coefficient matrix

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

of the system $Ax = b$ is invertible for all $b \in \mathbb{R}^3$ ($|A| = -6$)
(b) This follows from the fact that the coefficient matrix with rows the vectors $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$ is singular ■

Exercise 310
Show that

$$M_{22} = span\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$$

Solution.
Indeed, every $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written as

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}\blacksquare$$

Exercise 311
(a) Show that the vectors $\vec{v_1} = (2, -1, 0, 3), \vec{v_2} = (1, 2, 5, -1)$, *and* $\vec{v_3} = (7, -1, 5, 8)$ *are linearly dependent.*
(b) Show that the vectors $\vec{v_1} = (4, -1, 2)$ *and* $\vec{v_2} = (-4, 10, 2)$ *are linearly independent.*

Solution.
(a) Suppose that $\alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \alpha_3\vec{v_3} = \vec{0}$. This leads to the system

$$\begin{cases} 2\alpha_1 & + & \alpha_2 & + & 7\alpha_3 & = & 0 \\ -\alpha_1 & + & 2\alpha_2 & - & \alpha_3 & = & 0 \\ & & 5\alpha_2 & + & 5\alpha_3 & = & 0 \\ 3\alpha_1 & - & \alpha_2 & + & 8\alpha_3 & = & 0 \end{cases}$$

The augmented matrix of this system is

$$\begin{pmatrix} 2 & -1 & 0 & 3 & 0 \\ 1 & 2 & 5 & -1 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{pmatrix}$$

The reduction of this matrix to row-echelon form is carried out as follows.

Step 1: $r_1 \leftarrow r_1 - 2r_2$ and $r_3 \leftarrow r_3 - 7r_2$

$$\begin{pmatrix} 0 & -5 & -10 & 6 & 0 \\ 1 & 2 & 5 & -1 & 0 \\ 0 & -15 & -30 & 15 & 0 \end{pmatrix}$$

Step 2: $r_1 \leftrightarrow r_2$

$$\begin{pmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & -5 & -10 & 6 & 0 \\ 0 & -15 & -30 & 15 & 0 \end{pmatrix}$$

Step 2: $r_3 \leftarrow r_3 - 3r_2$

$$\begin{pmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & -5 & -10 & 6 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{pmatrix}$$

The system has a nontrivial solution so that $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly dependent.
(b) Suppose that $\alpha(4, -1, 2) + \beta(-4, 10, 2) = (0, 0, 0)$ this leads to the system

$$\begin{cases} 4\alpha_1 & - & 4\alpha_2 & = & 0 \\ -\alpha_1 & + & 10\alpha_2 & = & 0 \\ 2\alpha_2 & + & 2\alpha_2 & = & 0 \end{cases}$$

This system has only the trivial solution so that the given vectors are linearly independent ■

Exercise 312
Show that the $\{u, v\}$ is linearly dependent if and only if one is a scalar multiple of the other.

Solution.
Suppose that $\{u, v\}$ is linearly dependent. Then there exist scalars α and β not both zero such that $\alpha u + \beta v = 0$. If $\alpha \neq 0$ then $u = -\frac{\beta}{\alpha}v$, i.e. u is a scalar multiple of v. A similar argument if $\beta \neq 0$.
Conversely, suppose that $u = \lambda v$ then $1u + (-\lambda)v = 0$. This shows that $\{u, v\}$ is linearly dependent ■

Exercise 313
Let V be the vector of all real-valued functions with domain $\mathbb{R}$. If f, g, h are twice differentiable functions then we define $w(x)$ by the determinant

$$w(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

We call $w(x)$ the **Wronskian** *of f, g, and h. Prove that f, g, and h are linearly independent if and only if $w(x) \neq 0$.*

Solution.
Suppose that $\alpha f(x) + \beta g(x) + \gamma h(x) = 0$ for all $x \in \mathbb{R}$. Then this leads to the system

$$\begin{pmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Thus $\{f(x), g(x), h(x)\}$ is linearly independent if and only if the coefficient matrix of the above system is invertible and this is equivalent to $w(x) \neq 0$ ■

Exercise 314
Use the Wronskian to show that the functions e^x, xe^x, x^2e^x are linearly independent.

Solution.
Indeed,

$$w(x) = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix} = 2e^x \neq 0 ■$$

Exercise 315
Show that $\{1 + x, 3x + x^2, 2 + x - x^2\}$ is linearly independent in P_2.

Solution.
Suppose that $\alpha(1+x)+\beta(3x+x^2)+\gamma(2+x-x^2)=0$ for all $x \in \mathbb{R}$. That is, $(\beta-2\gamma)x^2+(\alpha+3\beta+\gamma)x+\alpha+2\gamma=0$ for all $x \in \mathbb{R}$. This leads to the system

$$\left\{\begin{array}{ccccccc} & & \beta & - & 2\gamma & = & 0\\ \alpha & + & 3\beta & + & \gamma & = & 0\\ \alpha & & & + & 2\gamma & = & 0\end{array}\right.$$

Solving this system by Gauss elimination (details omitted) one finds $\alpha=\beta=\gamma=0$. That is $\{1+x, 3x+x^2, 2+x-x^2\}$ is linearly independent ■

Exercise 316
Show that $\{1+x, 3x+x^2, 2+x-x^2\}$ *is a basis for* P_2.

Solution.
Since $dim(P_2)=3$ and $\{1+x, 3x+x^2, 2+x-x^2\}$ is linearly independent by the previous exercise then $\{1+x, 3x+x^2, 2+x-x^2\}$ is a basis by Theorem 58 ■

Exercise 317
Find a basis of P_3 *containing the linearly independent set* $\{1+x, 1+x^2\}$.

Solution.
Note first that any polynomial in the span of $\{1+x, 1+x^2\}$ is of degree at most 2. Thus, $x^3 \notin span\{1+x, 1+x^2\}$ so that the set $\{1+x, 1+x^2, x^3\}$ is linearly independent. Since 1 is not in the span $\{1, 1+x, 1+x^2, x^3\}$ is linearly independent. Since $dim(P_3)=4$ then $\{1, 1+x, 1+x^2, x^3\}$ is a basis of P_3 ■

Exercise 318
Let $\vec{v_1}=(1,2,1), \vec{v_2}=(2,9,0), \vec{v_3}=(3,3,4)$. *Show that* $S=\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ *is a basis for* $\mathbb{R}^3$.

Solution.
Since $dim(R^3)=3$ then by Theorem 58 it suffices to show that the given vectors are linearly independent, that is the matrix A whose rows consists of the given vectors is invertible which is the case since $|A|=-1\neq 0$ ■

Exercise 319
Let S *be a susbet of* $\mathbb{R}^n$ *with* $n+1$ *vectors. Is* S *linearly independent or linearly dependent?*

Solution.
By Theorem 53 S is linearly dependent ■

Exercise 320
Find a basis for the vector space M_{22} *of* 2×2 *matrices.*

Solution.
We have already shown that

$$M_{22} = span\left\{M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, M_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

Now, if $\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = \mathbf{0}$ then

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and this shows that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Hence, $\{M_1, M_2, M_3, M_4\}$ is a basis for M_{22} ■

Exercise 321
(a) Let U, W be subspaces of a vector space V. Show that the set $U + W = \{u + w : u \in U \text{ and } w \in W\}$ is a subspace of V.
(b) Let M_{22} be the collection of 2×2 matrices. Let U be the collection of matrices in M_{22} whose second row is zero, and W be the collection of matrices in M_{22} whose second column is zero. Find $U + W$.

Solution.
(a) Let $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$ be two vectors in $U + W$ and $\alpha \in \mathbb{R}$. Then $\alpha v_1 + v_2 = (\alpha u_1 + u_2) + (\alpha w_1 + w_2) \in U + W$ since $\alpha u_1 + u_2 \in U$ and $\alpha w_1 + w_2 \in W$. Hence, $U + W$ is a subspace of V.
(b) $U + W = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R}\right\}$ ■

Exercise 322
Let $S = \{(1,1)^T, (2,3)^T\}$ and $S' = \{(1,2)^T, (0,1)^T\}$ be two bases of $\mathbb{R}^2$. Let $\vec{u} = (1,5)^T$ and $\vec{v} = (5,4)^T$.

(a) Find the coordinate vectors of $\vec{u}$ and $\vec{v}$ with respect to the basis S.
(b) What is the transition matrix P from S to S'?
(c) Find the coordinate vectors of $\vec{u}$ and $\vec{v}$ with respect to S' using P.
(d) Find the coordinate vectors of $\vec{u}$ and $\vec{v}$ with respect to S' directly.
(e) What is the transition matrix Q from S' to S?
(f) Find the coordinate vectors of $\vec{u}$ and $\vec{v}$ with respect to S using Q.

Solution.
(a) Since $\vec{u} = -7(1,1)^T + 4(2,3)^T$ then $[\vec{u}]_S = \begin{pmatrix} -7 \\ 4 \end{pmatrix}$. Similarly, $[\vec{v}]_S = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$.
(b) Since

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, the transition matrix from S to S' is

$$P = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

(c)

$$[\vec{u}]_{S'} = P[\vec{u}]_S = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -7 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and

$$[\vec{v}]_{S'} = P[\vec{v}]_S = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$$

(d) The equation

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

implies $\alpha = 5$ and $\beta = 3$ so

$$[\vec{u}]_{S'} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

implies $\alpha = 4$ and $\beta = -6$ so that

$$[\vec{v}]_{S'} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$$

(e)

$$Q = P^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

(f)

$$[\vec{u}]_S = Q[\vec{u}]_{S'} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

and

$$[\vec{v}]_S = Q[\vec{v}]_{S'} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ -6 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}\blacksquare$$

Exercise 323

Suppose that $S' = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ *is a basis of* $\mathbb{R}^3$, *where* $\vec{u_1} = (1,0,1), \vec{u_2} = (1,1,0), \vec{u_3} = (0,0,1)$. *Let* $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$. *Suppose that the transition matrix from* S *to* S' *is*

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

Determine S.

Solution.
We have

$$\vec{v_1} = \vec{u_1} + 2\vec{u_2} - \vec{u_3} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

$$\vec{v_2} = \vec{u_1} + \vec{u_2} - \vec{u_3} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v_3} = 2\vec{u_1} + \vec{u_2} + \vec{u_3} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \blacksquare$$

Exercise 324
Show that if A is not a square matrix then either the row vectors of A or the column vectors of A are linearly dependent.

Solution.
Let A be an $n \times m$ matrix with $n \neq m$. Suppose that the row vectors of A are linearly independent. Then $rank(A) = n$. If the column vectors of A are linearly independent then $rank(A^T) = m$. But $rank(A) = rank(A^T)$ and this implies $n = m$ a contradiction. Thus, the column vectors of A are linearly dependent $\blacksquare$

Exercise 325
Prove that the row vectors of an $n \times n$ invertible matrix A form a basis for $\mathbb{R}^n$.

Solution.
Suppose that A is an $n \times n$ invertible matrix. Then $rank(A) = n$. By Theorem 71 the rows of A are linearly independent. Since $dim(\mathbb{R}^n) = n$ then by Theorem 55 the rows of A form a basis for $\mathbb{R}^n$ $\blacksquare$

Exercise 326
Compute the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

and find a basis for the row space of A.

Solution.
The reduction of this matrix to row-echelon form is as follows.

Step 1: $r_2 \leftarrow r_2 - 3r_1$ and $r_3 \leftarrow r_3 - r_1$

$$\begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

Step 2: $r_3 \leftarrow r_3 - r_2$

$$\begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, $rank(A) = 2$ and $\{(1,2,2,-1),(0,0,1,-3)\}$ is a basis for the row space of A ■

Exercise 327

Let U and W be subspaces of a vector space V. We say that V is the **direct sum** *of U and W if and only if $V = U + W$ and $U \cap W = \{0\}$. We write $V = U \oplus W$. Show that V is the direct sum of U and W if and only if every vector v in V can be written uniquely in the form $v = u + w$.*

Solution.

Suppose that $V = U \oplus W$. Let $v \in V$ such that $v = u_1 + w_1$ and $v = u_2 + w_2$. Then $u_1 - u_2 = w_2 - w_1$. This says that $u_1 - u_2 \in U \cap W = \{0\}$. That is, $u_1 - u_2 = 0$ or $u_1 = u_2$. Hence, $w_1 = w_2$.

Conversely, suppose that every vector in $v \in V$ can be written uniquely as the sum of a vector in $u \in U$ and a vector in $w \in W$. Suppose that $v \in U \cap W$. Then $v \in U$ and $v \in W$. But $v = v + 0$ so that $v = 0$ ■

Exercise 328

Let V be an inner product space and W is a subspace of V. Let $W^\perp = \{u \in V :< u, w >= 0, \forall w \in W\}$.

(a) Show that $W^\perp$ is a subspace of V.

(b) Show that $V = W \oplus W^\perp$.

Solution.

(a) Let $u_1, u_2 \in W^\perp$ and $\alpha \in \mathbb{R}$. Then $< \alpha u_1 + u_2, w >= \alpha < u_1, w > + < u_2, w >= 0$ for all $w \in W$. Hence, $\alpha u_1 + u_2 \in W$ and this shows that $W^\perp$ is a subspace of V.

(b) We first show that $W \cap W^\perp = \{0\}$. Indeed, if $0 \neq w \in W \cap W^\perp$ then $w \in W$ and $w \in W^\perp$ and this implies that $< w, w >= 0$ which contradicts the fact that $w \neq 0$. Hence, we must have $W \cap W^\perp = \{0\}$. Next we show that $V = W + W^\perp$. Let $\{v_1, v_2, \cdots, v_n\}$ be an orthonormal basis of W. Then by Exercise 266 (d), every vector $v \in V$ can be written as a sum of the form $v = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^\perp$ ■

Exercise 329

Let U and W be subspaces of a vector space V. Show that $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.

Solution.

If $U \cap W = \{0\}$ then we are done. So assume that $U \cap W \neq \{0\}$. Then $U \cap W$ has a basis, say $\{z_1, z_2, \cdots, z_k\}$. Extend this to a basis $\{z_1, z_2, \cdots, z_k, u_{k+1}, \cdots, u_m\}$ of U and to a basis $\{z_1, z_2, \cdots, z_k, w_{k+1}, \cdots, w_n\}$ of W. Let $v \in U + W$. Then $v = u + w = (\alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_k z_k + \alpha_{k+1} u_{k+1} + \cdots + \alpha_m u_m) + (\beta_1 z_1 +$

$\beta_2 z_2 + \cdots + \beta_k z_k + \beta_{k+1} w_{k+1} + \cdots + \beta_n w_n) = \gamma_1 z_1 + \gamma_2 z_2 + \cdots + \gamma_k z_k + \gamma_{k+1} u_{k+1} + \cdots + \gamma_m u_m + \delta_{k+1} w_{k+1} + \cdots + \delta_n w_n$. This says that $U + W = span\{z_1, z_2, \cdots, z_k, u_{k+1}, \cdots, u_m, w_{k+1}, \cdots, w_n\}$. Now, suppose that $\gamma_1 z_1 + \gamma_2 z_2 + \cdots + \gamma_k z_k + \gamma_{k+1} u_{k+1} + \cdots + \gamma_m u_m + \delta_{k+1} w_{k+1} + \cdots + \delta_n w_n = 0$. Then $\delta_{k+1} w_{k+1} + \cdots + \delta_n w_m = -(\gamma_1 z_1 + \cdots + \gamma_k z_k + \gamma_{k+1} u_{k+1} + \cdots + \gamma_m u_m \in U$. Hence, $\delta_{k+1} w_{k+1} + \cdots + \delta_n w_n \in U \cap W$ and this implies that $\delta_{k+1} w_{k+1} + \cdots + \delta_n w_n = \sigma_1 z_1 + \sigma_2 z_2 + \cdots + \sigma_k z_k$ or $\delta_{k+1} w_{k+1} + \cdots + \delta_n w_n - \sigma_1 z_1 - \sigma_2 z_2 - \cdots - \sigma_k z_k = 0$. Since $\{z_1, z_2, \cdots, z_k, w_{k+1}, \cdots, w_n\}$ is linearly independent then $\delta_{k+1} = \cdots = \delta_n = 0$. This implies that $\gamma_1 z_1 + \gamma_2 z_2 + \cdots + \gamma_k z_k + \gamma_{k+1} u_{k+1} + \cdots + \gamma_m u_m = 0$. Because $\{z_1, z_2, \cdots, z_k, u_{k+1}, \cdots, u_m\}$ is linearly independent then we must have $\gamma_1 = \cdots = \gamma_k = 0$. Thus, we have shown that $\{z_1, z_2, \cdots, z_k, u_{k+1}, \cdots, u_m, w_{k+1}, \cdots, w_n\}$ is a basis of $U + W$. It follows that $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$ ■

Exercise 330
Show that $\cos 2x \in span\{\cos^2 x, \sin^2 x\}$.

Solution.
This follows from the trigonometric identity $\cos(2x) = \cos^2 x - \sin^2 x$ ■

Exercise 331
If X *and* Y *are subsets of a vector space* V *and if* $X \subset Y$, *show that* $spanX \subset spanY$.

Solution.
First of all note that $X \subset Y \subset spanY$. Let $u \in spanX$. Then $u = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ where $x_1, x_2, \cdots, x_n \in X$. But for each i, $\alpha_i x_i \in spanY$. Since $spanY$ is a subspace then $u \in spanY$ ■

Exercise 332
Show that the vector space $F(\mathbb{R})$ *of all real-valued functions defined on* $\mathbb{R}$ *is an infinite-dimensional vector space.*

Solution.
We use the proof by contradiction. Suppose that $dim(F(\mathbb{R})) = n$. Then any S of $m > n$ functions must be linearly dependent. But $S = \{1, x, \cdots, x^n\} \subset F(\mathbb{R})$ is linearly independent set with $n + 1$ functions, a contradiction. Thus, $F(\mathbb{R})$ must be an infinite-dimensional vector space ■

Exercise 333
Show that $\{\sin x, \cos x\}$ *is linearly independent in the vector space* $F([0, 2\pi])$.

Solution.
Suppose that α and β are scalars such that $\alpha \sin x + \beta \cos x = 0$ for all $x \in [0, 2\pi]$. In particular, taking $x = 0$ yields $\beta = 0$. By taking, $x = \frac{\pi}{2}$ we find $\alpha = 0$ ■

Exercise 334
Find a basis for the vector space V *of all* 2×2 *symmetric matrices.*

Solution.
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$ then we must have $b = c$. Hence

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

That is,

$$V = span\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

It remains to show that the set $S = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ is linearly independent. Indeed, if

$$\alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then $\alpha = \beta = \gamma = 0$. Thus, S is a basis and $dim(V) = 3$ ■

Exercise 335
Find the dimension of the vector space M_{mn} of all $m \times n$ matrices.

Solution.
For $1 \le i \le m$ and $1 \le j \le n$ let A_{ij} be the matrix whose (i,j)th entry is 1 and 0 otherwise. Let $S = \{A_{ij} : 1 \le i \le m \text{ and } 1 \le j \le n\}$. We will show that S is a basis of M_{mn}. Indeed, if $A = (a_{ij}) \in M_{mn}$ then $A = \sum_{i=1}^m \sum_{j=1}^n a_{ij}A_{ij}$. That is, $M_{mn} = spanS$. To show that S is linearly independent, we suppose that $\sum_{i=1}^m \sum_{j=1}^n \alpha_{ij}A_{ij} = \mathbf{0}$. Then the left-hand side is an $m \times n$ matrix with entries α_{ij}. By the equality of matrices we conclude that $\alpha_{ij} = 0$ that is S is linearly indepndent and hence a basis for M_{mn}. Also, $dim(M_{mn}) = mn$ ■

Exercise 336
Let A be an $n \times n$ matrix. Show that there exist scalars $a_0, a_1, a_2, \cdots, a_{n^2}$ not all 0 such that

$$a_0I_n + a_1A + a_2A^2 + \cdots + a_{n^2}A^{n^2} = \mathbf{0}. \tag{7.1}$$

Solution.
Since $dim(M_{nn}) = n^2$ then the $n^2 + 1$ matrices $I_n, A, A^2, \cdots, A^{n^2}$ are linearly dependent. So there exist scalars $a_0, a_1, \cdots, a_{n^2}$ not all 0 such that (7.1) holds ■

Exercise 337
Show that if A is an $n \times n$ invertible skew-symmetric matrix then n must be even.

Solution.
Since A is skew-symmetric then $A^T = -A$. Taking the determinant of both sides to obtain $|A| = |A^T| = (-1)^n|A|$. Since A is invertible then $|A| \neq 0$ and consequently $(-1)^n = 1$. That is, n is even ■

Chapter 5

Exercise 363
Show that $\lambda = -3$ is an eigenvalue of the matrix

$$A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$$

and then find the corresponding eigenspace V_{-3}.

Solution.
The characteristic equation of the matrix A is

$$\begin{pmatrix} \lambda - 5 & -8 & -16 \\ -4 & \lambda - 1 & -8 \\ -4 & -4 & \lambda + 11 \end{pmatrix}$$

Expanding the determinant and simplifying we obtain

$$(\lambda + 3)^2(\lambda - 1) = 0.$$

Thus, $\lambda = -3$ is an eigenvalue of A.
A vector $x = (x_1, x_2, x_3)^T$ is an eigenvector corresponding to an eigenvalue λ if and only if x is a solution to the homogeneous system

$$\left\{ \begin{array}{ccccccc} (\lambda-5)x_1 & - & 8x_2 & - & 16x_3 & = & 0 \\ -4x_1 & + & (\lambda-1)x_2 & - & 8x_3 & = & 0 \\ 4x_1 & + & 4x_2 & + & (\lambda+11)x_3 & = & 0 \end{array} \right. \tag{7.2}$$

If $\lambda = -3$, then (7.2) becomes

$$\left\{ \begin{array}{ccccccc} -8x_1 & - & 8x_2 & - & 16x_3 & = & 0 \\ -4x_1 & - & 4x_2 & - & 8x_3 & = & 0 \\ 4x_1 & + & 4x_2 & + & 8x_3 & = & 0 \end{array} \right. \tag{7.3}$$

Solving this system yields

$$x_1 = -s - 2t, x_2 = s, x_3 = t$$

The eigenspace corresponding to $\lambda = -3$ is

$$V_{-3} = \left\{ \begin{pmatrix} -2t - s \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\} = span \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} ■$$

Exercise 364
Find the eigenspaces of the matrix

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

Solution.
The characteristic equation of the matrix A is

$$\begin{pmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{pmatrix}$$

Expanding the determinant and simplifying we obtain

$$(\lambda - 3)^2(\lambda + 1) = 0.$$

Thus, $\lambda = 3$ and $\lambda = -1$ are the eigenvalues of A.
A vector $x = (x_1, x_2)^T$ is an eigenvector corresponding to an eigenvalue λ if and only if x is a solution to the homogeneous system

$$\left\{ \begin{array}{ccccc} (\lambda - 3)x_1 & & & = & 0 \\ -8x_1 & + & (\lambda + 1)x_2 & = & 0 \end{array} \right. \tag{7.4}$$

If $\lambda = 3$, then (7.4) becomes

$$\left\{ \begin{array}{ccccc} -8x_1 & + & 4x_2 & = & 0 \end{array} \right. \tag{7.5}$$

Solving this system yields

$$x_1 = \frac{1}{2}s, x_2 = s$$

The eigenspace corresponding to $\lambda = 3$ is

$$V_3 = \left\{ \begin{pmatrix} \frac{1}{2}s \\ s \end{pmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \right\}$$

If $\lambda = -1$, then (7.4) becomes

$$\left\{ \begin{array}{ccc} -4x_1 & = & 0 \\ -8x_1 & = & 0 \end{array} \right. \tag{7.6}$$

Solving this system yields

$$x_1 = 0, x_2 = s$$

The eigenspace corresponding to $\lambda = -1$ is

$$V_{-1} = \left\{ \begin{pmatrix} 0 \\ s \end{pmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \blacksquare$$

Exercise 365
Find the characteristic polynomial, eigenvalues, and eigenspaces of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$$

Solution.
The characteristic equation of the matrix A is

$$\begin{pmatrix} \lambda - 2 & -1 & -1 \\ -2 & \lambda - 1 & 2 \\ 1 & 0 & \lambda + 2 \end{pmatrix}$$

Expanding the determinant and simplifying we obtain

$$(\lambda - 3)^2(\lambda + 1)^2 = 0.$$

Thus, $\lambda = 3$ and $\lambda = -1$ are the eigenvalues of A.
A vector $x = (x_1, x_2, x_3)^T$ is an eigenvector corresponding to an eigenvalue λ if and only if x is a solution to the homogeneous system

$$\left\{ \begin{array}{rcrcrcl} (\lambda - 2)x_1 & - & x_2 & - & x_3 & = & 0 \\ -2x_1 & + & (\lambda - 1)x_2 & + & 2x_3 & = & 0 \\ x_1 & & & + & (\lambda + 2)x_3 & = & 0 \end{array} \right. \qquad (7.7)$$

If $\lambda = 3$, then (7.7) becomes

$$\left\{ \begin{array}{rcrcrcl} x_1 & - & x_2 & - & x_3 & = & 0 \\ -2x_1 & + & 2x_2 & + & 2x_3 & = & 0 \\ x_1 & & & + & 5x_3 & = & 0 \end{array} \right. \qquad (7.8)$$

Solving this system yields

$$x_1 = -5s, x_2 = -6s, x_3 = s$$

The eigenspace corresponding to $\lambda = 3$ is

$$V_3 = \left\{ \begin{pmatrix} -5s \\ -6s \\ s \end{pmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix} \right\}$$

If $\lambda = -1$, then (7.7) becomes

$$\left\{ \begin{array}{rcrcrcl} -3x_1 & - & x_2 & - & x_3 & = & 0 \\ -2x_1 & - & 2x_2 & + & 2x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 0 \end{array} \right. \qquad (7.9)$$

Solving this system yields

$$x_1 = -s, x_2 = 2s, x_3 = s$$

The eigenspace corresponding to $\lambda = -1$ is

$$V_{-1} = \left\{ \begin{pmatrix} -s \\ 2s \\ s \end{pmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\} \blacksquare$$

Exercise 366
Find the bases of the eigenspaces of the matrix

$$A = \begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$$

Solution.
The characteristic equation of the matrix A is

$$\begin{pmatrix} \lambda+2 & & -1 \\ 6 & \lambda+2 & 0 \\ -19 & -5 & \lambda+4 \end{pmatrix}$$

Expanding the determinant and simplifying we obtain

$$(\lambda+8)(\lambda^2+1) = 0.$$

Thus, $\lambda = -8$ is the only eigenvalue of A.
A vector $x = (x_1, x_2, x_3)^T$ is an eigenvector corresponding to an eigenvalue λ if and only if x is a solution to the homogeneous system

$$\left\{ \begin{array}{rcrcrcl} (\lambda+2)x_1 & & & - & x_3 & = & 0 \\ 6x_1 & + & (\lambda+2)x_2 & & & = & 0 \\ -19x_1 & - & 5x_2 & + & (\lambda+4)x_3 & = & 0 \end{array} \right. \qquad (7.10)$$

If $\lambda = -8$, then (7.10) becomes

$$\left\{ \begin{array}{rcrcrcl} -6x_1 & & & - & x_3 & = & 0 \\ 6x_1 & - & 6x_2 & & & = & 0 \\ -19x_1 & - & 5x_2 & - & 4x_3 & = & 0 \end{array} \right. \qquad (7.11)$$

Solving this system yields

$$x_1 = -\frac{1}{6}s, x_2 = -\frac{1}{6}s, x_3 = s$$

The eigenspace corresponding to $\lambda = -8$ is

$$V_{-8} = \left\{ \begin{pmatrix} -\frac{1}{6}s \\ -\frac{1}{6}s \\ s \end{pmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{pmatrix} -\frac{1}{6} \\ -\frac{1}{6} \\ 1 \end{pmatrix} \right\} \blacksquare$$

Exercise 367
Show that if λ is a nonzero eigenvalue of an invertible matrix A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Solution.
Let x be an eigenvector of A corresponding to the nonzero eigenvalue λ. Then $Ax = \lambda x$. Multiplying both sides of this equality by A^{-1} and then dividing the resulting equality by λ to obtain $A^{-1}x = \frac{1}{\lambda}x$. That is, x is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{\lambda}$ ■

Exercise 368
Show that if λ is an eigenvalue of a matrix A then λ^m is an eigenvalue of A^m for any positive integer m.

Solution.
Let x be an eigenvector of A corresponding to the eigenvalue λ. Then $Ax = \lambda x$. Multiplying both sides by A to obtain $A^2x = \lambda Ax = \lambda^2 x$. Now, multiplying this equality by A to obtain $A^3x = \lambda^3 x$. Continuing in this manner, we find $A^m x = \lambda^m x$ ■

Exercise 369
(a) Show that if D is a diagonal matrix then D^k, where k is a positive integer, is a diagonal matrix whose entries are the entries of D raised to the power k.
(b) Show that if A is similar to a diagonal matrix D then A^k is similar to D^k.

Solution.
(a) We will show by induction on k that if

$$D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$$

then

$$D^k = \begin{pmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{pmatrix}$$

Indeed, the result is true for $k = 1$. Suppose true up to $k - 1$ then

$$\begin{aligned} D^k &= D^{k-1}D = \begin{pmatrix} d_{11}^{k-1} & 0 & \cdots & 0 \\ 0 & d_{22}^{k-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn}^{k-1} \end{pmatrix} \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix} \\ &= \begin{pmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{pmatrix} \end{aligned}$$

(b)Suppose that $D = P^{-1}AP$. Then $D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}AP^2$. Thus, by induction on k one finds that $D^k = P^{-1}A^kP$ ■

Exercise 370
Show that the identity matrix I_n has exactly one eigenvalue. Find the corresponding eigenspace.

Solution.
The characteristic equation of I_n is $(\lambda - 1)^n = 0$. Hence, $\lambda = 1$ is the only eigenvalue of I_n. The corresponding eigenspace is $V_1 = span\{e_1, e_2, \cdots, e_n\} = \mathbb{R}^n$ ■

Exercise 371
Show that if $A \sim B$ then
(a) $A^T \sim B^T$.
(b) $A^{-1} \sim B^{-1}$.

Solution.
(a) Suppose that $A \sim B$ and let P be an invertible matrix such that $B = P^{-1}AP$. Taking the transpose of both sides we obtain $B^T = (P^T)^{-1}A^TP^T$; that is, $A^T \sim B^T$.
(b) Suppose that A and B are invertible and $B = P^{-1}AP$. Taking the inverse of both sides we obtain $B^{-1} = P^{-1}A^{-1}P$. Hence $A^{-1} \sim B^{-1}$ ■

Exercise 372
If A is invertible show that $AB \sim BA$ for all B.

Solution.
Suppose that A is an $n \times n$ invertible matrix. Then $BA = A^{-1}(AB)A$. That is $AB \sim BA$ ■

Exercise 373
Let A be an $n \times n$ **nilpotent** *matrix, i.e. $A^k = \mathbf{0}$ for some positive ineteger k.*

(a) Show that $\lambda = 0$ is the only eigenvalue of A.
(b) Show that $p(\lambda) = \lambda^n$.

Solution.
(a) If λ is an eigenvalue of A then there is a nozero vector x such that $Ax = \lambda x$. By Exercise 368, λ^k is an eigenvalue of A^k and $A^kx = \lambda^kx$. But $A^k = \mathbf{0}$ so $\lambda^kx = \mathbf{0}$ and since $x \neq \mathbf{0}$ we must have $\lambda = 0$.
(b) Since $p(\lambda)$ is of degree n and 0 is the only eigenvalue of A then $p(\lambda) = \lambda^n$ ■

Exercise 374
Suppose that A and B are $n \times n$ similar matrices and $B = P^{-1}AP$. Show that if λ is an eigenvalue of A with corresponding eigenvector x then λ is an eigenvalue of B with corresponding eigenvector $P^{-1}x$.

Solution.
Since λ is an eigenvalue of A with corresponding eigenvector x then $Ax = \lambda x$. Postmultiply B by P^{-1} to obtain $BP^{-1} = P^{-1}A$. Hence, $BP^{-1}x = P^{-1}Ax = \lambda P^{-1}x$. This says that λ is an eigenvalue of B with corresponding eigenvector $P^{-1}x$ ■

Exercise 375
Let A be an $n \times n$ matrix with n odd. Show that A has at least one real eigenvalue.

Solution.
The characteristic polynomial is of degree n. The Fundamental Theorem of Algebra asserts that such a polynomial has exactly n roots. A root in this case can be either a complex number or a real number. But if a root is complex then its conjugate is also a root. Since n is odd then there must be at least one real root ■

Exercise 376
Consider the following $n \times n$ matrix

$$A = \left(\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{array}\right)$$

Show that the characterisitc polynomial of A is given by $p(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$. Hence, every monic polynomial (i.e. the coefficient of the highest power of λ is 1) is the characteristic polynomial of some matrix. A is called the **companion matrix** *of $p(\lambda)$.*

Solution.
The characterisitc polynomial of A is

$$p(\lambda) = \left|\begin{array}{cccc} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ a_0 & a_1 & a_2 & \lambda + a_3 \end{array}\right|$$

Expanding this determinant along the first row we find

$$\begin{array}{rcl} p(\lambda) & = & \lambda\left|\begin{array}{ccc} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_1 & a_2 & \lambda + a_3 \end{array}\right| + \left|\begin{array}{ccc} 0 & -1 & 0 \\ 0 & \lambda & -1 \\ a_0 & a_2 & \lambda + a_3 \end{array}\right| \\ & = & \lambda[\lambda(\lambda^2 + a_3\lambda + a_2) + a_1] + a_0 \\ & = & \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \blacksquare \end{array}$$

Exercise 377
Find a matrix P that diagonalizes

$$A = \left(\begin{array}{ccc} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{array}\right)$$

Solution.
From Exercise 346 the eigenspaces corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 2$ are

$$V_1 = span\left\{\left(\begin{array}{c} -2 \\ 1 \\ 1 \end{array}\right)\right\}$$

and

$$V_2 = span\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Let $\vec{v_1} = (-2, 1, 1)^T, \vec{v_2} = (-1, 0, 1)$, and $\vec{v_3} = (0, 1, 0)^T$. It is easy to verify that these vectors are linearly independent. The matrices

$$P = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

satisfy $AP = PD$ or $D = P^{-1}AP$ ■

Exercise 378
Show that the matrix A is diagonalizable.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$$

Solution.
From Exercise 5.13, the eigenvalues of A are $\lambda = 4, \lambda = 2 + \sqrt{3}$ and $\lambda = 2 - \sqrt{3}$. Hence, by Theorem 80 A is diagonalizable ■

Exercise 379
Show that the matrix A is not diagonalizable.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$$

Solution.
By Exercise 365, the eigenspaces of A are

$$V_{-1} = \left\{\begin{pmatrix} -s \\ 2s \\ s \end{pmatrix} : s \in \mathbb{R}\right\} = span\left\{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right\}$$

and

$$V_3 = \left\{\begin{pmatrix} -5s \\ -6s \\ s \end{pmatrix} : s \in \mathbb{R}\right\} = span\left\{\begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix}\right\}$$

Since there are only two eigenvectors of A then A is not diagonalizable ■

Exercise 380
Show that if A is diagonalizable then the rank of A is the number of nonzero eigenvalues of A.

Solution.
Suppose that A is diagonalizable. Then there exist matrices P and D such that $D = P^{-1}AP$, where D is diagonal with the diagonal entries being the eigenvalues of A. Hence, $rank(D) = rank(P^{-1}AP) = rank(AP) = rank(A)$ by Theorem 70. But $rank(D)$ is the number of nonzero eigenvalues of A ■

Exercise 381
Show that A is diagonalizable if and only if A^T is diagonalizable.

Solution.
Suppose that A is diagonalizable. Then there exist matrices P and D such that $D = P^{-1}AP$, with D diagonal. Taking the transpose of both sides to obtain $D = D^T = P^T A^T (P^{-1})T = Q^{-1}A^T Q$ with $Q = (P^{-1})^T = (P^T)^{-1}$. Hence, A^T is diagonalizable. Similar argument for the converse ■

Exercise 382
Show that if A and B are similar then A is diagonalizable if and only if B is diagonalizable.

Solution.
Suppose that $A \sim B$. Then there exists an invertible matrix P such that $B = P^{-1}AP$. Suppose first that A is diagonalizable. Then there exist an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$. Hence, $B = P^{-1}QDQ^{-1}$ and this implies $D = (P^{-1}Q)^{-1}B(P^{-1}Q)$. That is, B is diagonalizable. For the converse, repeat the same argument using $A = (P^{-1})^{-1}BP^{-1}$ ■

Exercise 383
Give an example of two diagonalizable matrices A and B such that $A+B$ is not diagonalizable.

Solution.
Consider the matrices

$$A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

The matrix A has the eigenvalues $\lambda = 2$ and $\lambda = -1$ so by Theorem 80, A is diagonalizable. Similar argument for the matrix B. Let $C = A + B$ then

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

This matrix has only one eigenvalue $\lambda = 1$ with corresponding eigenspace (details omitted)

$$V_1 = span\{(1, 0)^T\}$$

Hence, there is only one eigenvector of C and by Theorem 78, C is not diagonalizable ■

Exercise 384
Show that the following are equivalent for a symmetric matrix A.
(a) A is orthogonal.
(b) $A^2 = I_n$.
(c) All eigenvalues of A are ± 1.

Solution.
(a) $\Rightarrow$ (b): Suppose that A is orthogonal. Then $A^{-1} = A^T$. This implies $I_n = A^{-1}A = A^TA = AA = A^2$.
(b) $\Rightarrow$ (c): Suppose that $A^2 = I_n$. By Theorem 81, there exist an orthogonal matrix P and a diagonal matrix D such that $D = P^TAP$. This implies that $PDP^T = A$. Hence, $I_n = A^2 = PDP^TPDP^T = PD^2P^T$ or $D^2 = I_n$ since P is orthogonal. If λ_i is an eigenvalue of A then λ_i is on the main diagonal of D and since $D^2 = I_n$ then $\lambda_i^2 = 1$. It follows that $\lambda_i = \pm 1$.
(c) $\Rightarrow$ (a): Suppose that all the eigenvalues of A are ± 1. Since A is symmetric then $D = P^TAP$. Thus, $I_n = D^2 = P^TAPP^TAP = P^TA^2P$. Thus, $I_n = A^2 = A^TA$ and by Theorem 20, A is invertible with $A^{-1} = A^T$. This means that A is orthogonal ■

Exercise 385
A matrix that we obtain from the identity matrix by writing its rows in a different order is called **permutation matrix**. *Show that every permutation matrix is orthogonal.*

Solution.
‘The columns of a permutation matrix is a rearrangement of the orthonormal set $\{e_1, e_2, \cdots, e_n\}$. Thus, by Theorem 63 a permutation matrix is always orthogonal ■

Exercise 386
Let A be an $n \times n$ skew symmetric matrix. Show that
(a) $I_n + A$ is nonsingular.
(b) $P = (I_n - A)(I_n + A)^{-1}$ is orthogonal.

Solution.
(a) We will show that the homogeneous system $(I_n + A)x = \mathbf{0}$ has only the trivial solution. So let $x \in \mathbb{R}^n$ such that $(I_n + A)x = \mathbf{0}$. Then $0 =< x, x + Ax >=< x, x > + < x, Ax >=< x, x > + x^TAx$. But $x^TAx \in \mathbb{R}$ so that $(x^TAx)^T = x^TAx$, i.e. $x^TA^Tx = x^TAx$ and since $A^T = -A$ then $x^TAx = 0$. Hence, $< x, x >= 0$. This leads to $x = 0$.

(b) It is easy to check that $(I_n - A)(I_n + A) = (I_n + A)(I_n - A)$. Then, $PP^T = (I_n - A)(I_n + A)^{-1}(I_n - A)^{-1}(I_n + A) = (I_n - A)[(I_n - A)(I_n + A)]^{-1}(I_n + A) = (I_n - A)([(I_n + A)(I_n - A)]^{-1}(I_n + A) = (I_n - A)(I_n - A)^{-1}(I_n + A)^{-1}(I_n + A) = I_n$. Thus, P is orthogonal ■

Exercise 387
We call square matrix E a **projection matrix** *if $E^2 = E = E^T$.*

(a) If E is a projection matrix, show that $P = I_n - 2E$ is orthogonal and symmetric.
(b) If P is orthogonal and symmetric, show that $E = \frac{1}{2}(I_n - P)$ is a projection matrix.

Solution.
(a) $P^T = I_n - 2E^T = I_n - 2E$ so that P is symmetric. Moreover, $PP^T = (I_n - 2E)(I_n - 2E) = I_n - 4E + 4E^2 = I_n - 4E + 4E = I_n$. Hence, P is orthogonal.
(b) Suppose P is a symmetric and orthogonal $n \times n$ matrix. Then $E^T = \frac{1}{2}(I_n - P^T) = \frac{1}{2}(I_n - P) = E$ and $E^2 = \frac{1}{4}(I_n - P)(I_n - P) = \frac{1}{4}(I_n - 2P + P^2) = \frac{1}{4}(2I_n - 2P) = E$ ■

Chapter 6

Exercise 416
Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, x - 2y, 3x)$ is a linear transformation.

Solution.
Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then

$$\begin{array}{rcl} T(\alpha(x_1, y_1) + (x_2, y_2)) & = & T(\alpha x_1 + x_2, \alpha y_1 + y_2) \\ & = & (\alpha x_1 + x_2 + \alpha y_1 + y_2, \alpha x_1 + x_2 - 2\alpha y_1 - 2y_2, 3\alpha x_1 + 3x_2) \\ & = & \alpha(x_1 + y_1, x_1 - 2y_1, 3x_1) + (x_2 + y_2, x_2 - 2y_2, 3x_2) \\ & = & \alpha T(x_1, y_1) + T(x_2, y_2) \end{array}$$

Hence, T is a linear transformation ■

Exercise 417
(a) Show that $D : P_n \longrightarrow P_{n-1}$ given by $D(p) = p'$ is a linear transformation.
(b) Show that $I : P_n \longrightarrow P_{n+1}$ given by $I(p) = \int_0^x p(t)dt$ is a linear transformation.

Solution.
(a) Let $p, q \in P_n$ and $\alpha \in \mathbb{R}$ then

$$\begin{array}{rcl} D[\alpha p(x) + q(x)] & = & (\alpha p(x) + q(x))' \\ & = & \alpha p'(x) + q'(x) = \alpha D[p(x)] + D[q(x)] \end{array}$$

Thus, D is a linear transformation.

(b) Let $p, q \in P_n$ and $\alpha \in \mathbb{R}$ then

$$\begin{array}{rcl} I[\alpha p(x) + q(x)] & = & \int_0^x (\alpha p(t) + q(t))dt \\ & = & \alpha \int_0^x p(t)dt + \int_0^x q(t)dt = \alpha I[p(x)] + I[q(x)] \end{array}$$

Hence, I is a linear transformation.■

Exercise 418
If $T : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *is a linear transformation with* $T(3,-1,2) = 5$ *and* $T(1,0,1) = 2$. *Find* $T(-1,1,0)$.

Solution.
Suppose that $(-1,1,0) = \alpha(3,-1,2) + \beta(1,0,1)$. This leads to a linear system in the unknowns α and β. Solving this system we find $\alpha = -1$ and $\beta = 2$. Since T is linear then

$$T(-1,1,0) = -T(3,-1,2) + 2T(1,0,1) = -5 + 4 = -1 \blacksquare$$

Exercise 419
Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ *be the transformation* $T(x,y,z) = (x,y)$. *Show that* T *is linear. This transformation is called a* **projection.**

Solution.
Let $(x_1, y_1, z_1 \in \mathbb{R}^3, (x_2, y_2, z_2) \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T(\alpha(x_1,y_1,z_1) + (x_2,y_2,z_2)) &= T(\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2) \\ &= (\alpha x_1 + x_2, \alpha y_1 + y_2) = \alpha(x_1,y_1) + (x_2,y_2) \\ &= \alpha T(x_1,y_1,z_1) + T(x_2,y_2,z_2) \end{aligned}$$

Hence, T is a linear transformation ■

Exercise 420
Let θ *be a given angle. Define* $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *by*

$$T\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right)$$

Show that T *is a linear transformation. Geometrically,* Tv *is the vector that results if* v *is rotated counterclockwise by an angle* θ. *We call this transformation the* **rotation of** $\mathbb{R}^2$ **through the angle** θ

Solution.
Let $\left(\begin{array}{c} x_1 \\ y_1 \end{array}\right)$ and $\left(\begin{array}{c} x_2 \\ y_2 \end{array}\right)$ be two vectors in $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T\left(\alpha\left(\begin{array}{c} x_1 \\ y_1 \end{array}\right) + \left(\begin{array}{c} x_2 \\ y_2 \end{array}\right)\right) &= T\left(\left(\begin{array}{c} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \end{array}\right)\right) \\ &= \left(\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right)\left(\begin{array}{c} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \end{array}\right) \\ &= \left(\begin{array}{c} (\alpha x_1 + x_2)\cos\theta - (\alpha y_1 + y_2)\sin\theta \\ (\alpha x_1 + x_2)\sin\theta + (\alpha y_1 + y_2)\cos\theta \end{array}\right) \\ &= \alpha\left(\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right)\left(\begin{array}{c} x_1 \\ y_1 \end{array}\right) + \left(\begin{array}{c} x_2 \\ y_2 \end{array}\right) \\ &= \alpha T\left(\alpha\left(\begin{array}{c} x_1 \\ y_1 \end{array}\right)\right) + T\left(\alpha\left(\begin{array}{c} x_2 \\ y_2 \end{array}\right)\right) \end{aligned}$$

This shows that T is linear ■

Exercise 421
Show that the following transformations are not linear.

(a) $T : M_{nn} \longrightarrow \mathbb{R}$ *given by* $T(A) = |A|$.
(b) $T : M_{mn} \longrightarrow \mathbb{R}$ *given by* $T(A) = rank(A)$.

Solution.
(a) Since $|A + B| \neq |A| + |B|$ in general (See Exercise 173), then the given transformation is not linear.
(b) Suppose $m > n$. let A be the $m \times n$ matrix such that $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$. Let $B \in M_{mn}$ such that $b_{ii} = -1$ and $b_{ij} = 0$. Then $rank(A) = rank(B) = n$. Also, $A+B = \mathbf{0}$ so that $rank(A+B) = 0$. Hence, T is not linear ■

Exercise 422
If $T_1 : U \longrightarrow V$ *and* $T_2 : V \longrightarrow W$ *are linear transformations, then* $T_2 \circ T_1 : U \longrightarrow W$ *is also a linear transformation.*

Solution.
Let $u_1, u_1 \in U$ and $\alpha \in \mathbb{R}$. Then

$$\begin{array}{rcr} (T_2 \circ T_1)(\alpha u_1 + u_2) & = & T_2(T_1(\alpha u_1 + u_2)) \\ & = & T_2(\alpha T_1(u_1) + T_1(u_2)) \\ & = & \alpha T_2(T_1(u_1)) + T_2(T_1(u_2)) \\ & = & \alpha(T_2 \circ T_1)(u_1) + (T_2 \circ T_1)(u_2) \end{array} \blacksquare$$

Exercise 423
Let $\{v_1, v_2, \cdots, v_n\}$ *be a basis for a vector space* V*, and let* $T : V \longrightarrow V$ *be a linear transformation. Show that if* $T(v_i) = v_i$*, for* $1 \leq i \leq n$ *then* $T = id_V$*, i.e.* T *is the identity transformation on* V*.*

Solution.
Let $v \in V$. Then there exist scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. Since T is linear then $T(v) = \alpha T v_1 + \alpha T v_2 + \cdots + \alpha_n T v_n = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = v$ ■

Exercise 424
Let T *be a linear transformation on a vector space* V *such that* $T(v - 3v_1) = w$ *and* $T(2v - v_1) = w_1$*. Find* $T(v)$ *and* $T(v_1)$ *in terms of* w *and* w_1*.*

Solution.
Consider the system in the unknowns $T(v)$ and $T(v_1)$

$$\left\{ \begin{array}{rcrcr} T(v) & - & 3T(v_1) & = & w \\ 2T(v) & - & 2T(v_1) & = & w_1 \end{array} \right.$$

Solving this system to find $T(v) = \frac{1}{5}(3w_1 - w)$ and $T(v_1) = \frac{1}{5}(w_1 - 2w)$ ■

Exercise 425

Let $T : M_{mn} \to M_{mn}$ be given by $T(X) = AX$ for all $X \in M_{mn}$, where A is an $m \times m$ invertible matrix. Show that T is both one-one and onto.

Solution.

We first show that T is linear. Indeed, let $X, Y \in M_{mn}$ and $\alpha \in \mathbb{R}$. Then $T(\alpha X + Y) = A(\alpha X + Y) = \alpha AX + AY = \alpha T(X) + T(Y)$. Thus, T is linear. Next, we show that T is one-one. Let $X \in ker(T)$. Then $AX = \mathbf{0}$. Since A is invertible then $X = \mathbf{0}$. This shows that $ker(T) = \{0\}$ and thus T is one-one. Finally, we show that T is onto. Indeed, if $B \in R(T)$ then $T(A^{-1}B) = B$. This shows that T is onto ■

Exercise 426

Consider the transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $T(x) = Ax$, where $A \in M_{mn}$.

(a) Show that $R(T) = span\{c_1, c_2, \cdots, c_n\}$, where $c_1, c_2, \cdots, c_n$ are the columns of A.
(b) Show that T is onto if and only if $rank(A) = m$(i.e. the rows of A are linearly independent).
(c) Show that T is one-one if and only if $rank(A) = n$ (i.e. the columns of A are linearly independent).

Solution.

(a) We have

$$\begin{aligned} R(T) = \{Ax : x \in \mathbb{R}^n\} &= \left\{ \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right\} \\ &= \{x_1c_1 + x_2c_2 + \cdots + x_nc_n : x_i \in \mathbb{R}\} \\ &= span\{c_1, c_2, \cdots, c_n\} \end{aligned}$$

(b) Suppose that T is onto. Then $R(T) = span\{c_1, c_2, \cdots, c_n\} = \mathbb{R}^m$. Thus, $rank(A) = dim(span\{c_1, c_2, \cdots, c_n\}) = dim(\mathbb{R}^m) = m$. Since $dim(span\{c_1, c_2, \cdots, c_n\}) = dim(span\{r_1, r_2, \cdots, r_m\})$ then the vectors $\{r_1, r_2, \cdots, r_m\}$ are linearly independent.

Conversely, suppose that $rank(A) = m$. Then $dim(span\{c_1, c_2, \cdots, c_n\}) = dim(\mathbb{R}^m)$. By Exercise 265, $R(T) = \mathbb{R}^m$. That is, T is onto.

(c) Suppose that T is one-one. Then $ker(T) = \{0\}$. By Theorem 91, $dim(R(T)) = rank(A) = dim(\mathbb{R}^n) = n$. An argument similar to (b) shows that the columns of A are linearly independent. Conversely, if $rank(A) = n$ then by Theorem 91, $nullity(T) = 0$ and this implies that $ker(T) = \{0\}$. Hence, T is one-one ■

Exercise 427

Let $T : V \longrightarrow W$ be a linear transformation. Show that if the vectors

$$T(v_1), T(v_2), \cdots, T(v_n)$$

are linearly independent then the vectors $v_1, v_2, \cdots, v_n$ are also linearly independent.

Solution.
Suppose that $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$. Then $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = T(0) = 0$. Since the vectors $T(v_1), T(v_2), \cdots, T(v_n)$ are linearly independent then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This shows that the vectors $v_1, v_2, \cdots, v_n$ are linearly independent ■

Exercise 428
Show that the projection transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x, y)$ is not one-one.

Solution.
Since $(0, 0, 1) \in \ker(T)$ then by Theorem 89, T is not one-one ■

Exercise 429
Let M_{nn} be the vector space of all $n \times n$ matrices. Let $T : M_{nn} \to M_{nn}$ be given by $T(A) = A - A^T$.
(a) Show that T is linear.
(b) Find $ker(T)$ and $R(T)$.

Solution.
(a) Let $A, B \in M_{nn}$ and $\alpha \in \mathbb{R}$. Then $T(\alpha A + B) = (\alpha A + B - (\alpha A + B)^T = \alpha(A - A^T) + (B - B^T) = \alpha T(A) + T(B)$. Thus, T is linear.
(b) Let $A \in ker(T)$. Then $T(A) = \mathbf{0}$. That is $A^T = A$. This shows that A is symmetric. Conversely, if A is symmetric then $T(A) = \mathbf{0}$. It hollows that $ker(T) = \{A \in M_{nn} : A \ is \ symmetric\}$. Now, if $B \in R(T)$ and A is such that $T(A) = B$ then $A - A^T = B$. But then $A^T - A = B^T$. Hence, $B^T = -B$,i.e. B is skew-symmetric. Conversely, if B is skew-symmetric then $B \in R(T)$ since $T(\frac{1}{2}B) = \frac{1}{2}(B - B^T) = B$. We conclude that $R(T) = \{B \in M_{nn} : B \ is \ skew - symmetric\}$ ■

Exercise 430
Let $T : V \to W$. Prove that T is one-one if and only if $dim(R(T)) = dim(V)$.

Solution.
Suppose that T is one-one. Then $ker(T) = \{0\}$ and therefore $dim(ker(T)) = 0$. By Theorem 91, $dim(R(T)) = dimV$. The converse is similar ■

Exercise 431
Show that the linear transformation $T : M_{nn} \to M_{nn}$ given by $T(A) = A^T$ is an isomorphism.

Solution.
If $A \in ker(T)$ then $T(A) = \mathbf{0} = A^T$. This implies that $A = \mathbf{0}$ and consequently $ker(T) = \{\mathbf{0}\}$. So T is one-one. Now suppose that $A \in M_{mn}$. Then $T(A^T) = A$ and $A^T \in M_{nn}$. This shows that T is onto. It follows that T is an isomorphism ■

Exercise 432
Let $T : P_2 \to P_1$ be the linear transformation $Tp = p'$. Consider the standard ordered bases $S = \{1, x, x^2\}$ and $S' = \{1, x\}$. Find the matrix representation of T with respect to the basis S and S'.

Solution.
One can easily check that $p(1) = 0(1)+0x, p(x) = 1+0x$, and $p(x^2) = 0(1)+2x$. Hence, the matrix representation of T with respect to S and S' is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\blacksquare$$

Exercise 433
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

(a) Find the matrix representation of T with respect to the standard basis S of $\mathbb{R}^2$.
(b) Let

$$S' = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$$

Find the matrix representation of T with repspect to the bases S and S'.

Solution.
(a) We have

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

So the matrix representation of T with respect to the standard basis is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) One can easily check that

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So the matrix representation of T with respect to the standard basis is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\blacksquare$$

Exercise 434
Let V be the vector space of continuous functions on $\mathbb{R}$ with the ordered basis $S = \{\sin t, \cos t\}$. Find the matrix representation of the linear transformation $T : V \to V$ defined by $T(f) = f'$ with respect to S.

Solution.
Since $T(\sin t) = \cos t$ and $T(\cos t) = -\sin t$ then the matrix representation of T with respect to S is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \blacksquare$$

Exercise 435
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix representation with the respect to the standard basis of $\mathbb{R}^3$ is given by

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Find

$$T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

.

Solution.
We have

$$\begin{aligned} T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 5 \\ 5 \end{pmatrix} \blacksquare \end{aligned}$$