# A HITCHHIKER'S GUIDE TO BUSINES MATH - II ©

MOHAMED SAMI ELBIALY

[1] Comments and examples included in these notes are not sufficient for a complete course on elementary differential equations. They do not replace the lecture notes nor the textbook. They are mere elaborations on some of the concepts, ideas, techniques, examples and exercises that are discussed in the text and lectures. You are expected to attend all the lectures and take notes and study them and read the textbook and do all the recommended exercise.

Contents



*Date*: June 14, 2005.
[1]Mathematics Department
University of Toledo
Toledo, Ohio 43606
U.S.A.

## 1. Review

Do the problem in parenthesis in class. The others as a homework.

**§5.1:** (1, 7, 11), 15, 17, 21, 31, 41.
**§5.2:** (1, 11), 5, 15, 27, 37.
**§5.3:** (23, 29), 31.
**§5.4:** (7, 15), 11, 17.
**§5.5:** (5, 7 , 13).
**§3.2:** (3-7), 11, 25-28, 29.
**§3.3:** 7-15.
**§3.4:** 3-11.
**§3.5:** 7, 15, 19, 27, 29, 33, 35.

## 2. Graphs and Derivatives

### 2.1. **Increasing and decreasing functions.**

2.1.1.

***Example.*** For each functions find

(1) the critical points.
(2) The critical values (numbers).
(3) The open intervals on which the function is increasing.
(4) The open intervals on which the function is decreasing.

$$f(x) = 4x^3 - 9x^2 - 30x + 6$$

We find the derivative $f'(x)$ then solve the equation $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 12x^2 - 18x - 30 \\ &= 6(2x^2 - 3x - 5) \\ &= 6(x+1)(2x-5) \end{aligned}$$

We have two critical points

$$x_1 = -1, \qquad x_2 = \frac{5}{2}$$

We have two critical values

$$y_1 = f(-1) = ---, \quad y_2 = f(\frac{5}{2}) = ---$$

The function is increasing on the open interval(s)

( ...... , ...... )

The function is decreasing on the open interval(s)

( ...... , ...... )

The slope of the tangent line is positive on the open interval(s)

( ...... , ...... )

The slope of the tangent line is negative on the open interval(s)

( ...... , ...... )

At the point $x_1 = -1$, the tangent line is __________ .

At the point $x_1 = 5/2$, the tangent line is __________ .

2.1.2. ***Critical points and critical values.*** We study a continuous function

$$y = f(x), \qquad a \le x \le b$$

**Recall:** $f(x)$ is continuous means that

(1) The graph of $f(x)$ does not have any jumps.
(2) Avoid

$$\frac{1}{0}\ ,\ \ln 0\ ,\ \ln(-)\ ,\ \sqrt{-}\ ,\ \cdots$$

the graph of $f(x)$ does not have any jumps.

**Recall:** $f'(c)$ does not exist means that the graph of $f(x)$ has a corner.

| |
|---|
| $c$ is a critical point<br>$\Updownarrow$<br>$f'(c) = 0$ or $f'(c)$ does not exist. |

2.2. **Relative extrema.** Let

$$a \le c \le b$$

| |
|---|
| $f(c)$ has a local (relative) maximum<br>$\Updownarrow$<br>$f(x) \le f(c)$ for all $x$ near $c$ |

| |
|---|
| $f(c)$ has a local (relative) minimum<br>$\Updownarrow$<br>$f(x) \ge f(c)$ for all $x$ near $c$ |

| |
|---|
| In either case<br>$c$ is called a critical point<br>$f(c)$ is called a critical value |

2.2.1.

***Example.*** Find all critical points and values and classify them as local max or local min. Then Sketch the function.

$$f(x) = f(x) = x^3 + 3x^2 - 24x + 2$$

**Solution:**

$$\begin{aligned} f'(x) &= 3x^2 + 6x - 24 \\ &= 3(x-2)(x+4) \end{aligned}$$

We have two critical points

$$x_1 = -4, \qquad x_2 = 2$$

The corresponding critical values ar

$$\begin{aligned} y_1 &= f(-4) = ------- \\ y_2 &= f(2) = ------- \end{aligned}$$

$$\begin{aligned} f'(x) &> 0 \quad \text{on} \quad (-\infty, -4) \\ f'(-4) &= 0 \\ f'(x) &< 0 \quad \text{on} \quad (-4, 2) \\ f'(2) &= 0 \\ f'(x) &> 0 \quad \text{on} \quad (2, \infty) \end{aligned}$$

Thus

$$(-4, \cdots\cdots) \quad \text{is a local} \ \text{———-}$$

$(2, \cdots\cdots)$ is a local ———-

**Sketch** the function.

### 2.3. **Higher derivatives, concavity, and the second derivative test.**

#### 2.3.1. ***Second derivative.***

$$\begin{aligned}
f(x) &= 2x^4 - x^3 + 7x^2 + 5x + 12 \\
f'(x) &= 8x^3 - 3x^2 + 14x + 5 \\
f''(x) &= 24x^2 - 6x + 14 \\
f^{(3)}(x) &= 48x - 6 \\
f^{(4)}(x) &= 48 \\
f^{(5)}(x) &= 0 \\
f^{(6)}(x) &= 0 \\
&\vdots
\end{aligned}$$

Another example

$$\begin{aligned}
g(x) &= x^{2/3} - 4x^2 + 7x \\
g'(x) &= (2/3)x^{-1/3} - 8x + 7 \\
g''(x) &= (2/3)(-1/3)x^{-4/3} - 8 \\
f'''(x) &= (2/3)(-1/3)(-4/3)x^{-7/3} \\
g^{(4)}(x) &= (2/3)(-1/3)(-4/3)(-7/3)x^{-10/3} \\
&= (-56/81)x^{-10/3} \\
&\vdots
\end{aligned}$$

#### 2.3.2. ***Concavity.***

$$f''(x) < 0 \quad \Updownarrow \quad \text{graph is concave downward}$$

This is because

$$(f'(x))' = f''(x) < 0$$

means that the slope of the tangent line is decreasing, which means that the tangent line is turning clockwise and the graph is concave downwards.

$$f''(x) > 0 \quad \Updownarrow \quad \text{graph is concave upward}$$

This is because

$$(f'(x))' = f''(x) > 0$$

means that the slope of the tangent line is increasing, which means that the tangent line is turning counter-clockwise and the graph is concave upwards.

A point of inflection $(c, f(c))$
is a point where
the concavity of the graph changes ,

$$f''(c) = 0$$

**Example**

$$\begin{aligned}
f(x) &= -x^3 - 12x^2 - 45x + 2 \\
f'(x) &= -3x^2 - 24x - 45 \\
&= -3(x^2 + 8x + 15) \\
&= -3(x+3)(x+5) \\
f''(x) &= -3(2x+8) = -6(x+4)
\end{aligned}$$

$$\begin{cases} > 0, & x < -4 \\ = 0, & x = 4 \\ < 0, & x > -4 \end{cases}$$

It follows that The graph of $f(x)$ is

(1) concave upward when $x < -4$,
(2) has an inflection point when $x = 0$,

(3) concave downward when $x > -4$.

**2.3.3. *Point of diminishing return*.** In economics, a point of inflection ia called a point of diminishing return.

### 2.4. Curve Sketching.

- Critical points and local max and min.
- concavity and points of inflection.
- Vertical asymptotes, horizontal asymptotes.
- $y$ and $x$ intercept.

(1) $f(x) = x^3 - 6x^2 + 12x - 11$
(2) $f(x) = x + 2/x$
(3) $f(x) = -2/(x^2 - x - 6)$
(4) $f(x) = x/(x^2 + 3)$

## 3. Applications of the derivative

**3.1. Absolute extrema.** Find the absolute extrema if they exist.

$$f(x) = (1/3)x^3 - (1/2)x^2 - 6x + 3, \quad [-4, 4]$$
$$f(x) = x^4 - 32x^2 - 7, \qquad [-5, 6]$$

**3.2. Application of extrema.** §7.2: 8, 14, 16.

**3.3. Implicit differentiation.** §7.4: 4, 6, 20.

**3.4. Related rates.** §7.5: 10, 13, 14, 4.

**3.5. Differentials, linear approximation.** §7.6: 4, 10, 14, 18.

## 4. Integration

4.1. **Antiderivatives.** §8.1: 10, 20, 34, 36, 38, 40

4.2. **Integration by substitution.** §8.2: 8, 10, 16, 20,

4.3. **Area and the definite integral.** §8.3: 8, 13, 18,

4.4. **The fundamental theorem of calculus.** If $f(x)$ is continuous on the closed interval $a \le x \le b$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$$

Examples: §8.4: 4, 20, 42

4.5. **The area between two curves.** Examples: §8.5: 6, 10, 22.

## 5. Techniques and applications of integration

5.1. **Integration by parts.** and column integration.

(1) $\displaystyle\int xe^{3x}\,dx$

(2) $\displaystyle\int x^2e^{-5x}\,dx$

(3) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{3+x^2}}\,dx$

(4) $\displaystyle\int \ln x\,dx$

(5) $\displaystyle\int x^3 \ln x\,dx$

5.2. **Volume and average value.** Examples: §9.2: 26, 30.

5.3. **Improper integrals.** On the same graph sketch the regions whose areas are given by the following two integrals

$$(1)\int_2^\infty \frac{1}{x^{2/3}}\,dx, \qquad (2)\int_2^\infty \frac{1}{x^{5/4}}\,dx$$

On the same graph sketch the regions whose areas are given by the following two integrals

$$(3)\int_0^3 \frac{1}{x^{3/4}}\,dx, \qquad (4)\int_0^3 \frac{1}{x^{1.001}}\,dx$$

> When we integrate a continuous function
> (continuous = no $1/0$, $\sqrt{(-)}, \cdots$),
> over a finite interval $a \le x \le b$,
> we will always get a finite result.
> Even if we couldn't evaluate the integral,
> we know we would get a finite answer
> had we been successful.

**Example**

$$\int_0^\infty \frac{1}{(2x+1)^3}\,dx$$

Notice that the function

$$f(x) = \frac{1}{(2x+1)^3}, \qquad a \le x \le b$$

is continuous on any interval $a \le x \le b$, as long as $-1/2 \lneqq a$, and $b \lneqq \infty$.
For example the integral

$$\int_{-3/4}^{22345} \frac{1}{(2x+1)^3}\,dx$$

Also notice that when $0 \lll x \lessapprox \infty$, we have

$$\frac{1}{(2x+1)^3} \lessapprox \frac{1}{8x^3}$$

**Example**

$$\int_{-1/2}^\infty \frac{1}{(2x+1)^3}\,dx$$

We can break the integral into two parts

$$\int_{-1/2}^\infty = \int_{-1/2}^0 + \int_0^\infty$$

We know that the second integral is finite.

But when $x \searrow -1/2$, $(2x+1) \nearrow \infty$.

If we make a substitution

$$u = 2x + 1, \qquad du = 2dx$$

The first integral becomes

$$\int_0^1 \frac{1}{2u^3}\, du = -\frac{1}{4u^2}|_0^1 = -\frac{1}{4} + \infty = \infty$$

## 6. System of linear equations and matrices

**Example**

$$\begin{aligned} 4x + y &= 9 \\ 3x - y &= 5 \end{aligned}$$

**Example**

$$\begin{aligned} 12s - 5t &= 9 \\ 3s - 8 &= -18 \end{aligned}$$

**Example**

$$\begin{aligned} 4x - y + 3z &= -2 \\ 3x + 5y - z &= 15 \\ -2x + y + 4z &= 14 \end{aligned}$$

**Example**

$$\begin{aligned} 5x + 3y + 4z &= 19 \\ 3x - y + z &= -4 \end{aligned}$$

## 7. Exercises

Problems below are labelled in three different ways:

(1) **Assignment:** These are the types of problems you are responsible for.
(2) **GW:** These are problems that you work on in classroom in groups of 3 to 4.
(3) **HW:** These are problems that you should hand in individually. However, you may work on them in groups in or outside the classroom.

**§6.1: Assignment:** 1-8, 9-28, 34-37.
**GW:** 1-8; 11, 13, 15; 35, 36.
**§6.2: Assignment:** 1-8, 9-28, 29-30,.
**GW:** 1, 3, 5, 7, 11, 15, 29.
**§6.3: Assignment:** 1-16, 17-26, 27-44, 59-62.
**GW:** 3, 29, 35, 59
**HW:** 3, 9, 11, 19, 29, 30, 35, 39, 59, 61.
**§6.4: Assignment:** 3-24, 25-27
**GW:** 5, 15, 17, 25
**HW:** 5, 7, 15, 17, 19, 25, 27.
**§7.1: Assignment:** 1-8, 10-23.
**GW:** 1,7, 11, 15
**HW:** 1,5, 7, 11, 13, 15.
**§7.2: Assignment:** 7, 13, 17, 19, 23.
**GW:** 7, 17, 23.
**HW:** 7, 13, 17, 19, 23.
**§7.4: Assignment:** 1-16, 17-22, 23-28
**GW:** 5, 9, 19, 37
**HW:** 3, 5, 9, 19, 21, 37, 38.
**§7.5: Assignment:** 1-8, 9-14.
**GW:** 3, 9-14
**§7.6: Assignment:** 1-8, 7-16, 17-22
**GW:** 3, 9, 17, 21
**HW:** 3, 7, 9, 13, 17, 19, 21
**§8.1: Assignment:** 5-42, 45-52, 53-56
**GW:** 11, 13, 18, 31, 39, 47, 55
**HW:** 11, 13, 18, 19, 29, 31, 39
**§8.2: Assignment:** 1, 2, 3-34, 37-40.
**GW:** 2(c), 7, 9, 13, 29, 37, 39
**HW:** 2(c), 3, 7, 9, 13, 17, 18, 25, 29, 37, 39
**§8.3: Assignment:** 6-13, 14-16, 17-20, 23-25.
**GW:** 9, 15, 17, 23.
**HW:** 9, 11, 15, 17, 23.
**§8.4: Assignment:** 1-30, 31-40,41-43, 53-54.
**GW:** 11, 23, 37, 43, 53.
**HW:** 11, 23, 27, 37, 40, 43, 53, 54.
**§8.5:** 1-22, 25-36.
**GW:** 5, 11, 21, 29.
**HW:** 5, 9, 11, 15, 21, 25, 29, 31.
**§9.1:** 1-10, 11, 12, 13-22, 35.
**HW:** 3, 5, 9, 14, 15, 19, 35.
**§9.2:** 24-29, 34-37.
**HW:** 25, 30, 31, 37.
**§9.4:** 1-30, 31-36, 42-50.
**GW:** 3, 7, 8, 11, 23, 33.
**HW:** 3, 7, 8, 11,13, 23, 33.
**§10.1: Assignment:** 1-16, 19-22, 23-26, 28-31, 34-45.
**GW:** 11, 21, 25, 29, 39.
**HW:** 11, 15, 19, 21,23, 25, 29, 31, 39, 45.