A Hitchhiker's Guide to Elementary Differential Equations©

Mohamed Sami ElBialy

March 30, 2011

Contents

Comments and examples included in these notes are not sufficient for a complete course on elementary differential equations. They do not replace the lecture notes nor the textbook. They are mere elaborations on some of the concepts, ideas, techniques, examples and exercises that are discussed in the text and lectures. You are expected to attend all the lectures and take notes and study them and read the textbook and do all the recommended exercise.

1 Basic Concepts

1.1 Examples

1.1.1. Example Solve The *initial value problem* (IVP)

$$\frac{d^2x}{dt^2} = t^2 + 3t \qquad \text{(DE)}$$

$$x(1) = 2, \quad \dot{x}(1) = -1 \qquad \text{(IC)}$$

Remark. An IVP consists of a differential equation (DE) and initial conditions (IC).
Solution. Integrating twice we obtain

$$\dot{x}(t) = \frac{1}{3}t^3 + \frac{3}{2}t^2 + a \qquad (*)$$

$$x(t) = \frac{1}{12}t^4 + \frac{1}{2}t^3 + at + b \qquad (**)$$

1. Each time we integrate we need a constant. As a result we obtain infinitely many solutions, one for each choice (a, b). in fact $x(t)$ is *two-parameter family of solutions*. It is called the *general solution* of the DE.

2. The initial conditions (IC) allow us to determine (a, b) uniquely and obtain a unique solution to the IVP.

3. From (*) we obtain

$$2 = \dot{x}(1) = \frac{1}{3} + \frac{3}{2} + a \longrightarrow a = \frac{1}{6}$$

   Thus we have a one-parameter family of solution to the DE that satisfies $\dot{x}(1) = 2$. Namely

$$x_1(t) = \frac{1}{12}\, t^4 + \frac{1}{2}\, t^3 + \frac{1}{6}\, t + b$$

   Now we determine the parameter b from the second IC:

$$-1 = x(1) = \frac{1}{12} - \frac{1}{2} - \frac{1}{6} + b \longrightarrow b = -\frac{5}{12}$$

   Thus, we have a unique solutio to the IVP:

$$q(t) = \frac{1}{12}\, t^4 + \frac{1}{2}\, t^3 + \frac{1}{6}\, t - \frac{5}{12}$$

□

1.1.2. Exercise. An object weighs lb is thrown up vertically with speed of 20 ft/h from the top of a building that is 300 ft high. When does the object hit the ground.

1.1.3. Example. Solve the following *initial value problem* (IVP) and sketch the solution

$$\dot{x} = -3x, \qquad x(1) = 5$$

Solution. We know that

$$\frac{d}{dt}e^{at} = ae^{at}$$

Try a solution

$$x(t) = e^{-3t}$$

But

$$x(1) = e^{-2} \neq 5$$

Let's try the one-parameter family of solutions

$$x(t) = Ae^{-3t}$$

If we insist that

$$x(1) = Ae^{-3} = 5$$

Then

$$A = 5e^{3}$$

And the solution to our IVP is

$$q(t) = 5e^{3}e^{-3t} = 5e^{-3(t-1)}$$

□

1.1.4. Independent variables and dependent variables

The variable t is called the independent variable and we may think of it as time.
The variable x is called the dependent variable and we may think of it as position of a particle moving on the x-axis. In other words, $x(t)$ tells us *position* of a particle moving on the x-axis at time t.

Exercise 1.1.5. Determine the independent variable and the dependent variable in each of the following:

1. $y'' + (\sin x)y' = xy$, where $y' = dy/dx, \cdots$.
2. $uv'' + u^2vv' + 3v = 0$, where $v' = dv/du, \cdots$.
3. $y'' + z^2y' = 3y^2 + 5z$.
4. $y^{(4)} + x^2y'' = 3y^6 + 5x$.

1.1.6. Terminology and observations.

An **initial value problem (IVP)** consists of
a DE and
as many initial conditions as the order of the DE.

The order of a DE
is
the order of the highest derivative that appears in the DE.

Each time we integrate we need a constant. Therefore,

The number of parameters in the general solution
must equal
the order the DE.

The general solution of a DE of order n
is a solution that satisfies the following:
(1) It has n parameters.
(2) Given any set of n appropriate initial conditions,
we can use these initial conditions
to uniquely determine the n parameters
and obtain a unique solution for the IVP.

1.1.7. Example. Solve the following (IVP) and sketch the solution

$$\ddot{x} = 5x, \qquad x(0) = -2, \qquad \dot{x}(0) = 4 \tag{1.1}$$

Solution. We are looking for a function whose second derivative is the function itself multiplied by 5. We know two

$$x_1(t) = e^{\sqrt{5}\,t}, \qquad x_2(t) = e^{-\sqrt{5}\,t} \tag{1.2}$$

Neither of them can be made to satisfy the IC. for example

$$x(t) = -2e^{\sqrt{5}\,t}$$

satisfies the first IC $x(1) = -2$, but does not satisfy the second IC.

Let's try a linear combination

$$x(t) = c_1 x_1(t) + c_2 x_2(t) \tag{1.3}$$

$$\text{that is,} \qquad x(t) = c_1 e^{\sqrt{5}\,t} + c_2 e^{-\sqrt{5}\,t} \tag{1.4}$$

We need to find constants (parameters) $\mathbf{c} = (c_1, c_2)$ such that $x(t)$ satisfies the IC.

$$\begin{aligned} c_1 + c_2 &= -2 \\ \sqrt{5}c_1 - \sqrt{5}c_2 &= 4 \end{aligned}$$

In matrix notation

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4/\sqrt{5} \end{pmatrix}$$

Thus

$$\begin{aligned} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 4/\sqrt{5} \end{pmatrix} \\ &= \begin{pmatrix} -1 + 2/\sqrt{5} \\ -1 - 2/\sqrt{5} \end{pmatrix} \end{aligned}$$

And the solution to the IVP is

$$q(t) = (-1 + 2/\sqrt{5})e^{\sqrt{5}\,t} - (1 + 2/\sqrt{5})e^{-\sqrt{5}\,t}, \qquad -\infty < t < \infty$$

Sketch the solution.

1.1.8. Terminology. Notice that the two solutions $x_1(t)$ and $x_2(t)$ given in (1.2) are not multiple of each other. In this case we say that they are *linearly independent*.

> We call $x_1(t) = e^{\sqrt{5}t}$ and $x_2(t) = e^{-\sqrt{5}t}$ given in (1.2)
>
> fundamental solutions of the
> linear homogenous DE $\ddot{x} = 5x$.

> $x(t) = c_1 e^{\sqrt{5}t} + c_2 e^{-\sqrt{5}t}$ given in (1.3)
>
> is the general solution of the DE $\ddot{x} = 5x$.
>
> Notice that it is a two-parameter family of solutions.

Notice that

#(parameters) = #(fundamental solutions)= order of the DE.

1.1.9. Remark: The inverse of a 2×2 **matrix** We used the fact that the inverse of a mtrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad \text{provided that} \qquad ad - bc \neq 0$$

1.1.10. Example Solve the following (IVP) and sketch the solution

$$\ddot{x} = -3x, \qquad x(0) = 3, \qquad \dot{x}(0) = -2$$

Solution. The main difference between this example and example 1.1.7 is the minus sign on the righthand side.

So, again we are looking for a function whose second derivative is the function itself multiplied by -3. We know two

$$x_1(t) = \cos\sqrt{3}\,t, \qquad x_2(t) = \sin\sqrt{3}\,t \tag{1.5}$$

Neither of them can be made to satisfy the IC. For example

$$x(t) = \frac{-2}{\sqrt{3}} \sin\sqrt{3}t$$

satisfies the second IC but not the first.

As in 1.1.7 let's try a linear combination

$$\begin{aligned} x(t) &= c_1 x_1(t) + c_2 x_2(t) \qquad (**) \\ x(t) &= c_1 \cos(\sqrt{3}\,t) + c_2 \sin(\sqrt{3}\,t) \end{aligned}$$

We need to find constants (parameters) $\mathbf{c} = (c_1, c_2)$ such that $x(t)$ satisfies the IC.

$$c_1 = 3, \qquad \sqrt{3}c_2 = -2$$

And the solution to the IVP is

$$q(t) = 3\cos(\sqrt{3}\,t) - \frac{2}{\sqrt{3}} \sin(\sqrt{3}\,t), \qquad -\infty < t < \infty$$

Sketch the solution. Fill in the space for the previous example.

The fundamental solutions are

______________ and ______________

The general solution of the DE is

___________________________.

It is a ....-parameter family of solutions.

1.1.11. Example Solve the following (IVP) and sketch the solution

$$\ddot{x} = -3x + 5\sin\sqrt{2}t \qquad \text{(DE)}$$
$$x(0) = 3, \qquad \dot{x}(0) = -2 \qquad \text{(IC)}$$

1.2 Classifying DE's

1.2.1. Order of a DE. See above.

1.2.2. Nonlinear verses linear equations

The canonical for of a linear equation of order n is

$$a_n(t)x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots a_1(t)x' + a_o(t)x = g(t)$$

If $g(t) = 0$, it is said to be homogenous.
If $g(t) \neq 0$, it is said to be non-homogenous.

If $x_1(t), x_2(t), \cdots, x_n(t)$
are linearly independent solutions of the linear homogenous DE
$a_n(t)x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots a_1(t)x' + a_o(t)x = 0$
they are called fundamental solutions.

Exercise. Which of the following is a linear DE? For the nonlinear ones, determines all the nonlinear terms.

$$(t^2+1)y'' + 3ty' - 5t^3y = t$$
$$yy'' + 3y' - 5y = t$$
$$x'' + (x')^2 - tx = \sin t$$
$$3x'' + 5x' = \sin x$$
$$\ln(1+t^2)y'' + 3t^3y' - \frac{1}{2+\cos t}y = t$$
$$3x'' + 5x' = \sin t$$
$$t^3x'' + x' - tx = t + \cos x$$

Theorem 1.2.3. *Consider the homogenous linear system*

$$a_n(t)x^{(n)} + \cdots a_1(t)x' + a_o(t)x = 0 \tag{H}$$

Suppose we have n linearly independent solutions

$$x_1(t), x_2(t), \cdots, x_n(t)$$

Then, the general solution of (H) takes the form
$$x(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t)$$

1.2.4. Ordinary (ODE) verses partial (PDE) differential equations:

1.3 Exercises

1. For each of the following:

   (a) Classify each of the following DE's according to order and linearity.

   (b) Determine the independent and dependent variables.

   (a) $\dfrac{dy}{dx} = 3y^2 + 5x.$

   (b) $\dfrac{dx}{dy} = 3y^2 + 5x.$
   What are the differences between between problem (a) and (b)?

   (c) $y'' + x^2y' = 3y^2 + 5x.$

   (d) $y'' + z^2y' = 3y^2 + 5z.$

   (e) $v'''v' + v'^2u = 3vu^2 + \sin v.$

   (f) $y^{(4)} + x^2y'' = 3y^6 + 5x.$

2. Show that $x(t) = e^{-5t}$ is a solution for the DE

$$x'' + 3x' - 10 = 0$$

Find another solution for the DE.

3. For what values of α is $u(t) = e^{\alpha t}$ a solution of the following DE?

$$\ddot{u} - 4\dot{u} + 3u = 0$$

Solve the IVP

$$\ddot{u} - 4\dot{u} + 3u = 0, \qquad u(0) = 5, \dot{u}(0) = -2$$

4. For what values of β is $x(t) = e^{\beta t}$ a solution of the following DE?

$$\ddot{x} - \dot{x} - 6x = 0$$

Solve the IVP

$$\ddot{x} - \dot{x} - 6x = 0, \qquad x(0) = 2, \dot{x}(0) = -3$$

5. After you answer the following two questions, comment on the difference between the two equations:

   (a) For what values of β is $x(t) = e^{\omega t}$ a solution of the following DE?

   $$\ddot{x} - 5x = 0$$

   (b) For what values of β is $x(t) = \sin \beta t$ a solution of the following DE?

   $$\ddot{x} + 3x = 0$$

   Find another solution for the DE.

6. Solve each of the following the following IVP's and sketch the solution

$$\begin{aligned} &\dot{x} + 5x = 0, \qquad x(1) = -2 \\ &\ddot{x} + 5x = 0, \qquad x(0) = -2, \quad \dot{x}(0) = 7 \end{aligned}$$

7. Show that $y(x) = \dfrac{3}{4} + \dfrac{c}{t^2}$ is the general solution for the DE

$$2xy' + 4y = 3$$

Solve the IVP

$$2xy' + 4y = 3, \qquad y(2) = -3$$

8. Verify that the given functions are solutions to the DE. Then, find the unique solution to the DE that satisfies the given IC when one is given.

(a) $x'' + 2x' - 3x = 0,$
$x_1(t) = e^{-3t}, x_2(t) = e^t$
$x(t) = ae^{-3t} + be^t$
(IC) $x(0) = 2, x'(0) = -1$

(b) $2t^2y'' + 3ty' - y = 0, t > 0;$
$y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1}$, $y(t) = at^{1/2} + bt^{-1}$
(IC) $y(1) = 2, y'(1) = -1$
Definition Equations of this form are called linear.

9. Show that $y(x) = \displaystyle\int_1^x \frac{\sin x}{x}\, dx$ is a solution to the DE $xy' - \sin x = 0$.
Hint: Recall the *fundamental theorem of calculus.*

10. Find the value of r for which the given DE has a solution of the form e^{rt}, then solve the IVP.

(a) $\ddot{x} + \dot{x} - 6x = 0$
$x(0) = -2, \dot{x}(0) = 3.$

(b) $y'' + 5y' + 6y = 0,$
$y(0) = 3, y'(0) = -1.$

(c) $y''' - 3y'' + 2y' = 0$
$y(0) = 3, y'(0) = -1, y''(0) = 1.$

Definition Equations of this form are called linear with constant coefficients.

11. Find the value of r for which the given DE has a solution of the form x^rthen solve the IVP.

(a) $x^2y'' + xy' = y,$
$y(1) = -1, y'(1) = 2.$

(b) $x^2y'' - xy' - 15y = 0,$
$y(1) = 2, y'(1) = -1.$

Definition Equations of this form are called linear. However, they don't have constant coefficients.

12. Find the trivial solutions of the following DE's:

(a) $\dot{x} = (x - 1)(x + 2)$

(b) $\dot{x} = (x - 8)\ln(x - 5)$

13. Solve the IVP

$$\dot{x} = x - 2, \qquad x(0) = a$$

Describe the long term behaviour of the solution.

Hint: The long term behaviour is $\lim_{t\to\infty}$ and $\lim_{t\to-\infty}$

2 First order homogenous linear DE's

2.1 Separation of variables

One of the simplest types of DE are equations that can be reduced to the form

$$f(x)dx = f(y)dy \tag{2.1}$$

where the left-hand-side depends on x only and the right-hand-side on y only. therefore, the problems amounts to two integrals.

2.1.1. Example The following example shows that the interval of definition of a solution might be finite and depends on the initial conditions.

$$\frac{dy}{dx} = -\frac{5x}{3y}, \quad y \neq 0, \quad y(1) = -3$$

This DE is not defined for $y = 0$ because there is a y in the denominator.
We Can rewrite the DE in the form

$$3ydy = -5xdx$$

Integrating each side separately and combining the two constants, we obtain

$$3y^2 + c_1 = -5x^2 + c_2$$

$$3y^2 + 5x^2 = c$$

Now we determine c from IC ($y = -3$ when $x = 1$) and obtain

$$3y^2 + 5x^2 = 32, \qquad y \neq 0 \tag{*}$$

This is a vertical ellipse.
Important Remark: As a geometric object, this ellipse is defined for $y = 0$, i.e. includes the two points $(\pm\sqrt{32/5}, 0)$. However, the DE is not defined on the x-axis, $y = 0$ (Why?). In other words, the x-axis, $y = 0$, does not have a physical meaning for the system that this DE models. Thus, as far as the DE is concerned, we have to remove the two points $(x_\pm, 0), x_\pm = \pm\sqrt{32/5}$, from the ellipse. Now we can write (*) as

$$\phi_\pm(x) = \pm\sqrt{\frac{32}{3} - \frac{5}{3}x^2}$$
$$x_- = -\sqrt{32/5} < x < \sqrt{32/5} = x_+$$

But these are two different functions $\phi_\pm(x)$. Not one.
Which one is the unique solution to our IVP? $\phi_+(x)$ or $\phi_-(x)$?

1. The solution of the IVP is the largest part of the ellipse that
   (a) *contains the* IC and

(b) *does not intersect x-axis* $\{y = 0\}$.

2. Since $y(1) = -3 < 0$, it follows that the part of the ellipse that satisfies these two requirements is lower part of the ellipse. Therefore, the unique solution to the IVP is given by
$$\phi_-(x) = -\sqrt{\frac{32}{3} - \frac{5}{3}x^2}$$
$$x_- = -\sqrt{32/5} < x < \sqrt{32/5} = x_+$$

The interval
$x_- < x < x_+$
is called <u>*the interval of definition of the solution*</u>
or <u>*the interval of existence of the solution*</u>.

Notice that the inequalities that define the interval of definition are strict inequalities because the two points $(\pm\sqrt{32/5}, 0)$ do not lie on the solution curve.

3. If we interpret the independent variable x as time and the dependent variable y as the position of a particle moving on the y-axis, then the solution $\phi_-(x)$ means that a particle appears at time x_- at the point $y = 0$, moves down on the y-axis towards the point $y_- = -\sqrt{32/3}$ and reaches it at time $x = 0$. The particles stops instantaneously, then moves up again towards the point $y = 0$ and reaches it at time x_+ and disappears there.

4. If we look for the solution with IC $y(1) = 3$, we obtain
$$\phi_+(x) = +\sqrt{\frac{32}{3} - \frac{5}{3}x^2},$$
$$x_- = -\sqrt{32/5} < x < \sqrt{32/5} = x_+$$
We can interpret the solution $\phi_+(x)$ in a similar fashion, but now we have a particle that moves upward on the y-axix to the point $y_+ = +\sqrt{32/3}$ and back to $y = 0$ where it disappears.

5. Together, the two solutions can be interpreted as two particles appearing at the origin $y = 0$, travelling in opposite direction with the same speed, stoping at time $x = 0$, then returning to the origin and annihilating each other.

6. **Important Remark:**

The Interval of existence of a solution depends on the IC.

We can see this form example 2.1.1:

(a) There is nothing special about the initial condition ($y(1) = -3$) that we used. We can see that any *initial condition* $y(x_o) = y_o$ will yields an ellipse of the form

$$5x^2 + 3y^2 = c^2$$

where

$$c^2 = 5x_o^2 + 3y_o^2$$

and the positive constant c^2 will depend on the initial condition .

(b) If $y_o > 0$, the solution is the upper half of the ellipse

$$\phi_+ = +\sqrt{\frac{c^2 - 5x^2}{3}},$$
$$x_- = -\sqrt{\frac{c^2}{5}} < x < +\sqrt{\frac{c^2}{5}} = x_+$$

(c) If $y_o < 0$, the solution is the lower half of the ellipse

$$\phi_- = -\sqrt{\frac{c^2 - 5x^2}{3}},$$
$$x_- = -\sqrt{\frac{c^2}{5}} < x < +\sqrt{\frac{c^2}{5}} = x_+$$

(d) Notice that in either case, the interval of existence of the solution is

$$x_- = -\sqrt{\frac{c^2}{5}} < x < +\sqrt{\frac{c^2}{5}} = x_+$$

depends on $c^2 = 5x_o^2 + 3y_o^2$ which is obtained form the IC. Thus, it depends on the IC. In physical terms, the life span of these two particles depends on the IC.

2.1.2. Exercise Describe and interpret all solutions of the differential equation

$$\frac{dy}{dx} = -\frac{bx}{ay}$$

where a and b are two positive constants.

2.1.3. Exercise Find the general solution of the DE

$$y' = -2ty^2$$

Find the unique solution that satisfies each of the following IC and find its interval of definition:

1. $y(3) = \frac{1}{2}$

2. $y(-3) = -\dfrac{1}{2}$
3. $y(1) = 4$
4. $y(1) = \dfrac{1}{3}$
5. Sketch the four solutions on the same graph.
6. Suppose that t is time and y is the position of a particle moving on the y-axis. Describe the behaviour of the particle in both cases.

2.1.4. Exercise Find the general solution of the DE

$$y' = -ty^3$$

1. Find the unique solutions to the IVP's with IC's $y(\pm 3) = \dfrac{1}{2}$, $y(\pm 3) = \dfrac{1}{7}$ and $y(\pm 3) = \dfrac{1}{\sqrt{5}}$.
2. Find the unique solutions to the IVP's with IC's $y(\pm 3) = -\dfrac{1}{2}$, $y(\pm 3) = -\dfrac{1}{7}$ and $y(\pm 3) = -\dfrac{1}{\sqrt{5}}$.
3. Sketch all these solutions on one graph.

2.1.5. Exercises Solve the IVP, find the interval of existence of the solution and sketch the solution. Suppose that t is time and x is the position of a particle moving on the x-axis. Give a physical interpretation to the four solutions together.

1. $\dot{x} = (1 - 2t)x^2, x(0) = -1/2$.
2. Describe the dependence of interval of definition on the initial condition $x(0) = r$ and $x(1) = s$.

2.1.6. Exercise Consider the DE

$$\dot{x} = \frac{1 - 2t}{2x} \qquad x \neq 0$$

1. Find the four solutions that satisfy the following IC's and give the interval of definition for each solution:

$$x(0) = \pm\sqrt{\frac{2}{9}} \tag{i}$$

$$x(1) = \pm\sqrt{\frac{2}{9}} \tag{ii}$$

$$x(1/2) = -1/36 \tag{iii}$$

2. Sketch the six solutions on the same graph.

3. What is the difference between these solutions?

4. Suppose that t is time and x is the position of a particle moving on the x-axis. Give a physical interpretation to the four solutions together.

2.1.7. Exercise Solve the following IVP and determine the interval of definition of the solution.

$$y' = xy^3(1+x^2)^{-1/2}, \qquad y(0) = -1$$

2.1.8. Exercise Solve the following IVP and determine the interval of definition of the solution.

$$y' = \frac{2x}{y + x^2 y}, \qquad y(0) = -2$$

Example 2.1.9. Consider the separable DE

$$\frac{dy}{dx} = -xy, \qquad y(a) = b \tag{2.2}$$

1. We can think of the independent variable x as time and the dependent variable y as the position of a test particle moving on the y-axis.

2. Notice that the initial condition $y(a) = 0$ leads to the equilibrium solution

$$\phi_o(x) = 0, \qquad x \in \mathbb{R}$$

3. Separating the variables we obtain

$$\int \frac{1}{y}\,dy = -\int x\,dx$$

Integrating,

$$\begin{aligned} \ln|y| &= -\frac{x^2}{2} + c \qquad -\infty < x < \infty \\ |y(x)| &= e^c e^{-x^2/2}, \qquad -\infty < x < \infty \\ y(x) &= \pm e^c e^{-x^2/2}, \qquad -\infty < x < \infty \end{aligned}$$

We can write the constant $\pm e^c = A$ Thus, the general solution of the IVP is

$$y(x) = Ae^{-x^2/2}, \qquad -\infty < x < \infty$$

Notice that this general solution works also for the equilibrium solution. How?

4. How do you interpret these solutions physically?

Example 2.1.10. Solve the IVP, find the interval of definition of the solution and sketch the solution.

$$y' = \frac{x(x^2-1)}{4y^3}, y \neq 0, \qquad y(0) = -\frac{1}{\sqrt{2}}$$

This is a separable DE:

$$\int 4y^3\,dy = \int (x^3 - x)\,dx$$
$$y^4 = \frac{1}{4}x^4 - \frac{1}{2}x^2 + c$$

Determine c from IC:

$$\frac{1}{4} = c$$

Solution to the IVP

$$\begin{aligned} y &= -\frac{1}{\sqrt{2}}(x^4 - 2x^2 + 1)^{\frac{1}{4}} \\ x^4 &- 2x^2 + 1 > 0 \\ &= -\frac{1}{\sqrt{2}}((x^2 - 1)^2)^{\frac{1}{4}} \\ &= -\frac{1}{\sqrt{2}}\sqrt{|x^2 - 1|}, \qquad x^2 < 1 \end{aligned}$$

Solution to the IVP

$$\phi(x) = -\frac{1}{\sqrt{2}}\sqrt{|x^2 - 1|}, \qquad -1 < x < 1$$

Notice that $\sqrt{|x^2 - 1|}$ is defined when $x = 1$. But $y = 0$ does not have a physical meaning because we are dividing by y^3 in the DE.
To sketch the solution, notice that if we square it (and write y for $\phi(x)$) and rearrange the equation we get

$$y^2 = \frac{1}{2}|x^2 - 1|, \qquad x \neq \pm 1$$

But $0 \leq x^2 < 1$, that is $x^2 - 1 < 0$. Thus, $|x^2 - 1| = 1 - x^2$. It follows that

$$y^2 = \frac{1}{2}(1 - x^2), \qquad x \neq \pm 1$$
$$x^2 + \frac{y^2}{\frac{1}{2}} = 1, \qquad x \neq \pm 1$$

which is a horizontal ellipse.
Question. Which part of this ellipse is the graph of the solution?

2.1.11. Exercise Solve the following IVP's :

1. $\dfrac{dy}{dx} + 2y = f(x), \qquad y(0) = 1$

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 3, \\ -1 & x \geq 3. \end{cases}$$

2. $\dfrac{dy}{dx} + y = f(x), \qquad y(0) = 1$

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 3, \\ -1, & x \geq 3. \end{cases}$$

Notice that $\phi_o(x) \equiv 1, 0 \leq x \leq 3$, is a solution for the first 3 seconds.

3. Consider the IVP

$$\dot{x} = (2 - x)\tan t, \qquad x(0) = b$$

   (a) Solve the IVP. Don't forget to find the interval of definition.

   (b) Sketch and describe the behaviour of solutions away from these critical values.

   (c) What is the unique solution that satisfies $x(0) = 0$?

   (d) Let $b_1 < b_2$. What happen to the line segment $b_1 \leq x \leq b_2$ as t varies?
   Notation: Denote the unique solution that satisfies the IVP by $\phi_b(t)$.

3 First order Linear O D E

3.1 Method of integrating factor or variation of parameter

$$\dot{x} + p(t)x = g(t) \tag{NH}$$

where $p(t)$ and $g(t)$ are **continuous** in some open interval of $a < t < b$.

3.1.1. The Homogenous Case: $g(t) = 0$. In this case we have the DE

$$\dot{x} + p(t)x = 0 \tag{H}$$

which is also separable since

$$\frac{dx}{x} = -p(t)dt$$

Integrating, we obtain the general solution of differential equation (H) given by

$$x_h(t) = A\exp[-\int p(t)dt] \tag{3.1}$$

with parameter A. Nothing new so far. But what about solving the nonhomogeneous differential equation (NH). For this we need what is called the *variation of parameter* method.

3.1.2. The Variation of Parameter Method (AKA integrating factor method) for the nonhomogeneous DE (NH).

The steps:

1. **Write the DE in the standard form**

$$\text{(NH)} \qquad x' + p(t)x = g(t)$$

2. **Find a fundamental solution of (H):**

$$\text{(H)} \qquad x' + p(t)x = 0$$

Calculate: $\int p(t)\,dt$

Fundamental solution of (H)

$$\Phi_h(t) = \exp[-\int p(t)\,dt]$$

Simplify. (This is very important).

3. **Calculate**

$$\boxed{u(t) = \int \frac{g(t)}{\Phi_h(t)}\,dt}$$

4. **The general solution of (NH)**

$$\boxed{x(t) = \Phi_h(t)\left(C + \int \frac{g(t)}{\Phi_h(t)}\,dt\right)}$$

5. **The constant C:** Determine the constant C from the initial conditions.

6. **Interval of existence:** Find the interval of definition of the solution to the initial value problem.

7. Once we solve (NH) we should study *dominant term* and *transient terms* in the solution.

3.1.3. The term "integrating factor". This is just different terminlogy and notation. The function

$$\mu(t) = \Phi_h^{-1}(t)^{-1} = \frac{1}{\Phi_h(t)}$$

is called *an integrating factor*.

Thus
The general solution of (NH)
can be written in the form

$$x(t) = \frac{C + \int g(t)\mu(t)dt}{\mu(t)}$$

Exercises 3.1.4. Solve the following, find interval of definition and sketch the solution. Determine the dominant and transient terms at both ends of the interval of definition.

1. (a) $ty' + 2y = 4t^2, \quad y(1) = 2$
   (b) $ty' + 2y = 4t^2, \quad y(1) = -16$

2. $y' - 2y = t^2 e^{2t}, \quad y(1) = 2$

3. $tx' + 2x = t^2 - t + 1, \quad t > 0, \quad y(1) = 1/2$
4. $y' + 3y = t + e^{-2t}, \quad y(1) = 2$
5. $y' + 2ty = 2te^{-t^2}, \quad y(1) = -1$
6. $ty' + y = 3t\cos 2t, \quad t > 0, \quad y(1) = 2$
7. $(1 + t^2)y' + 4ty = 1/(1 + t^2)^2, \quad y(1) = -2$
8. $(1 + t^2)y' + 4ty = 1/(1 + t^3)^2, \quad y(1) = -2$
9. $y' + (2/t)y = (\cos t)/t^2, \quad t > 0, \quad y(\pi/3) = 2$
10. $y' + (\tan x)y = \cos^2 x, \quad y(\pi) = -1$
11. $uv' - u^2 \sin u = v$
12. $(1 + x^2)y' + 2xy = f(x), \qquad y(0) = 1$

$$f(x) = \begin{cases} x, & 0 \le x \le 3, \\ -5 & x \ge 3. \end{cases}$$

13. $y' + (2/t)y = M(t), \quad t > 0, \quad y(\pi/3) = 2$

$$M(x) = \begin{cases} (\sin t)/t^2, & 0 \le x \le 3, \\ 1/t^3 & x \ge 3. \end{cases}$$

Justification of the method:

1. The first step is to solve the homogenous DE (H) and obtain the general solution given by (3.1). We rewrite this solution as

$$x_h(t) = A\Phi_h(t) \tag{3.2}$$
$$\Phi_h(t) = \exp[-\int p(t)\,dt] \tag{3.3}$$

and A is our parameter. Notice that if we set $A = 1$, the $\Phi_h(t)$ itself is a solution of (H). That is

$$\Phi_h'(t) + p(t)\Phi_h(t) = 0$$

2. Now, we want to solve the non-homogenous equation (NH) ($g(t) \neq 0$). The *variation of parameter (A) method* says that *in order to account for the effect of the external force $g(t)$ we replace the parameter A by a function of $w(t)$*. That is, we look for a solution for the non-homogenous equation (NH) in the form

$$x(t) = \Phi_h(t)w(t)$$

If we substitute this $x(t)$ in (NH) we obtain

$$\begin{aligned}\Phi_h(t)w'(t) + \Phi_h'(t)w(t)\\ + p(t)\Phi_h(t)w(t) = g(t)\end{aligned}$$

factorizing $w(t)$, we obtain

$$\Phi_h(t)w'(t) + [\Phi_h'(t) + p(t)\Phi_h(t)]w(t) = g(t)$$

But we know that

$$\Phi_h'(t) + p(t)\Phi_h(t) = 0 \qquad (*)$$

Substituting in (*) we see that $w(t)$ satisfies the separable DE

$$\begin{aligned} w' =& \Phi_h(t)^{-1}g(t) = \frac{g(t)}{\Phi_h(t)} \qquad (3.4)\\ dw =& \frac{g(t)}{\Phi_h(t)}\, dt \end{aligned}$$

Integrating, we obtain

$$\begin{aligned} w(t) =& C + \int \Phi_h(t)^{-1}g(t)\, dt\\ =& C + \int \frac{g(t)}{\Phi_h(t)}\, dt \end{aligned}$$

3. **The general solution to (NH) takes the form**

$$\begin{aligned} x(t) =& \Phi_h(t)\left(C + \int \frac{g(t)}{\Phi_h(t)}\, dt\right) \qquad (3.5)\\ \Phi_h(t) =& \exp\left[-\int p(t)\, dt\right] \qquad (3.6) \end{aligned}$$

Remarks 3.1.5. We wrote the general solution of (NH) in several forms (3.5-3.7). We would like to emphasize the following:

1. The form (3.7) tells us that *the general solution of the non-homogenous* DE (NH) is the sum of *the general solution of the homogenous* DE (H) ($x_h(t) = \Phi_h(t)$) and *any particular solution* ($x_p(t)$) *of the nonhomogenous* DE (NH). This is a consequence of the linearity of the DE (NH).

2. The form (3.5) says that *the particular solution of the nonhomogenous* DE (NH), $x_p(t)$, can be constructed by replacing the parameter c in the general solution of the homogenous DE (H) by a function $w(t)$. This is what's known as the *variation of parameter method*.

3. In this way, we replace the the nonhomogenous DE (NH) by two separable equations, (H) and (3.4).

4. We can also write the general solution of (NH) as follows:

$$x(t) = C\Phi_h(t) + \Phi_h(t) \int \frac{g(t)}{\Phi_h(t)}\, dt \tag{3.7}$$

4 Second order homogenous linear DE's

Abbreviation: LHDE = linear homogenous DE. The simplest example of a HLDE is the motion of an undamped free spring. Undamped means there is no friction or viscosity in its ambiance. Free means that there is no external force affecting it.

4.1 Free undamped Spring-Mass

Consider *undamped* (no friction) *free* (no external force) motion.

$$\begin{aligned}
l &= \text{natural length of spring} \\
s &= \text{elongation of spring} \\
&\quad \text{in the downward (positive) direction} \\
&\quad \text{caused by the mass } m \\
L &= \text{ length during vibration.} \\
L &= l + s = \text{ the point of equilibrium.} \\
x &= L - (l + s) = \text{deviation from equilibrium point.} \\
m\ddot{x} &= F \qquad \text{Newton's equation} \\
F(x) &= \text{restoring force} + \text{ weight} \\
&+ \qquad \text{friction} + \text{ external force} \\
\text{friction} &= 0 \qquad \text{undamped motion.} \\
\text{external force} &= 0 \qquad \text{free motion.}
\end{aligned}$$

$$\begin{aligned}
\text{restoring force} &= -\,k(\text{elongation}) \\
F(x) &= -k(x + s) + mg \\
x &= 0, \text{equilibrium position } \Longrightarrow \\
F(0) &= 0 \Longrightarrow \\
ks &= mg \Longrightarrow \qquad \text{(I)} \\
m\ddot{x} &= -ks \Longrightarrow \\
&\qquad \text{undamped free motion} \\
\ddot{x} &= -\omega^2 x, \qquad (*) \\
\omega^2 &= \frac{k}{m} = \frac{g}{s}
\end{aligned}$$

The general solution of (*) is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t \tag{4.1}$$

We take $x = 0$ at the equilibrium position. We take the positive direction of x to be the downward direction. Therefore, if the mass is moving downward, its velocity is positive and if it is moving upward its velocity is negative.

4.2 Fundamental and general solutions.

A second order linear DE with constant coefficients takes the form

$$\text{(H)} \qquad a\ddot{x} + b\dot{x} + cx = 0, \qquad a \neq 0$$

The constants a, b and c are all real numbers. (What are the numbers that are not real numbers?)

Fundamental set of solutions

Two solutions $x_1(t)$ and $x_2(t)$
form a fundamental set of solutions for (H)
iff
any IVP
$a\ddot{x} + b\dot{x} + cx = 0, \quad x(t_o) = x_o, \dot{x}(t_o) = \dot{x}_o$
has a unique solution
which is a linear combination
$x(t) = c_1x_1(t) + c_2x_2(t)$

4.2.1. The characteristic equation. We saw earlier that the DE $\ddot{x} - \omega^2 x = 0$ has **two fundamental solutions** of the forms

$$x_1(t) = e^{\omega t}, \qquad x_2(t) = e^{-\omega t} \tag{4.2}$$

We also saw that DE $\ddot{x} + \omega^2 x = 0$ has **two fundamental solutions** of the forms

$$x_1(t) = \cos\omega t, \qquad x_2(t) = \sin\omega t \tag{4.3}$$

The two types of solutions are related by the **Euler formula**

$$e^{irt} = \cos rt + i \sin rt$$

It is reasonable then to look for solutions of (H) of the form e^{rt}. If we substitute $x(t) = e^{rt}$ in the HLDE (H) and simplify we obtain

$$e^{rt}(ar^2 + br + c) = 0$$

Since $e^{rt} \neq 0$ for all $t \in \mathbb{R}$, we divide by e^{rt} and obtain

the **characteristic equation**

$$\text{(CE)} \qquad ar^2 + br + c = 0$$

This is a quadratic equation with solutions given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Recall that the constants a, b and c are all real numbers. (What are the numbers that are not real numbers?) Thus solutions are either real or complex conjugates to each other. (What is the complex conjugate of $3 + 5i$?)

We have three different cases:

1. (CE) has two different real roots
$$r_1 \neq r_2$$
2. (CE) has repeated roots
$$r_1 = r_2 = \delta$$
3. (CE) has two complex conjugate roots
$$r = \alpha \pm i\beta, \quad \beta > 0$$

4.2.2. General solutions a HLDE. We have three cases

Case 1: (CE) has two different real roots $r_1 \neq r_2$:

$$\begin{aligned} \text{Fundamental solutions} \qquad & x_1(t) = e^{r_1 t}, \quad x_2(t) = e^{r_2 t} \\ \text{General solution} \qquad & x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \end{aligned} \tag{4.4}$$

Case 2: (CE) has one repeated real root $r_1 = r_2 = \delta$:

$$\begin{aligned} \text{Fundamental solutions} \qquad & x_1(t) = e^{\delta t}, \quad x_2(t) = te^{\delta t} \\ \text{General solution} \qquad & x(t) = c_1 e^{\delta t} + c_2 te^{\delta t} \\ & \quad = e^{\delta t}(c_1 + c_2 t) \end{aligned} \tag{4.5}$$

Case 3: (CE) has two complex conjugate roots $r = \alpha \pm i\beta, \beta \neq 0$:

$$\begin{aligned} \text{Fundamental solutions} \qquad & x_1(t) = e^{\alpha t}\cos\beta t, \qquad x_2(t) = e^{\alpha t}\sin\beta t \\ \text{General solution} \qquad & x(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t \\ & \quad = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t) \end{aligned} \tag{4.6}$$

4.3 Exercises

1. Show that in the three cases above, $x_1(t)$ and $x_2(t)$ are linearly independent. That is, they are not multiple of each other.
2. The graph of the solution of the DE $y'' + 4y' + 4y = 0$ goes through the two points $(1, e^{-2})$ and $(2, -e^{-4})$.

(a) Find $y(0)$ and $y'(0)$.

(b) Find the unique solution of the given DE that satisfy the initial conditions that you found.

3. Write each of the complex numbers in the form $\lambda + \mu i$.

$$3e^{i\pi/6}, \quad -4e^{-i\frac{\pi}{3}}, \quad (4-5i)e^{-i\sqrt{3}\,t},$$

4. Solve the following IVP's and

(a) Sketch the solution.

(b) Describe its long term behaviour.

(c) Assume that t is time and $y(t)$ is the position of a particle on the y-axis at time t.

i. For case 1 and 2 find the following when they exist.

A. The time the particle passes through the origin.

B. Its highes and lowest points and the times the particle reaches them.

ii. For case 3 (2 complex conjugate roots):

A. Find the amplitude A, the frequency ω, the period T and time shift δ.

B. Write the solution in the form $y(t) = A\sin(\omega t + \delta)$.

(a) $y'' + 2y' - 3y = 0, \qquad y(0) = 2, y'(0) = 10.$

(b) $6y'' - 7y' - 3y = 0, \qquad y(0) = -2, y'(0) = 3.$

(c) $y'' - 5y' = 0, \qquad y(0) = 1, y'(0) = 2.$

(d) $y'' + 3y' + 2y = 0, \qquad y(0) = 3, y'(0) = -2.$

(e) $y'' + 2y' + y = 0, \qquad y(0) = 1, y'(0) = 2.$

(f) $4y'' + 12y' + 9y = 0, \qquad y(0) = 1, y'(0) = -2.$

(g) $y'' + 4y' + 4y = 0, \qquad y(0) = 3, y'(0) = 2.$

(h) $y'' + 3y = 0, \qquad y(0) = 1, y'(0) = 0.$
$y'' + 3y = 0, \qquad y(0) = 0, y'(0) = 1.$

(i) $y'' + 2y' + 2y = 0, \qquad y(0) = 0, y'(0) = 1.$
$y'' + 2y' + 2y = 0, \qquad y(0) = 1, y'(0) = 0.$

(j) $y'' + y' + y = 0, \qquad y(0) = -1, y'(0) = 2.$

(k) $y'' + 4y' + 5y = 0, \qquad y(0) = 3, y'(0) = -2.$

(l) $y'' + 4y' + 6.25y = 0, \qquad y(\pi/4) = -3, y'(\pi/4) = -1.$

(m) $y'' + y' + 1.25y = 0, \qquad y(0) = 3, y'(0) = 1.$

5 Nonhomogenous linear DE's with constant coefficients

5.1 The General solution

Consider the nonhomogeneous linear DE

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_o) = r, y'(t_o) = s \tag{NH}$$

- **Step 1. Find the general solution of the homogeneous DE (H).**

$$y'' + p(t)y' + q(t)y = 0 \tag{H}$$

  - Find two fundamental solutions of (H).

$$y_1(t) = \cdots \quad \& \quad y_2(t) = \cdots$$

  - The general solution of (H) is

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

- **Step 2. Find any particular solution of (NH).**

$$z(t) = \cdots$$

- **Step 3. The general solution of (NH) is**

$$y(t) = y_H(t) + z(t)$$
$$y(t) = c_1 y_1(t) + c_2 y_2(t) + z(t)$$

- **Step 4. Find c_1 and c_2 from initial the conditions.**

5.2 Method of undetermined coefficients

We use this method when the nonhomogeneous term $g(t)$ is of the form

$$g(t) = e^{3t}, te^{-2t}, e^{-5t}\cos(2t), \cdots$$

5.2.1. Example. Solve the IVP

$$y'' - y' - 6y = 7e^{\sqrt{2}t}, \qquad y(0) = 1, y'(0) = 3 \tag{NH}$$

- **Step 1. Find the general solution of the homogeneous DE (H).**

$$y'' - y' - 6y = 0 \tag{H}$$

  - Find two fundamental solutions of (H).

* *Characteristic equation*: $r^2 - r - 6 = 0 \longrightarrow (r+2)(r-3) = 0$
* *Roots of characteristic equation*: $r_1 = -2, r_2 = 3$.
* Fundamental solutions of (H):

$$y_1(t) = e^{-2t}, \qquad y_2(t) = e^{3t}$$

– The general solution of (H) is

$$H(t) = c_1 y_1(t) + c_2 y_2(t)$$
$$H(t) = c_1 e^{-2t} + c_2 e^{3t}$$

- **Step 2. Find any particular solution of (NH).** Since $e^{\sqrt{2}t} \neq y_1(t), y_2(t)$, we try a solution of the form

$$z(t) = Ae^{\sqrt{2}t}$$

Determine the constant A by substituting $z(t)$ in (NH).

Recall that if we substitute $x(t) = Ae^{st}$ in

$$L[x] = ax'' + bx' + cx$$

we obtain

$$L[Ae^{rt}] = Ae^{rt}(ar^2 + br + c)$$

Thus, if we substitute $z(t) = Ae^{\sqrt{2}t}$ in the left hand side of (Nh) we obtain

$$Ae^{\sqrt{2}t}((\sqrt{2})^2 - \sqrt{2} - 6 = 7e^{\sqrt{2}t}$$

Thus,

$$A = \frac{-7}{4+\sqrt{2}}$$

And we have a particular solution

$$z(t) = \frac{-7}{4+\sqrt{2}}\, e^{\sqrt{2}t}$$

- **Step 3. The general solution of (NH) is**

$$y(t) = H(t) + z(t)$$
$$y(t) = c_1 e^{-2t} + c_2 e^{3t} + \frac{-7}{4+\sqrt{2}}\, e^{\sqrt{2}t}$$

- **Step 4. Find c_1 and c_2 from initial the conditions.** Exercise.

5.2.2. Example. Solve the IVP

$$y'' - y' - 12y = 5e^{-3t}, \qquad y(0) = 1, y'(0) = 3 \qquad \text{(NH)}$$

- **Step 1. Find the general solution of the homogeneous DE (H).**

$$y'' - y' - 12y = 0 \tag{H}$$

  - Find two fundamental solutions of (H).
    - *Characteristic equation*: $r^2 - r - 12 = 0 \longrightarrow (r+3)(r-4) = 0$
    - *Roots of characteristic equation*: $r_1 = -3, r_2 = 4$.
    - Fundamental solutions of (H):

$$y_1(t) = e^{-3t}, \qquad y_2(t) = e^{4t}$$

  - The general solution of (H) is

$$y_H(t) = c_1 y_1(t) + c_2 y_2(t)$$
$$y_H(t) = c_1 e^{-3t} + c_2 e^{4t}$$

- **Step 2. Find any particular solution of (NH).** Since $e^{-3t} = y_1(t)$ but $e^{-3t} \neq y_2(t)$, we take

$$z(t) = Ate^{-3t}$$

Determine the constant A by substituting $z(t)$ in (NH).

Recall that if we substitute $x(t) = Ate^{rt}$ in

$$L[x] = ax'' + bx' + cx$$

we obtain

$$L[Ae^{bt}] = Ae^{st}[(ar^2 + br + c)t + (2ar + b)]$$

  - Notice that the coefficient of t is the characteristic polynomial

$$P(r) = ar^2 + br + c$$

  - If we substitute one of the roots in $P(r)$ we obtain zero. That is, in our example, if we substitute $r = -3$ in $r^2 - r + 12$ we should get zero.
  - Thus, if r is a root of the characteristic equation and we substitute $x(t) = Ate^{rt}$ in

$$L[x] = ax'' + bx' + cx$$

  we obtain

$$L[Ate^{rt}] = Ae^{st}(2ar + b)$$

Thus, in our example, if we substitute $z(t) = Ae^{-3t}$ in the left hand side of (NH) we obtain

$$Ate^{-3t}(2(-3) - 1) = 5e^{-3t}$$

Thus,

$$A = \frac{-5}{7}$$

And we have a particular solution

$$z(t) = \frac{-5}{7}\, te^{-3t}$$

- **Step 3. The general solution of (NH) is**

$$y(t) = y_H(t) + z(t)$$
$$y(t) = c_1 e^{-3t} + c_2 e^{4t} - \frac{5}{7}\, t e^{-3t}$$

- **Step 4. Find c_1 and c_2 from initial the conditions.** Exercise.

5.2.3. Exercises: Method of undetermined coefficients.

1. In each of the following find a particular solution $z(t)$. Then solve the IVP. (Find the roots and $z(t)$ and check them against the given ones).

   (a) $y'' - y' - 6y = 7e^{5t}$, $\quad y(0) = 2, y'(0) = -3$
   $r = -2, 3$, $\quad z(t) = (-7/10)e^{5t}$.

   (b) $y'' - y' - 6y = 7e^{-2t}$, $\quad y(0) = 2, y'(0) = -3$
   $z(t) = (-7/5)te^{-2t}$.

   (c) $y'' - y' - 6y = 8\cos 2t$, $\quad y(0) = 2, y'(0) = -3$
   $z(t) = (-10/13)\cos 2t + (-2/13)\sin 2t$.

   (d) $y'' - y' + y = 2\sin 3t$, $\quad y(0) = 2, y'(0) = -3)$
   $r = (1/2) \pm (\sqrt{3}/2)i$, $\quad z(t) = (6/73)\cos 3t + (-16/73)\sin 3t$.

   (e) $y'' - 2y' - 3y = 4x - 5 + 4xe^{2x}$, $\quad y(0) = -1, y'(0) = 2,$, $\quad (r = -1, 3)$
   $z(x) = -(4/3)x + (23/9) - (2x + 4/3)e^{2x}$.

2. Solve each of the following IVP's:

   (a) $y'' - y' - 6y = 7e^{\sqrt{2}t} - 5e^{4t}$, $\quad y(0) = 1, y'(0) = 3$, $\quad (r = -2, 3)$.

   (b) $y'' - y' - 6y = 7e^{-2t} - 6e^{4t}$, $\quad y(0) = 4, y'(0) = -1$.

   (c) $y'' - y' - 6y = 7\sin 5t$, $\quad y(0) = 4, y'(0) = -1$.

3. Solve each of the following IVP's:

   (a) $y'' + 7y' + 12y = 7e^{-4t}$, $\quad y(0) = 1, y'(0) = 3$, $\quad (r = -3, -4)$.

   (b) $y'' + 7y' + 12y = -6te^{-3t}$, $\quad y(0) = 4, y'(0) = -1$.

   (c) $y'' + 7y' + 12y = 7\sin 5t$, $\quad y(0) = 4, y'(0) = -1$.

   (d) $y'' + 7y' + 12y = 3t^2e^{5t}$, $\quad y(0) = -1, y'(0) = 2$.

4. Find the general solutions to the following:

   (a) $y'' + 4y' + 3y = 7e^{-2t} - 6e^{4t}$, $\quad (r = -1, -3)$.

   (b) $y'' - 6y' + 9y = 7e^{-2t} - 6e^{4t}$, $\quad (r = -3, -3)$.

   (c) $y'' - 6y' + 9y = 7e^{3t}$.

   (d) $y'' - 6y' + 9y = 7te^{3t}$.

5. In each of the following the general solution of the DE $\ddot{x} + b\dot{x} + cx = g(t)$ is given where c_1 and c_2 are constants. Find b, c and $g(t)$.

   (a) $x(t) = c_1 e^{-2t} + c_2 e^{3t} + 5e^{4t}$.

   (b) $x(t) = c_1 e^{-2t} + c_2 t e^{-2t} + 5e^{4t}$.

   (c) $x(t) = c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t) - 5te^{-t}$.

5.2.4. How to find the particular solution $z(t)$:

If $g(t) = (\textit{polynomial})e^{rt}$
take
$z(t) = t^k$(polynomial of the same degree)e^{rt}
where k is the smallest integer
so that
all terms are different from the fundamental solutions $y_1(t)$ and $y_2(t)$

If $g(t) = (\textit{polynomial})e^{\alpha t}\cos\beta t$
take
$z(t) = t^k$(polynomial of the same degree)$e^{\alpha t}\cos\beta t$
$+\ t^k$(polynomial of the same degree)$e^{\alpha t}\sin\beta t$
where k is the smallest integer
so that
all terms are different from the fundamental solutions $y_1(t)$ and $y_2(t)$

If $g(t) = (\textit{polynomial})e^{\alpha t}\sin\beta t$
take
$z(t) = t^k$(polynomial of the same degree)$e^{\alpha t}\cos\beta t$
$+\ t^k$(polynomial of the same degree)$e^{\alpha t}\sin\beta t$
where k is the smallest integer
so that
all terms are different from the fundamental solutions $y_1(t)$ and $y_2(t)$

5.3 Method of variation of parameters

Consider the second order non-homogenous linear DE

$$y'' + p(t) + y + q(t)y = g(t) \tag{NH}$$

Assume that we found a fundamental set of solutions $\{y_1(t), y_2(t)\}$ for the associated homogenous DE

$$y'' + p(t) + y + q(t)y = 0 \tag{H}$$

And assume that $p(t), q(t)$ and $g(t)$ are continuous in an open Interval $a < t < b$.
Then a particular solution to (NH) is of the form

$$z(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{5.1}$$

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1(t), y_2(t))}\, dt$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1(t), y_2(t))}\, dt$$

The general solution of (NH) is

$$\begin{aligned} y(t) &= H(t) + z(t) \\ y(t) &= (c_1 + u_1(t))\, y_1(t) + (c_2 + u_2(t))\, y_2(t) \end{aligned} \tag{5.2}$$

We can also write the general solution oin the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \int_{t_o}^{t} \frac{y_1(\lambda)y_2(t) - y_2(\lambda)(y_1(t)}{W(y_1(\lambda), y_2(\lambda))}\, g(\lambda)d\lambda \tag{5.3}$$

We can also write the general solution oin the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \int_{t_o}^{t} \frac{\begin{vmatrix} y_1(\lambda) & y_2(\lambda) \\ y_1(t) & y_2(t) \end{vmatrix}}{W(y_1(\lambda), y_2(\lambda))}\, g(\lambda)d\lambda \tag{5.4}$$

Question: What happens if we write $\displaystyle\int_{t_o}^{t} \cdots \, d\lambda$ in (5.1)?

5.3.1. Exercises.

1. In each of the following

   - Find the general solution.
   - If initial condition are give, solve the IVP.

- In some problems a set of solutions is given. In this case verify that they are solutions and that the set is a set of fundamental solutions.
- Some problems has more questions. Answer them.

(a) $x'' - 2x' + x = \dfrac{e^t}{t^2+1}$.

(b) $y'' + 9y = 5\tan(3t)$.

(c) $y'' + 4y' + 4y = t^{-2}e^{-2t}, \qquad t > 0$.

(d) $ty'' - (t+1)y' + y = t^2, \qquad y_1(t) = e^t, y_2(t) = t + 1$.

(e) $y'' + y = \sec t, \qquad -\pi/2 < t < \pi/2$.

   i. What about $y'' + y = \sec(2t)$?

   ii. What about $y'' + 9y = \sec(3t)$? What can you conclude from these three DE's?

(f) $y'' + 16y = \sec^2 4t, \qquad -\pi/8 < t < \pi/8$.
Why do we have to restrict the domain of t to the given one?

(g) $x^2y'' - x(x+2)y' + (x+2)y = 2x^3, \qquad x > 0$.
$y_1(x) = x, y_2(x) = xe^x$.

(h) $tx'' - (1+t)x' + x = t^2e^{2t}, \qquad t > 0$.
$x_1(t) = 1 + t, x_2(t) = e^t$.

(i) $y'' + 4y = 7\sec 2t, \qquad 0 < t < \pi/2$.
Why do we have to restrict the domain of t to the given one?

(j) $y'' + 2y' + y = e^{-t}\ln t$.

(k) $t^2y'' + 3ty' - 3y = \dfrac{1}{t^3}, \qquad t > 0$.
Hint: Look for fundamental solutions of the form t^k.

(l) $4ty'' + 2y' - y = 4\sqrt{t}\, e^{\sqrt{t}}, \qquad y_1(t) = e^{\sqrt{t}}, t_2(t) = e^{-\sqrt{t}}$.

(m) $y'' + y = \sin^2 t$.

(n) $t^2y'' - 3ty' + 4y = t^2\ln t, \qquad t > 0$.

(o) $y'' + y' = \ln t$.

(p) $t^2y'' - 2ty' + 2y = t^3, \qquad t > 0$.

   - What about $t^2y'' - 2ty' + 2y = t^r, \qquad t > 0$? That is, find the r for which the method of the variation of parameters gives us an explicit solution and find this solution.
   - Can we use the method of undetermined coefficients for this problem?

(q) $y'' + y' = \dfrac{1}{2+\sin t}$.

(r) $x^2y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\sin x, \quad x > 0$,
$y_1(x) = x^{-1/2}\sin x, y_2(x) = x^{-1/2}\cos x$.

2. Consider the IVP

$$y'' + \alpha y' + \beta y = g(t), \qquad y(0) = y_o, y'(0) = v_o$$

In each of the following the solution to IVP is given. Find α, β, y_o and v_o and.

(a) $y(t) = \frac{1}{2}\int_0^t \sin(2(t-\lambda))g(\lambda)\, d\lambda.$

(b) $y(t) = t + \int_0^t (t-\lambda)g(\lambda)\, d\lambda.$

6 General Linear DE's

6.1 Linear O D E

1. Recall that an equation of a plane in $\mathbb{R}^3$ is given by
$$cx + by + az = d \tag{6.1}$$
where a, b and c are constants.

2. If think of t as time and $x(t)$ as the position at time t of a particle moving on the x axis. Then,$x'(t)$ and $x''(t)$ are its velocity and speed respectively. Let's form a vector out of these three quantities:
$$< x(t), x'(t), x''(t) >$$

3. Set
$$y = x', \qquad z = x''$$
Suppose that the vector
$$< x(t), x'(t), x''(t) >$$
satisfies (6.1), that is,
$$cx + bx' + ax'' = d \tag{6.2}$$
This means that the particle moves on the x-axis in such a way that the vector $< x(t), x'(t), x''(t) >$ always lies on the plane (6.1).

4. In this case, the differential equation (6.2) is called a *linear differential equation* of order 2, provided that $a \neq 0$.

5. We can also allow a, b and c to be functions of time. In this case, the plane (6.1) changes with time.

The general form of a second order linear differential equation is
$$a(t)x'' + b(t)x' + c(t)x = f(t) \qquad a(t) \qquad \neq 0 \tag{6.3}$$
Since we are studying the DE (6.3) when $a(t) \neq 0$, we can divide the equation by $a(t)$ and write $p(t) = b(t)/a(t)$, $q(t) = c(t)/a(t)$ and $g(t) = f(t)/a(t)$.

6.1.1. The standard form of a second order linear DE's.

$$(NH) \quad \ddot{x} + p(t)\dot{x} + q(t)x = g(t)$$

It is called homogenous iff
$g(t) \equiv 0$

That is

$$(H) \quad \ddot{x} + p(t)\dot{x} + q(t)x = 0.$$

Question: For which initial conditions (t_o, x_o, x'_o) do we have a unique solution? The answer is given by the following theorem:

Theorem 6.1.2. *Consider the two LDE's*

$$\ddot{x} + p(t)\dot{x} + q(t)x = g(t) \tag{NH}$$
$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{H}$$
$$t_1 < t < t_2$$

Assume that $p(t), q(t)$ and $g(t)$ are continuous for $t_1 < t < t_2$. Let $t_1 < t_o < t_2$.

Then, each of the DE (NH) and (H) has a unique solution $x(t)$ for every IC $x(t_o) = x_o, \dot{x}(t_o) = \dot{x}_o$ and this solution is defined on the interval $t_1 < t < t_2$. Moreover, this solution is twice differentiable. That is, $\dot{x}(t)$ and $\ddot{x}(t)$ are both continuous.

6.1.3. Remark. Linear DE's of any higher order can be written and treated in a similar fashion.

6.1.4. Declaration. In all what follows we assume that $p(t)$, $q(t)$ and $g(t)$ are continuous in the interval $t_1 < t < t_2$.

6.1.5. Continuous functions. For all practical purposes in this course, to make sure that a function is *continuous* avoid the following:

1. Dividing by zero.
2. $\sqrt{\text{negative }\#}$, $\sqrt[4]{\text{negative }\#}$, $\sqrt[6]{\text{negative }\#}, \cdots,$ $\sqrt[2k]{\text{negative }\#}, \cdots$.
3. $\ln(\text{negative }\#)$.
4. $\ln 0$.
5. Jumps if the function is defined by two formulae.

6.1.6. Exercises Find the longest interval on which the IVP is certain to have a t unique twice differentiable solution:

1. Compare the following four cases:

   (a) $(t^2 - 9)y'' + \dfrac{3}{t}\, y' - (\sin t)y = e^{2t-3}, \qquad y(2) = 5, y'(2) = 7.$

   (b) $(t^2 - 9)y'' + \dfrac{3}{t}\, y' - y = e^{2t-3}, \qquad y(-2) = 5, y'(-2) = 7.$

(c) $(t^2-9)y'' + \dfrac{3}{t}\, y' - y = e^{2t-3}, \qquad y(4) = 5, y'(4) = 7.$

(d) $(t^2-9)y'' + \dfrac{3}{t}\, y' - y = e^{2t-3}, \qquad y(-6) = 5, y'(-6) = 7.$

2. $t(t-5)x'' + t^2\, x' - x = 7e^{2t-3}, \qquad x(3) = 35, x'(3) = 74.$
3. $(t-3)x'' + \ln|t|\, x' - x = 7e^{2t-3}, \qquad x(2) = 45, x'(2) = 29.$
4. $(x+7)(\tan 3x)u'' + \ln|t|\, u' - u = t^3 + 7t - 3, \qquad u(2) = 4, u'(2) = 9.$
5. $(x+7)(\tan 3x)u'' + \ln|t|\, u' - u = t^3 + 7t - 3, \qquad u(-4) = 4, u'(-4) = 9.$

6.2 Fundamental solutions and the Wronskian.

6.2.1. Linearly independent functions. Two continuous functions $f(t)$ and $g(t)$ with $t_1 < t < t_2$, are said to be linearly independent if the are not multiple of each other.

Theorem 6.2.2. *Let $f(t)$ and $g(t)$ be two differentiable functions defined on the interval $t_1 < t < t_2$.*

Then $f(t)$ and $g(t)$ are linearly independent iff there is $t_1 < t_o < t_2$ such that the determinant

$$\begin{vmatrix} f(t_o) & g(t_o) \\ f'(t_o) & g'(t_o) \end{vmatrix} \neq 0$$

6.2.3. The Wronskian. The determinant

$$W(f,g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)g'(t) - g(t)f'(t)$$

is called the Wronskian of $f(t)$ and $g(t)$.

6.2.4. Exercises In each of the following determine whether the given two functions are linearly independent in two different ways, first using the definition and second using the Wronskian:

1. e^{5t}, e^{-3t}.
2. e^{2t}, te^{2t}.
3. $\cos 3t, \sin 3t$.
4. $e^{5t}\cos 2t, e^{-3t}\sin 2t$.
5. x, e^{-3x}.

Theorem 6.2.5 (Abel's Theorem)**.** *Consider the LHDE (H). Assume that $p(t)$ and $q(t)$ are continuous for $t_1 < t < t_2$. Let $t_1 < t_o < t_2$. Let $x_1(t)$ and $x_2(t)$ be two solutions of (H). Then*

$$W(x_1(t), x_2(t)) = W(x_1(t_o), x_2(t_o)) \ \exp[-\int_{t_o}^{t} p(s)ds]$$

6.2.6. Fundamental solutions for LHDE's. Consider the LHDE

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0, \qquad t_1 < t < t_2 \tag{H}$$

Fundamental set of solutions
Two solutions $x_1(t)$ and $x_2(t)$ of(H)
forma fundamental set of solutions
iff
any solution of an IVP
$\ddot{x} + p(t)\dot{x} + q(t)x = 0$
$x(t_o) = x_o, \dot{x}(t_o) = \dot{x}_o$
has a unique solution which is a linear combination
$x(t) = c_1x_1(t) + c_2x_2(t)$

Theorem 6.2.7. *Consider the LHDE (H). Assume that $p(t)$ and $q(t)$ are continuous for $t_1 < t < t_2$. Let$t_1 < t_o < t_2$.*

Two solutions $x_1(t)$ and $x_2(t)$
are fundamental solutions of (H)
iff
$W(x_1, x_2)(t_o) \neq 0$
for some $t_1 < t_o < t_2$

6.3 The general solution of a LHDE.

Abbreviation: LHDE = linear homogenous DE.

6.3.1. The principle of superposition. Consider the LHDE

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{H}$$

If $x_1(t)$ and $x_2(t)$ are two solutions of (H)
then so is
$ax_1(t) + bx_2(t)$
where a and b are any two numbers.

6.3.2. Exercises

1. Verify that $x_1(t) = t^{-1}$ and $x_2(t) = t^2$ are solutions of the DE $t^2\ddot{x} - 2x = 0$. Then show that $at^{-1} + bt^2$ is also a solution for the DE.

2. Verify that $x_1(t) = t^{1/2}$ and $x_2(t) = 1$ are solutions of the DE $x\ddot{x} + (\dot{x})^2 = 0, t > 0$. Then show that $at^{-1} + bt^2$ is not a solution of the DE. Explain.

7 The Laplace Transform:

7.0.3. Table

$$\begin{aligned}
f(t) &\xrightarrow{\mathcal{L}} F(s), && s > a\\
f'(t) &\xrightarrow{\mathcal{L}} sF(s) - f(0), && s > a\\
f''(t) &\xrightarrow{\mathcal{L}} s(F(s) - f(0)) - f'(0), && s > a
\end{aligned}$$

$$\begin{aligned}
1 &\xrightarrow{\mathcal{L}} \frac{1}{s}, && s > 0\\
t &\xrightarrow{\mathcal{L}} \frac{1}{s^2}, && s > 0\\
t^n &\xrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}}, && s > 0\\
e^{at} &\xrightarrow{\mathcal{L}} \frac{1}{s-a}, && s > a\\
\cos bt &\xrightarrow{\mathcal{L}} \frac{s}{s^2+b^2}, && s > 0\\
\sin bt &\xrightarrow{\mathcal{L}} \frac{b}{s^2+b^2}, && s > 0\\
e^{at}\cos bt &\xrightarrow{\mathcal{L}} \frac{s-a}{(s-a)^2+b^2}, && s > a\\
e^{at}\sin bt &\xrightarrow{\mathcal{L}} \frac{b}{(s-a)^2+b^2}, && s > a\\
e^{ct}f(t) &\xrightarrow{\mathcal{L}} F(s-c), && s - c > a\\
tf(t) &\xrightarrow{\mathcal{L}} -\frac{dF}{ds}(s), && s > a\\
u_c(t) &\xrightarrow{\mathcal{L}} \frac{e^{-sc}}{s}, && c > 0, s > 0\\
u_c(t)f(t-c) &\xrightarrow{\mathcal{L}} e^{-sc}F(s), && c > 0, s > a
\end{aligned}$$

$$\begin{aligned}
\delta(t-c) &\xrightarrow{\mathcal{L}} e^{-sc}, && c > 0, s > 0\\
f(t)\delta(t-c) &\xrightarrow{\mathcal{L}} f(c)e^{-sc}, && c > 0, s > 0\\
\cosh bt &\xrightarrow{\mathcal{L}} \frac{s}{s^2-b^2}, && s > |b|\\
\sinh bt &\xrightarrow{\mathcal{L}} \frac{b}{s^2-b^2}, && s > |b|
\end{aligned}$$

$$\begin{array}{ccc}
f(t) & \xrightarrow{\mathcal{L}} & F(s) \\
\downarrow & & \downarrow \qquad\qquad s > ? \\
f'(t) & \xrightarrow{\mathcal{L}} & sF(s) - f(0)
\end{array}$$

$$\begin{array}{ccc} f(t) & \xrightarrow{\mathcal{L}} & F(s) \\ \big\downarrow & & \big\downarrow \\ u_c(t)f(t-c) & \xrightarrow{\mathcal{L}} & e^{-cs}F(s) \end{array} \qquad s>?$$

$$\begin{array}{ccc} f(t) & \xrightarrow{\mathcal{L}} & F(s) \\ \big\downarrow & & \big\downarrow \\ e^{ct}f(t) & \xrightarrow{\mathcal{L}} & F(s-c) \end{array} \qquad s-c>?$$

$$\begin{array}{ccc} f(t) & \xrightarrow{\mathcal{L}} & F(s) \\ \big\downarrow & & \big\downarrow \\ tf(t) & \xrightarrow{\mathcal{L}} & -\tfrac{dF}{ds}(s) \end{array} \qquad s>?$$

$$\begin{array}{ccc} f(t) & \xrightarrow{\mathcal{L}} & F(s) \\ \big\downarrow & & \big\downarrow \\ f(ct) & \xrightarrow{\mathcal{L}} & \tfrac{1}{c}F(\tfrac{s}{c}) \end{array} \quad c>0, \qquad \frac{s}{c}>?$$

7.0.4. Useful integrals

$$\int te^{at}\,dt = \frac{1}{a}te^{at} - \frac{1}{a^2}e^{at}dt + C$$

$$\int t^2e^{at}\,dt = \frac{1}{a}t^2e^{at} - \frac{2}{a^2}te^{at} + \frac{2}{a^3}e^{at} + C$$

$$\int t^ne^{at}\,dt = \frac{1}{a}t^ne^{at} - \frac{n}{a}\int t^{n-1}e^{at}dt + C$$

$$\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt) + C$$

$$\int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt) + C$$

8 Autonomous DE's

Consider the following differential equation :

$$\dot{x} = f(x), \qquad x(t_o) = a \tag{8.1}$$

where $\dot{x} = \dfrac{dx}{dt}$.

Notice that $f(x)$ does not depend on time.

In this case the system is called **autonomous**.

1. Think of the independent variable t as time and the dependent variable x as the position of a particle in space. In this way, $f(x)$ gives the instantaneous velocity of any particle passing through the point x.
2. This means that in an autonomous system, the velocity field (vector field) $f(x)$ does not depend on time. That is the velocity $\dot{x}$ depends only on the *position* x of the particle and not on *when* the particle passes through the point x.
3. Physically, this means that if we repeat an experiment at a latter time, we obtain exactly the same results if we reset our clocks.
4. If $f(x_o) = 0$, any particle at the point x_o will have velocity $\dot{x} = 0$. Therefore, it will never move. And the solution of (8.1) with IC $x(t_o) = x_o$ is $\phi(t) = x_o, t \in \mathbb{R}$.

Definition 8.0.5. 1. A point x_o at which

$$f(x_o) = 0$$

is called a rest point.

2. The solution $\phi(t) = x_o, t \in \mathbb{R}$ of (8.1), is called an equilibrium solution or a trivial solution.

Exercise 8.0.6. Find the equilibrium solution for the following DE's:

1. $\dot{x} = x(x-2)(x+3)$
2. $\dot{x} = \sin x$
3. $\dot{x} = (x-7)\ln(x-2)$

It is not a good idea to try to find a solution to an equation if there is none. If several people try to find the solution to the same equation, it will be nice if they all find the same solution.
The following Theorem 8.0.7 gives us conditions that guarantee the existence of a unique solution to an autonomous system.

Theorem 8.0.7 (Autonomous DE)**.** *Consider the autonomous system (8.1) and assume that $f(x)$ and $f'(x)$ are continuous in a certain (open) domain G.*

1. *Then for every $a \in G$ and $t_o \in \mathbb{R}$, there is a unique solution $x(t), t_- < t < t_+$, to the IVP (8.1) defined on a time-interval $t_- < t_o < t_+$.*

2. *Solution curves don't intersect. If two solution curves did intersect, there would be two solutions through the point of intersection. This would violate uniqueness.*

3. *It takes nonequilibrium solutions infinitely long to reach rest points.*
   That is,
   nonequilibrium solutions never reach rest points in finite time
   and hence they never cross rest points.

Remark 8.0.8. For the purpose of this course, in order to find the regions where a function is continuous, we need to avoid undefined quantities such as

$$\frac{1}{0}, \quad \ln(0), \quad \ln(negative\,number)$$

$$\sqrt[n]{negative\,number}, n = even\,integer$$

$$jumps$$

Example 8.0.9. Consider the following autonomous system on the line:

$$\frac{dx}{dt} = x - x^2, \qquad x(0) = x_o \tag{8.2}$$

Notice that if we consider the DE with IC

$$x(t_o) = 0, 1$$

then we obtain the two equilibrium solutions

$$\phi_1(t) = 0, \qquad t \in \mathbb{R}$$

and

$$\phi_2(t) = 1, \qquad t \in \mathbb{R}$$

respectively. For $x \neq 0, 1$, we proceed as follows:

$$\frac{dx}{x(1-x)} = dt$$

Using partial fractions, we obtain

$$\int \left(\frac{1}{x} + \frac{1}{1-x} \right) dx = \int dt$$

Integrating, we obtain

$$\ln|x| - \ln|1-x| = t + c$$

$$\ln\left|\frac{x}{1-x}\right| = t + c$$
$$\left|\frac{x}{1-x}\right| = e^c e^t$$
$$\frac{x}{1-x} = Ae^t, \qquad A = \pm e^c$$

The constant $\pm e^c$ depends on IC. We give it a simpler name $A = \pm e^c$.
To get the general solution, we need to simplify and get
We can find A from the IC $x(0) = x_o$. Now the general solution is

$$\begin{aligned} \phi_{x_o}(t) &= \frac{Ae^t}{1+Ae^t}, \\ A &= \frac{x_o}{1-x_o}, \quad x_o \neq 0, 1 \\ \phi_0(t) &= 0, \qquad -\infty < t < \infty, \quad x_o = 0 \\ \phi_1(t) &= 1, \qquad -\infty < t < \infty, \quad x_o = 1 \end{aligned} \tag{8.3}$$

Notice that the function
$$\ln\left|\frac{x}{1-x}\right|$$
is not defined at $x = 0, 1$. But we are avoiding these two points anyway.
Question. What happens to the general solution (8.3) as $x_o \to 0$ and $x_o \to 1$? Notice that

$$\begin{aligned} &\lim_{x_o \to 0} A = 0 \\ &\lim_{x_o \to 1^-} A = \infty, \qquad \lim_{x_o \to 1^+} A = -\infty \\ &\lim_{A \to \pm\infty} \phi_{x_o}(t) = 1 \end{aligned}$$

8.1 Phase portrait of an autonomous ODE

Now we try to generalize the ideas of Example 8.0.9 and consider the autonomous system,

$$\dot{x} = f(x) \tag{8.4}$$

1. Rest points and equilibrium solutions.
2. Classifying rest points and equilibrium solutions:
   (a) Stable rest point (asymptotically stable, attractor, sink).
   (b) Unstable rest point (repeller, source).
   (c) Semistable rest point (node).
3. Derivative test for classifying rest points.

4. Transversal and nontransversal intersections.

5. Basin of attraction of a stable rest point.

6. Phase portrait.

7. Long term behaviour of solutions.

8. Observable rest points.

Example 8.1.1. Consider the DE

$$\dot{x} = (x-1)(3-x)^5(6-x)^3(x-8)^2(x-9)(x-11)$$

1. Sketch the phase portrait of the differential equation.

2. Find and classify the rest points.

3. Find and classify equilibrium solutions.

4. Find the basin of attraction of each stable rest point.

5. Let $x(t)$ is the concentration of a certain substance at time t during a chemical reaction. What are the achievable (observable) concentrations in this experiment? Why?

6. Find the long term behaviour of the solutions with initial condition : initial conditions:
$$x(0) = 0.3, 1.5, 2.3, 3.7, 5.7, 7.1, 8.8, 9.4, 12.13$$

Example 8.1.2. The following autonomous DE has no critical points but has a singularity a t $x = 3$:

$$\frac{dx}{dt} = \frac{1}{3-x}, \qquad x \neq 3$$

1. Sketch the phase portrait and analyze the long term behaviour of solutions.

2. Show that every solution reach the singular point $x = 3$ in finite time.

3. If t is time and x is the position of a particle on a straight line, how do you describe the behaviour of solutions near the singular point $x = 3$?

Example 8.1.3. Consider the DE

$$\dot{x} = (x-7)\ln(x-4)$$

1. Where is the velocity field defined.

2. Find and classify rest points using the first derivative test.

3. Sketch the phase portrait.

4. Find the basin of attraction of each stable rest point.

Example 8.1.4. The following autonomous DE has a singularity at $x = -2$:

$$\frac{dx}{dt} = -x\ln(x+2), \qquad x > 2$$

1. Sketch the phase portrait and describe the long term behaviour of solutions.
2. What happens to particles with initial positions $-2 < x(0) < -1$?

Example 8.1.5 (A semisimple rest point)**.** Consider the following DE with a small parameter ε:

$$\dot{x} = (x-1)^2 + \varepsilon$$

1. Sketch the phase portrait for the three different cases $\varepsilon < 0, \varepsilon = 0$ and $\varepsilon > 0$.
2. Describe the long term behaviour of solutions in the three cases.
3. Study the long term behaviour of the point $x = 1$ in the three cases.

8.2 Attractors, repellers, basin of attraction

9 Population Dynamics

Let $x(t)$ be the number of rabbits in a forest at time t. Assume that there are no foxes nor French chefs around.

9.1 Exponential growth model

It is reasonable to assume that the rate of growth of rabbits is proportional to their number. That is,

$$\dot{x} = rx, \qquad r > 0 \tag{9.1}$$

The phase portrait shows that $x = 0$ is an unstable equilibrium and that if we start with any positive number rabbits

$$\lim_{t\to\infty} x(t) = \infty$$

This means that if we wait long enough, the whole universe will be filled with rabbits. But this is not what we see in reality to the disappointment of foxes and French chefs. The reason for that is that the forest has a limited amount of food. So, if the number of rabbits exceeds a certain number $k > 0$, which is called the carrying capacity, rabbits will decrease in numbers because of shorter life span due to starvation and because of lack of interest in romance. Therefore, we need another model that takes into account the carrying capacity of the habitat. Before getting into that notice equation (9.1) is a separable equation whose solution is

$$x(t) = x(0)e^{rt} \tag{9.2}$$

This is why the model (9.1) is called *exponential growth* model.

9.2 Logistic growth model

$$\dot{x} = rx(1 - \frac{x}{k})\,, \qquad r > 0, k > 0 \tag{9.3}$$

The phase portrait shows that we have two rest points:

1. $x_1 = 0$, which is unstable. Is this good or bad?
2. $x_2 = k$, which is stable.

9.3 Logistic growth model with harvesting

Now what happens if a French chef shows up with his pet-fox?

$$\dot{x} = rx(1 - \frac{x}{k}) - h\,, \qquad r > 0, k > 0 \tag{9.4}$$

$$\dot{x} = rx(1 - \frac{x}{k}) - hx\,, \qquad r > 0, k > 0 \tag{9.5}$$

9.4 Logistic growth model with a threshold

$$\dot{x} = rx(x-a)(1-\frac{x}{k}), \qquad r>0, k>0 \tag{9.6}$$

9.5 Logistic growth model with a threshold and harvesting

$$\dot{x} = rx(x-a)(1-\frac{x}{k}) - h \tag{9.7}$$
$$r>0, k>0, a>0$$

$$\dot{x} = rx(x-a)(1-\frac{x}{k}) - hx \tag{9.8}$$
$$r>0, k>0, a>0$$

10 Homogenous Planar Linear Systems of DE's

In this section we study *linear systems of first order DE.*

Example 10.0.1. Solve the IVP

$$\begin{aligned} \dot{x} &= 3x \\ \dot{y} &= -5y \\ x(2) = 1/2, &\qquad y(2) = 7 \end{aligned} \tag{10.1}$$

Solution. These two equations are decoupled. That is there is no interaction between x and y. We know that the general solutions are

$$\begin{aligned} x(t) &= e^{3t}c_1 \\ y(t) &= e^{-5t}c_2 \end{aligned} \tag{10.2}$$

We can find (c_1, c_2) from IC

$$c_1 = (1/2)e^{-6}, \qquad c_2 = 7e^{10}$$

And the solution to the IVP is

$$\begin{aligned} \xi(t) &= (1/2)e^{3(t-2)} \\ \eta(t) &= 7e^{-5(t-2)} \end{aligned}$$

Example 10.0.2. Solve the IVP

$$\begin{aligned} \dot{x} &= -4x - 2y \\ \dot{y} &= 3x + 3y \\ x(2) = -1, &\qquad y(2) = 3 \end{aligned} \tag{10.3}$$

Notice that the two equations are coupled. That is, the x and y infulance each other.

Finding a method.
We can still look for solutions that look like the solutions of Example 10.0.1:

$$\begin{aligned} x(t) &= e^{\lambda t}p \\ y(t) &= e^{\lambda t}q \end{aligned} \tag{10.4}$$

Notice that the exponent λ is also an unknown together with (p, q). Not just the parameters (p, q).

If we substitute the functions (10.4) in the DE (10.3) we obtain

$$\begin{aligned} \lambda e^{\lambda t}p &= e^{\lambda t}(-4p - 2q) \\ \lambda e^{\lambda t}q &= e^{\lambda t}(3p + 3q) \end{aligned}$$

Simplifying,

$$\begin{aligned} (-4-\lambda)p - 2q &= 0 \\ 3p + (3-\lambda)q &= 0 \end{aligned} \tag{10.5}$$

These are equations of two lines through the origin (because the y-intercepts = 0):

$$\begin{aligned} q &= \frac{-4-\lambda}{2}p \\ q &= \frac{-3}{3-\lambda}p \end{aligned}$$

The only time we have a non-zero solution is when the two lines are identical because both lines go through the origin. This happens when they have the same slope:

$$\frac{-4-\lambda}{-2} = \frac{3}{3-\lambda}$$

Step 1: Wiriting down the charachteristic equation

$$(-4-\lambda)(3-\lambda) = (-2)(3) \tag{10.6}$$

Simplifying, we obtain

$$\lambda^2 + \lambda - 6 = 0$$

This quadratic equation has two solutions

$$\lambda_1 = 2, \qquad \lambda_2 = -3 \tag{10.7}$$

Notice

$$\begin{aligned} &(-4-\lambda)(3-\lambda) - (-2)(3) \\ &= \begin{vmatrix} -4-\lambda & -2 \\ 3 & 3-\lambda \end{vmatrix} \\ &= \det(A - \lambda I) \\ A &= \begin{pmatrix} -4 & -2 \\ 3 & 3 \end{pmatrix} \end{aligned}$$

To obtain solutions of the form (10.4) we need to find a vector $\mathbf{E} = (p, q)^\intercal$ for each λ. These are called **eignvectors**

Step 2: Find the eigenvectors.

$\boxed{\lambda_1 = 2}$ We substitute $\lambda_1 = 2$ in the two equations in (10.5) and get the line

$$\begin{aligned} q &= \frac{-4-2}{2}p \Longrightarrow q = -3p \\ q &= \frac{-3}{3-2}p \Longrightarrow q = -3p \end{aligned} \tag{10.8}$$

So, both equations give the same line $q = -3p$. Any point on the line $q = -3p$ will do. We chose $p = 1$. Then $q = -3$.

Very important sub-step. Check to make sure that $(p, q) = (1, -3)$ satisfy the second equation of the system (10.5). If not, something is wrong. And we start again by checking the equation (10.6).
If $(p, q) = (1, -3)$ satisfy the second equation of the system (10.5), we take

$$\mathbf{E}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$\boxed{\lambda_2 = -3}$ We repeat the same with $\lambda_2 = -3$ and substitute in the first equation in (10.5) and get the line

$$q = \frac{-4+3}{2} p$$

Any point on the line will do. We chose $p = 2, q = -1$.

Very important sub-step. Check to make sure that $(p, q) = (2, -1)$ satisfy the second equation of the system (10.5). If not, something is wrong. And we start again by checking the equation (10.6).
If $(p, q) = (2, -1)$ satisfy the second equation of the system (10.5), we take

$$\mathbf{E}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Step 3: Find two fundamental solutions.

$$\begin{aligned} \mathbf{v}_1(t) &= e^{2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} e^{2t} \\ -3e^{2t} \end{pmatrix} \\ \mathbf{v}_2(t) &= e^{-3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-3t} \\ -e^{-3t} \end{pmatrix} \end{aligned} \tag{10.9}$$

Notice that neither $\mathbf{v}_1(t)$ nor $\mathbf{v}_2(t)$ satisfy the IC.

Step 4: **<u>The general solution</u>** is a linear combination of $\mathbf{v}_1(t)$ of $\mathbf{v}_2(t)$:

$$\begin{aligned} \mathbf{z}(t) &= c_1 \mathbf{v}_1(t) + c_2 \mathbf{v}_2(t) \\ \mathbf{z}(t) &= c_1 e^{2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= c_1 \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-3t} \\ -e^{-3t} \end{pmatrix} \\ \mathbf{z}(t) &= \begin{pmatrix} e^{2t} & 2e^{-3t} \\ -3e^{2t} & -e^{-3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

$$\mathbf{z}(t) = \mathbf{\Phi}(t)\mathbf{c}$$
$$\mathbf{z}(t) = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \mathbf{c}$$

Step 5: **Find** $(c_1, c_2)^\intercal$ **from IC.** For this we need:
The inverse of a matrix. If A is a 2×2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det A = ad - bc \neq 0$$

it has a multiplicative inverse given by

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{10.10}$$

Exercise. Check that
$$AA^{-1} = A^{-1}A = I$$

Back to finding $(c_1, c_2)^\intercal$ from IC:

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \mathbf{z}(2) = \mathbf{\Phi}(2)\mathbf{c} \tag{\textbf{Step 5}}$$
$$\mathbf{c} = \mathbf{\Phi}(2)^{-1} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} e^4 & 2e^{-6} \\ -3e^4 & -e^{-6} \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^4 & -2e^{-6} \\ -3e^4 & e^{-6} \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$= \frac{1}{-5e^{-2}} \begin{pmatrix} e^{-6} & 2e^{-6} \\ 3e^4 & e^4 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 3e^{-4}/5 \\ -e^6/5 \end{pmatrix}$$

Step 6: **Solution to the IVP.**

$$\mathbf{q}(t) = \begin{pmatrix} e^{2t} & 2e^{-3t} \\ -3e^{2t} & -e^{-3t} \end{pmatrix} \begin{pmatrix} 3e^{-4}/5 \\ -e^6/5 \end{pmatrix}$$
$$= \begin{pmatrix} 3e^{2(t-2)}/5 - 2e^{-3(t-2)}/5 \\ -9e^{2(t-2)}/5 + e^{-3(t-2)}/5 \end{pmatrix}$$

Definition 10.0.3. λ_1 and λ_2 are called eigenvalues and $\mathbf{E}_1$ and $\mathbf{E}_2$ are called the corresponding eigenvectors.

10.0.4. **The method.** Given a system

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \tag{H}$$

Step 1.

Find the eigenvalues by solving the characteristic equation

$$(a - \lambda)(d - \lambda) = bc \tag{10.11}$$

This equation corresponds to equation (10.7) in our example.

For the second step we have three cases.

The case $\lambda_1 \neq \lambda_2$ and real

Step 2.

1. **Find an eigenvector $\mathbf{E}_1$ for λ_1** by solving

$$(a - \lambda_1)p + bq = 0 \tag{10.12}$$
$$cp + (d - \lambda_1)q = 0 \tag{10.13}$$

   This system corresponds to system (10.8).

   Before you solve the system, check that

$$(a - \lambda_1)(d - \lambda_1) - bc = 0 \tag{10.14}$$

   If not, something is wrong. check your calculations from the start.

   If it is true, solve the first equation and make sure that your solution satisfies the second one.

   We obtain

$$\mathbf{E}_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

2. Repeat the same for λ_2: **Find an eigenvector $\mathbf{E}_2$ for λ_2** by solving

$$(a - \lambda_2)p + bq = 0 \tag{10.15}$$
$$cp + (d - \lambda_2)q = 0 \tag{10.16}$$

   Make sure that

$$(a - \lambda_2)(d - \lambda_2) - bc = 0 \tag{10.17}$$

   If true, we solve the system and we get

$$\mathbf{E}_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$$

Step 3.
Find two fundamental solutions

$$\begin{aligned} \mathbf{v}_1(t) &= e^{\lambda_1 t}\mathbf{E}_1 \\ \mathbf{v}_2(t) &= e^{\lambda_2 t}\mathbf{E}2 \end{aligned} \tag{10.18}$$

Step 4.
Find the general solution

$$\begin{aligned} \mathbf{z}(t) &= c_1\mathbf{v}_1(t) + c_2\mathbf{v}_2(t) \\ &= c_1 e^{\lambda_1 t}\mathbf{E}_1 + c_2 e^{\lambda_2 t}\mathbf{E}_2 \end{aligned} \tag{10.19}$$

Step 5.
Determine the parameters $(c_1, c_2)^\intercal$ **from IC.**

Step 6.
Solution to the IVP. Substitute the the parameters $((c_1, c_2)^\intercal$ that you found in **Step 5** back in the general solution to obtain the unique solution to the IVP.

10.0.5. The fundamental matrix solution
Notice that

The general solution of (H) $\dot{\mathbf{x}} = A\mathbf{x}$

$$\mathbf{z}(t) = \begin{pmatrix} \mathbf{v}_1(t) & \mathbf{v}_2(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Define

The fundamental matrix solution

$$\mathbf{\Phi}(t) = \begin{pmatrix} \mathbf{v}_1(t) & \mathbf{v}_2(t) \end{pmatrix}$$

Then

The general solution of (H) $\dot{\mathbf{x}} = A\mathbf{x}$

$$\mathbf{z}(t) = \mathbf{\Phi}(t)\mathbf{c}$$

When $\lambda_1 \neq \lambda_2$, real

$$\mathbf{\Phi}(t) = \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Then

The general solution of (H) $\dot{\mathbf{x}} = A\mathbf{x}$
when $\lambda_1 \neq \lambda_2$, real

$$\mathbf{z}(t) = \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Exercise 10.0.6. Find the fundamental matrix solutions for Examples 10.0.1, 10.0.2 and 10.0.7.

Example 10.0.7. Solve the IVP

$$\begin{aligned} \dot{x} &= 3x - 2y \\ \dot{y} &= -3x + 4y \\ x(0) &= -2, \qquad y(0) = 1 \end{aligned}$$

Solution
Step 1: Find the eigenvalues. Solve

$$\begin{aligned} (a-\lambda)(d-\lambda) - bc &= 0 \\ (3-\lambda)(4-\lambda) - 6 &= 0 \\ \lambda^2 - 7\lambda + 6 &= 0 \\ (\lambda - 1)(\lambda - 6) &= 0 \end{aligned}$$

Thus,

$$\lambda_1 = 1, \qquad \lambda_2 = 6$$

Step 2: Find the corresponding eigenvectors. This system has two distinct real eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 6$.

For each λ check

$$(a-\lambda)(d-\lambda)-bc=0$$

If true, solve

$$(a-\lambda)p+bq=0$$

For $\lambda_1 = 1$, solve

$$(3-1)p-2q=0$$

We get

$$\mathbf{E}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_1 = 6$, solve

$$(3-6)p-2q=0$$

We get

$$\mathbf{E}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Step 3: Find two fundamental solutions.

$$\mathbf{v}_1(t) = e^{\lambda_1 t}\mathbf{E}_1, \qquad \mathbf{v}_2(t) = e^{\lambda_2 t}\mathbf{E}_2$$

$$\mathbf{v}_1(t) = e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2(t) = e^{6t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Step 4: Find the general solution.

$$\mathbf{z}(t) = c_1\mathbf{v}_1(t) + c_2\mathbf{v}_2(t)$$

$$\mathbf{z}(t) = c_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Step 5: Find the parameters $(c_1, c_2)^{\intercal}$ form IC. Exercise.
Step 6: Find the unique solution to the IVP. Exercise.

10.0.8. Matrix notation. We use matrix notation to write the system of Example 10.0.7.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

We also write the system in the form

$$\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$A = \begin{pmatrix} 3 & -2 \\ -3 & 4 \end{pmatrix}$$
$$\dot{\mathbf{z}} = A\mathbf{z}$$
$$\mathbf{z}(0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Definition 10.0.9.

1. Two linearly independent functions $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ that are solutions for the system (H) are called *fundamental solutions*
2. The matrix $\mathbf{\Phi}(t) = \begin{pmatrix}\mathbf{u}_1(t) & \mathbf{u}_2(t)\end{pmatrix}$ is called a *fundamental matrix solution.* Notice that $\mathbf{\Phi}(t)$ satisfies the same linear DE. That is,

$$\dot{\mathbf{\Phi}}(t) = A\mathbf{\Phi}(t)$$

The two functions $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$, given by (10.18), are two linearly independent solutions of the system (H).

10.0.10. Phase portrait
Sketch phase portrait for all the systems that you solve.

10.0.11. DE with complex conjugate eigenvalues.
Consider the planar linear DE

$$\dot{\mathbf{z}} = A\mathbf{z} \tag{10.20}$$

Assume that A has two complex conjugate eigenvalues $\lambda = \alpha + i\beta$ and $\bar{\lambda}$.
Notice that all the (arithmetic) calculations we did in the case of two distinct real eigenvalues $\lambda_1 \neq \lambda_2$, depended only on the fact that $\lambda_1 \neq \lambda_2$ and could be carried out with complex numbers.

1. Thus, we find an eigenvector for $\lambda_1 = \alpha - i\beta, \beta > 0$.

$$\mathbf{E}_1 = \mathbf{E} = \mathbf{N}_1 + i\mathbf{N}_2$$

Since the matrix A is real, it follows that

$$\mathbf{E}_2 = \bar{\mathbf{E}} = \mathbf{N}_1 - i\mathbf{N}_2$$

is an eigenvector for $\lambda_2 = \bar{\lambda}$

2. Now, we have two *complex invariant-line solutions* in the two dimensional complex space which can be identified with a four dimensional real space. These two solutions are

$$\begin{aligned} \mathbf{q}_1(t) &= e^{(\alpha - i\beta)t}(\mathbf{N}_1 + i\mathbf{N}_2) \\ &= e^{\alpha t}(\cos\beta t - i\sin\beta t)(\mathbf{N}_1 + i\mathbf{N}_2) \\ &= e^{\alpha t}[\mathbf{N}_1\cos\beta t + \mathbf{N}_2\sin\beta t)] \\ &\quad + e^{\alpha t}[-\mathbf{N}_1\sin\beta t + \mathbf{N}_2\cos\beta t] \\ &= \mathbf{u}_1(t) + i\mathbf{u}_2(t) \\ \mathbf{q}_2(t) &= \bar{\mathbf{q}}_1(t) \\ &= \mathbf{u}_1(t) - i\mathbf{u}_2(t) \end{aligned} \tag{10.21}$$

where,

$$\begin{aligned} \mathbf{u}_1(t) &= e^{\alpha t}[\mathbf{N}_1\cos\beta t + \mathbf{N}_2\sin\beta t] \\ \mathbf{u}_2(t) &= e^{\alpha t}[-\mathbf{N}_1\sin\beta t + \mathbf{N}_2\cos\beta t] \end{aligned}$$

3. Notice that $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are real valued functions.

Exercise 10.0.12. Show that

$$\begin{pmatrix} \mathbf{u}_1(t) & \mathbf{u}_2(t) \end{pmatrix} = e^{\alpha t}\begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \end{pmatrix}\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}$$

Exercise 10.0.13. 1. Show that

$$\begin{aligned} \mathbf{u}_1(t) &= \frac{1}{2}(\mathbf{q}_1 + \mathbf{q}_2) \\ \mathbf{u}_2(t) &= \frac{1}{2i}(\mathbf{q}_1 - \mathbf{q}_2) \end{aligned}$$

2. Show that $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are solutions for (10.20).
3. Show that $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are linearly independent for all $t \in \mathbb{R}$.
4. It follows that $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are two fundamental solutions for the system (10.20).

Now the general solution is

$$\begin{aligned} \mathbf{z}(t) &= b_1\mathbf{q}_1(t) + b_2\mathbf{q}_2(t) && (10.22) \\ &= (b_1 + b_2)\mathbf{u}_1(t) + i(b_1 - b_2)\mathbf{u}_2(t) \\ &= c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) && (10.23) \end{aligned}$$

where b_1 and b_2 are complex constants but c_1 and c_2 are real constants.

Summary of the complex case:

If we have two complex conjugate eigenvalues

$$\lambda_1 = \alpha - i\beta, \qquad \lambda_2 = \bar{\lambda}_1 = \alpha + i\beta$$

we proceed as follows:

1. Find an eigenvector for $\lambda_1 = \alpha - i\beta, \beta > 0$

   $$\mathbf{E}_1 = \mathbf{N}_1 + i\mathbf{N}_2$$

   Then,

   $$\mathbf{E}_2 = \bar{\mathbf{E}}_1 = \mathbf{N}_1 - i\mathbf{N}_2$$

   is an eigenvector for $\lambda_2 = \bar{\lambda}_1$.

2. Thus, we have a real general solution

   $$\begin{aligned}\mathbf{z}(t) &= c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) \\ &= \begin{pmatrix} u_1(t) & u_2(t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= e^{\alpha t}\begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \end{pmatrix}\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}\mathbf{c} \\ &= e^{\alpha t}\mathbf{N}\mathbf{R}(\beta t)\mathbf{c} \\ &= \mathbf{M}\mathbf{L}(t)\mathbf{c} \\ &= \mathbf{\Phi}(t)\mathbf{c}\end{aligned} \tag{10.24}$$

   where c_1 and c_2 are real constants.

3. It is obvious that the fundamental matrix solution

   $$\mathbf{\Phi}(t) = e^{\alpha t}\mathbf{N}\mathbf{R}(\beta t)$$

   has an inverse

   $$e^{-\alpha t}\mathbf{R}(-\beta t)\mathbf{N}^{-1}$$

   Thus, we can determine the vector parameter $\mathbf{c}$ uniquely from any initial condition.

4. The matrix

   $$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   is a counter-clockwise rotation with angle θ.

Exercise 10.0.14. In each of the following:

1. Find and sketch the fundamental solutions.
2. Find the general solution .
3. Solve the IVP.
4. Sketch the phase portrait.

Solve the following IVP and sketch the solutions

1.

$$\begin{aligned}\dot{x} &= -5x + y\\ \dot{y} &= -2x - 3y\\ x(0) = 1&, \quad y(0) = 2\end{aligned}$$

2.

$$\begin{aligned}\dot{x} &= 2x - 5y\\ \dot{y} &= 4x - 2y\\ x(0) = 2&, \quad y(0) = 3\end{aligned}$$

3.

$$\begin{aligned}\dot{x} &= x - 2y\\ \dot{y} &= 2x + y\\ x(0) = -2&, \quad y(0) = 1\end{aligned}$$

10.0.15. DE with a repeated eigenvalue.
Consider the planar linear DE

$$\dot{\mathbf{z}} = A\mathbf{z} \tag{10.25}$$

Assume that A has a repeated eigenvalue

$$\mu = \frac{tr(A)}{2}$$

This happens iff

$$(tr(A))^2 = 4\det(A)$$

1. Find an eigenvector $\mathbf{E}_1$ as before. That is, solve the equation

$$\begin{aligned}[A - \mu I]\mathbf{E}_1 &= 0 \qquad (10.26)\\ A\mathbf{E}_1 &= \mu\mathbf{E}_1\end{aligned}$$

   Since we are considering a planar system, we will be able to find only one linearly independent eigenvector.

2. Now we look for another vector $\mathbf{E}_2$ satisfies

$$A\mathbf{E}_2 = \mu\mathbf{E}_2 + \mathbf{E}_1$$

   That is, $\mathbf{E}_2$ must satisfy

$$[A - \mu I]\mathbf{E}_2 = \mathbf{E}_1 \tag{10.27}$$

3. **Exercise.** Show that $\mathbf{E}_1$ and $\mathbf{E}_2$ are linearly independent.

4. We have one invariant-line solution
$$\mathbf{u}_1(t) = e^{\mu t}\mathbf{E}_1 \tag{10.28}$$

5. A second solution is
$$\mathbf{u}_2(t) = e^{\mu t}(\mathbf{E}_2 + t\mathbf{E}_1) \tag{10.29}$$

6. Notice that
$$\mathbf{r}(t) = \mathbf{E}_2 + t\mathbf{E}_1, \quad -\infty < t < \infty$$
is an equation of a straight line parallel to $\mathbf{E}_1$.

7. Then the general solution is
$$\begin{aligned}
\mathbf{z}(t) &= c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) \qquad (10.30)\\
&= \begin{pmatrix} \mathbf{u}_1(t) & \mathbf{u}_2(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\\
&= e^{\mu t} \begin{pmatrix} \mathbf{E}_1 & t\mathbf{E}_1 + \mathbf{E}_2 \end{pmatrix} \mathbf{c}\\
&= e^{\mu t} \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mathbf{c}\\
&= \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 \end{pmatrix} \begin{pmatrix} e^{\mu t} & te^{\mu t} \\ 0 & e^{\mu t} \end{pmatrix} \mathbf{c}\\
&= \mathbf{M}\mathbf{L}(t)\mathbf{c}\\
&= \mathbf{\Phi}(t)\mathbf{c}
\end{aligned}$$

Definition 10.0.16. The vector $\mathbf{E}_2$ is called a *generalized eigenvector.*

Exercise 10.0.17. Show that $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are fundamental solutions.

Remark: This question can be rephrased as follows:
Show that $\mathbf{\Phi}(t)$ is a fundamental matrix solution.
In either case we need to show that $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are solutions and are linearly independent. To show that we need to show that the matrix $\mathbf{\Phi}(t) = \mathbf{M}\mathbf{L}(t)$ has an inverse. But this will be true if we show that each of $\mathbf{M}$ and $\mathbf{L}(t)$ has an inverse.

Exercise 10.0.18. Consider the case $\lambda < 0$. Show that as $t \to \infty$, the solution $\mathbf{u}_2(t) \to \mathbf{0}$ tangential to the the invariant line solution $\mathbf{u}_1(t)$.
Then show that as $t \to -\infty$, $\|\mathbf{u}_2(t)\| \to \infty$ in such a way that $\mathbf{u}_2(t)$ is almost parallel to $\mathbf{u}_1(t)$.
Hint: What are the dominant terms as $t \to \pm\infty$?

Exercise 10.0.19. Sketch and describe each of the two fundamental solutions (10.28) and (10.29) for the three cases $\lambda < 0, \lambda = 0$ and $\lambda > 0$.

Exercise 10.0.20. In each of the following:

1. Find and sketch the fundamental solutions.

2. Find the general solution .

3. Solve the IVP.

4. Sketch the phase portrait.

Solve the following IVP and sketch the solutions

1.
$$\begin{aligned} \dot{x} &= -2x + y \\ \dot{y} &= -x - 4y \\ x(0) = 1&, \quad y(0) = 2 \end{aligned}$$

2.
$$\begin{aligned} \dot{x} &= x - 2y \\ \dot{y} &= 2x + 5y \\ x(0) = 2&, \quad y(0) = 3 \end{aligned}$$

3.
$$\begin{aligned} \dot{x} &= 7x + y \\ \dot{y} &= -4x + 3y \\ x(0) = -2&, \quad y(0) = 1 \end{aligned}$$

10.0.21. Summary of the three cases:
To find the general solution to the real planar system

$$\dot{\mathbf{z}} = A\mathbf{z}$$

a Find the *eigenvalues* of the matrix A.

b Find the corresponding *eigenvectors* or *generalized eigenvectors.*

c Write down the *fundamental matrix solution* and the *general solution.*

To do that, we write down the characteristic equation

$$\lambda^2 - tr(A)\lambda + \det(A) = 0 \tag{10.31}$$

The quadratic equation (10.31) has two solutions. There are three cases:

1. Two distinct real roots $\lambda_1 \neq \lambda_2$:
   In this case we can find two linearly independent eigenvectors $\mathbf{E}_1$ and $\mathbf{E}_2$. Then, the *fundamental matrix solution* and the *general solution* are given by (10.19).

2. Two complex roots $\lambda_{\pm} = \alpha \pm i\beta$:
   In this case we can find two complex conjugate eigenvectors $\mathbf{E}_1 = \mathbf{N}_1 + i\mathbf{N}_2$ and $\mathbf{E}_2 = \mathbf{N}_1 - i\mathbf{N}_2$. The the real *fundamental matrix solution* and the *general solution* are given by (10.24).

3. One repeated real root $\lambda = tr(A)/2$:
   In this case we have only one *eigenvector* $\mathbf{E}_1$. Then we can find a *generalized eigenvector* $\mathbf{E}_2$ by solving

   $$[A - \lambda I]\mathbf{E}_2 = \mathbf{E}_1$$

   The general solution is given by (10.30).

10.0.22. Shortcuts for finding eigenvectors in the planar case.
Short cut for finding eigenvectors when $\lambda_1 \neq \lambda_2$, in 2-D: Here is a shortcut for finding eigenvectors that works only for 2 matrices. The explanation is given after Theorem 10.0.24.

Compute

$$\boxed{A - \mu_1 I = \begin{pmatrix}\mathbf{b}_1 & \mathbf{c}_1\end{pmatrix}}$$

$$\boxed{A - \mu_2 I = \begin{pmatrix}\mathbf{b}_2 & \mathbf{c}_2\end{pmatrix}}$$

Then take

$$\boxed{\mathbf{E}_1 = \mathbf{b}_2, \qquad \mathbf{E}_2 = \mathbf{b}_1}$$

This is not a typo.
The indices in both cases are switched.
Short cut for finding eigenvectors for two complex conjugate eigenvalues, in 2-D:

1. The **shortcut** that we used for the case $\lambda_1 \neq \lambda_2$ works for the case of two complex conjugate eigenvalues because $\mu \neq \bar{\mu}$.

2. So, to find $\mathbf{E}_1$ that corresponds to $\lambda_1 = \mu = \alpha - i\beta, \beta > 0$, we compute

   $$\boxed{A - (\alpha + i\beta)I = \begin{pmatrix}\mathbf{b}_2 & \mathbf{c}_2\end{pmatrix}}$$

   And then take

   $$\boxed{\mathbf{E}_1 = \mathbf{b}_2}$$

   Now, since we are dealing with complex eigenvalue, $\mathbf{b}_2$ will be complex too. So, we write

   $$\boxed{\mathbf{E}_1 = \mathbf{E} = \mathbf{N}_1 + i\mathbf{N}_2, \qquad \mu = \lambda_1 = \alpha - i\beta}$$

3. Now, since $\lambda_2 = \bar{\mu}$, the complex conjugate of μ, we expect the eigenvector of $\bar{\mu}$ to be

   $$\boxed{\mathbf{E}_2 = \bar{\mathbf{E}} = \mathbf{N}_1 - i\mathbf{N}_2}$$

Short cut for finding eigenvectors for a repeated eigenvalue, in 2-D:

1. Pick your favourite vector in $\mathbb{R}^2$ and call it $\mathbf{E}_2$.

2. Then, compute $\mathbf{E}_1$ from

   $$\boxed{\mathbf{E}_1 = (A - \alpha I)\mathbf{E}_2}$$

3. There is a remote possibility that you get $E_1 = \mathbf{0}$. If this happens, just pick another E_2 that is not parallel to your first pick. It is enough to change one component. Now compute E_1 again.

4. Couldn't be easier.

10.0.23. Why these shortcuts work:

Theorem 10.0.24. *[Cayley-Hamilton] A square matrix A satisfies its characteristic equation.*

Consider a 2×2 real matrix A.
Suppose that A has two distinct eigenvalues $\lambda_1 \neq \lambda_2$.

1. In this case the characteristic equation
$$\lambda^2 - tr(A)\lambda + \det(A) = 0$$
can be factorized to
$$(\lambda - \mu_1)(\lambda - \mu_2) = 0 \tag{10.32}$$
The Cayley-Hamilton Theorem 10.0.24 says that
$$(A - \mu_1 I)(A - \mu_2 I) = 0$$
which can be written as
$$A(A - \mu_2 I) = \mu_1(A - \mu_2 I) \tag{10.33}$$
Now, view $(A - \mu_2 I)$ as two column vectors
$$A - \mu_2 I = \begin{pmatrix} B_2 & C_2 \end{pmatrix}$$
We know that $\det(A - \mu_2 I) = 0$, which means that
$$C_2 = cB_2$$
for some scalar $c \neq 0$.

2. Now equation (10.33) takes the form
$$A\begin{pmatrix} B_2 & C_2 \end{pmatrix} = \mu_1 \begin{pmatrix} B_2 & C_2 \end{pmatrix}$$
Thus,
$$\begin{pmatrix} AB_2 & cAC_2 \end{pmatrix} = \begin{pmatrix} \mu_1 B_2 & c\mu_1 B_2 \end{pmatrix}$$
This means that
$$AB_2 = \mu_1 B_2 \tag{10.34}$$

3. **What does (10.34) mean?** It means that B_2 is an eigenvector corresponding to μ_1. It follows that B_2 is also an eigenvector corresponding to μ_1. Why?

4. **The moral of the story:** In order to find an eigenvector corresponding to μ_1, we compute $(A - \mu_2 I)$ and take either of its two columns.

5. **Exercise:** How do you find an eigenvector corresponding to μ_2 in the same fashion?

Suppose that A has a repeated eigenvalue μ.

1. In this case the characteristic equation

$$\lambda^2 - tr(A)\lambda + \det(A) = 0$$

can be factorized to

$$(\lambda - \mu)^2 \equiv 0 \tag{10.35}$$

The Cayley-Hamilton Theorem 10.0.24 says that

$$(A - \mu I)^2 = 0$$

2. Recall that we are looking for two vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ satisfying

$$(A - \mu I)\mathbf{E}_1 = 0 \tag{a}$$
$$(A - \mu I)\mathbf{E}_2 = \mathbf{E}_1 \tag{b}$$

Thus,

$$(A - \mu I)^2\mathbf{E}_2 = 0$$

But we know from (10.35) that

$$(A - \mu I)^2 \equiv 0$$

Since we are dealing with a planar system, any vector $\mathbf{E}_2 \in \mathbb{R}^2$ will do. But then, (b) tells us that we can take

$$\mathbf{E}_1 = [A - \lambda I]\mathbf{E}_2$$

11 Nonhomogenous Planar Linear Systems

In this section we study the nonhomogenous planar system

$$\dot{\mathbf{z}} = A\mathbf{z} + \mathbf{f}(t) \tag{NH}$$

The corresponding homogenous system is

$$\dot{\mathbf{z}} = A\mathbf{z} \tag{H}$$

The difference between (NH) and (H):

a The homogenous DE (H) represents the evolution of the system under the influence of *its intrinsic forces.*

b The nonhomogenous DE (NH) represents the evolution of the system when it is subjected to some *external excitation* $\mathbf{f}(t)$.

We know that the general solution to the homogenous (H) is

$$\mathbf{z}_H(t) = \mathbf{\Phi}(t)\mathbf{c} \tag{SH}$$

with the vector-parameter $\mathbf{c}$.

11.1 Variation of parameter method:

1. We conjecture that the general solution of the nonhomogenous (NH) takes the form

$$\mathbf{z}(t) = \mathbf{\Phi}(t)\mathbf{u}(t) \tag{SNH}$$

   In other words, to take into account the effect of the *external excitation* $\mathbf{f}(t)$, we replace the vector-parameter $\mathbf{c}$ by a function of $\mathbf{u}(t)$. Of course $\mathbf{u}(t)$ will depend on the *external excitation* $\mathbf{f}(t)$.

2. To try to find whether this is possible, we substitute (SNH) into (NH) and obtain

$$\begin{aligned}
\mathbf{\Phi}(t)\dot{\mathbf{u}}(t) &= \mathbf{f}(t) \\
\dot{\mathbf{u}}(t) &= \mathbf{\Phi}(t)^{-1}\mathbf{f}(t) \\
\mathbf{u}(t) &= \mathbf{c} + \int_a^t \mathbf{\Phi}(s)^{-1}\mathbf{f}(s)\,ds \\
\mathbf{u}(t) &= \mathbf{c} + \int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)\,dt \\
\mathbf{u}(t) &= \mathbf{c} + \int_{t_o}^t \mathbf{\Phi}(s)^{-1}\mathbf{f}(s)\,ds
\end{aligned}$$

   Thus,

$$\mathbf{z}(t) = \mathbf{\Phi}(t)[\mathbf{c} + \int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)\,dt] \tag{11.1}$$

$$= \mathbf{\Phi}(t)[\mathbf{c} + \int_a^t \mathbf{\Phi}(s)^{-1}\mathbf{f}(s)\,ds]$$
$$= \mathbf{\Phi}(t)[\mathbf{c} + \int_{t_o}^t \mathbf{\Phi}(s)^{-1}\mathbf{f}(s)\,ds]$$

where a and t_o can be any real numbers. As usual, we express $\mathbf{z}(t)$ in several equivalent forms, each of which is useful in a way. Notice that the vector-parameter $\mathbf{c}$ varies from one expression to the other.

3. To show that (11.1) is really the general solution, we need to show that *given any initial condition* $z(t_o) = z_o$, *we can determine the vector-parameter* $\mathbf{c}$ *uniquely from* (11.1). The crucial point in showing that is that *the fundamental matrix solution* $\mathbf{\Phi}(t)$ *has an inverse for all* $t \in \mathbb{R}$. Now,

$$\mathbf{c} = \mathbf{\Phi}(t_o)^{-1}z_o - \int_a^{t_o} \mathbf{\Phi}(s)^{-1}\mathbf{f}(s)\,ds$$

$$\begin{array}{ccc} f(t) & \xrightarrow{\mathcal{L}} & F(s) \\ \text{left}\big\downarrow & & \big\downarrow\text{right} \\ e^{ct}f(t) & \xrightarrow{\text{abcd}} & F(s-c) \\ & \xrightarrow{\mathcal{L}} & \end{array}$$