Pre-Class Problems 18 for Wednesday, April 4

These are the type of problems that you will be working on in class.

You can go to the solution for each problem by clicking on the problem letter.

1. Solve the following system of equations.

a. $\begin{aligned} x^2 + y^2 &= 9 \\ y &= x \end{aligned}$

b. $\begin{aligned} y &= 2x^2 \\ y &= 5x + 12 \end{aligned}$

c. $\begin{aligned} 3x^2 + y^2 &= 76 \\ x + 2y &= 3 \end{aligned}$

d. $\begin{aligned} x^2 + (y + 6)^2 &= 16 \\ y &= 4 - x^2 \end{aligned}$

e. $\begin{aligned} 3x^2 - 4y^2 &= 25 \\ x^2 + 6y^2 &= 12 \end{aligned}$

2. Solve the following systems of equations using Gaussian elimination.

a. $\begin{aligned} x - 3y - 2z &= -1 \\ 3x + y + 5z &= 32 \\ -4x + 6y - z &= -29 \end{aligned}$

b. $\begin{aligned} 5x + 4y - 3z &= -36 \\ 3x - 2y + 7z &= -15 \\ -2x - 6y + 9z &= 21 \end{aligned}$

c. $\begin{aligned} 2x - 7z &= 19 \\ 3y + z &= 9 \\ 4x - 5y &= -24 \end{aligned}$

Problems available in the textbook: Page 532 … 3b – 14b, 15 – 36, 37 – 46, 48 – 56. and Examples 1 – 5 starting on page 528. Page 573 … 41 – 64 and Examples 1 – 5 starting on page 564.

SOLUTIONS:

1a. $\begin{aligned} x^2 + y^2 &= 9 \\ y &= x \end{aligned}$

Back to Problem 1.

Notice in the second equation, the variable of *y* is solved for *x*. Thus, replace the *y* variable in the second equation of $x^2 + y^2 = 9$ by *x* and then solve for *x*:

$$x^2 + y^2 = 9 \Rightarrow x^2 + x^2 = 9 \Rightarrow 2x^2 = 9 \Rightarrow x^2 = \frac{9}{2} \Rightarrow x = \pm\frac{3}{\sqrt{2}} =$$

$$x = \pm\frac{3\sqrt{2}}{2}$$

Now, use the equation $y = x$ to solve for *y*.

When $x = \frac{3\sqrt{2}}{2}$, $y = x = \frac{3\sqrt{2}}{2}$. Thus, the ordered pair $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ is a solution to the system of equations.

When $x = -\frac{3\sqrt{2}}{2}$, $y = x = -\frac{3\sqrt{2}}{2}$. Thus, the ordered pair $\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ is a solution to the system of equations.

Answer: $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

1b. $\begin{aligned} y &= 2x^2 \\ y &= 5x + 12 \end{aligned}$

Back to Problem 1.

Replace the *y* variable in the second equation by $2x^2$:

$$2x^2 = 5x + 12 \Rightarrow 2x^2 - 5x - 12 = 0 \Rightarrow (x - 4)(2x + 3) = 0 \Rightarrow$$

$$x = 4,\ x = -\frac{3}{2}$$

You can use either the first or second equation to find *y* when $x = 4$:

Using the first equation of $y = 2x^2$ and $x = 4$: $y = 2(16) = 32$.

Using the second equation of $y = 5x + 12$, $x = 4$: $y = 20 + 12 = 32$.

Thus, the ordered pair $(4,\ 32)$ is a solution to the system of equations.

It is easier to use the first equation of $y = 2x^2$ to find *y* when $x = -\frac{3}{2}$:

$$y = 2x^2 \text{ and } x = -\frac{3}{2} \Rightarrow y = 2\left(\frac{9}{4}\right) = \frac{9}{2}$$

Thus, the ordered pair $\left(-\frac{3}{2}, \frac{9}{2}\right)$ is a solution to the system of equations.

Answer: $(4,\ 32),\ \left(-\frac{3}{2},\ \frac{9}{2}\right)$

1c. $\begin{aligned} 3x^2 + y^2 &= 76 \\ x + 2y &= 3 \end{aligned}$

Back to Problem 1.

Use the second equation to solve for x in terms of y and substitute into the first equation.

$$x + 2y = 3 \Rightarrow x = 3 - 2y$$

$3x^2 + y^2 = 76$ and $x = 3 - 2y \Rightarrow 3(3 - 2y)^2 + y^2 = 76 \Rightarrow$

$$3(9 - 12y + 4y^2) + y^2 = 76 \Rightarrow 27 - 36y + 12y^2 + y^2 = 76 \Rightarrow$$

$$13y^2 - 36y + 27 = 76 \Rightarrow 13y^2 - 36y - 49 = 0 \Rightarrow$$

$$(y + 1)(13y - 49) = 0 \Rightarrow y = -1,\ y = \frac{49}{13}$$

$x = 3 - 2y$ and $y = -1 \Rightarrow x = 3 + 2 = 5$

The ordered pair $(5,\ -1)$ is a solution to the system of equations.

$x = 3 - 2y$ and $y = \frac{49}{13} \Rightarrow x = 3 - \frac{98}{13} = \frac{39}{13} - \frac{98}{13} = -\frac{59}{13}$

The ordered pair $\left(-\frac{59}{13},\ \frac{49}{13}\right)$ is a solution to the system of equations.

Answer: $(5, -1), \left(-\frac{59}{13}, \frac{49}{13}\right)$

1d. $\begin{aligned} x^2 + (y+6)^2 &= 16 \\ y &= 4 - x^2 \end{aligned}$

Back to Problem 1.

If we replace *y* in the first equation by $4 - x^2$ since $y = 4 - x^2$ as given in the second equation, we obtain the equation $x^2 + (4 - x^2 + 6)^2 = 16$.

$x^2 + (4 - x^2 + 6)^2 = 16 \Rightarrow x^2 + (10 - x^2)^2 = 16 \Rightarrow$

$x^2 + 100 - 20x^2 + x^4 = 16 \Rightarrow x^4 - 19x^2 + 84 = 0$. This last equation is quadratic in x^2. This equation can be factored.

$x^4 - 19x^2 + 84 = 0 \Rightarrow (x^2 - 7)(x^2 - 12) = 0$

$x^2 - 7 = 0 \Rightarrow x^2 = 7 \Rightarrow x = \pm\sqrt{7}$

$x^2 - 12 = 0 \Rightarrow x^2 = 12 \Rightarrow x = \pm\sqrt{12}$

$y = 4 - x^2$ and $x = \sqrt{7} \Rightarrow y = 4 - 7 = -3$

Thus, the point $(\sqrt{7}, -3)$ is a solution to the system of equations.

$y = 4 - x^2$ and $x = -\sqrt{7} \Rightarrow y = 4 - 7 = -3$

Thus, the point $(-\sqrt{7}, -3)$ is a solution to the system of equations.

$y = 4 - x^2$ and $x = \sqrt{12} \Rightarrow y = 4 - 12 = -8$

Thus, the point $(2\sqrt{3}, -8)$ is a solution to the system of equations.

$y = 4 - x^2$ and $x = -\sqrt{12} \Rightarrow y = 4 - 12 = -8$

Thus, the point $(-2\sqrt{3}, -8)$ is a solution to the system of equations.

The system of equations $\begin{array}{l} x^2 + (y+6)^2 = 16 \\ y = 4 - x^2 \end{array}$ has an easier solution. Use the second equation of $y = 4 - x^2$ to solve for x^2 and substitution in the first equation.

$y = 4 - x^2 \Rightarrow x^2 = 4 - y$

$x^2 + (y+6)^2 = 16$ and $x^2 = 4 - y \Rightarrow 4 - y + (y+6)^2 = 16$

$4 - y + (y+6)^2 = 16 \Rightarrow 4 - y + y^2 + 12y + 36 = 16 \Rightarrow$

$y^2 + 11y + 40 = 16 \Rightarrow y^2 + 11y + 24 = 0 \Rightarrow (y+3)(y+8) = 0 \Rightarrow$

$y = -3, -8$

$x^2 = 4 - y$ and $y = -3 \Rightarrow x^2 = 7 \Rightarrow x = \pm\sqrt{7}$

Thus, the points $(\sqrt{7}, -3)$ and $(-\sqrt{7}, -3)$ are solutions to the system of equations.

$x^2 = 4 - y$ and $y = -8 \Rightarrow x^2 = 12 \Rightarrow x = \pm\sqrt{12} = \pm 2\sqrt{3}$

Thus, the points $(2\sqrt{3}, -8)$ and $(-2\sqrt{3}, -8)$ are solutions to the system of equations.

This system of equations can also be solved by the addition method.

$$\begin{aligned} x^2 + (y+6)^2 &= 16 \\ y &= 4 - x^2 \end{aligned} \Rightarrow \begin{array}{r} x^2 + (y+6)^2 = 16 \\ \underline{x^2 + \quad y \qquad = \ 4} \\ (y+6)^2 - y = 12 \end{array}$$

$$(y+6)^2 - y = 12 \Rightarrow y^2 + 12y + 36 - y = 12 \Rightarrow y^2 + 11y + 24 = 0$$

We solved this quadratic equation above.

Answer: $(\sqrt{7}, -3)$, $(-\sqrt{7}, -3)$, $(2\sqrt{3}, -8)$, $(-2\sqrt{3}, -8)$

1e. $$\begin{aligned} 3x^2 - 4y^2 &= 25 \\ x^2 + 6y^2 &= 12 \end{aligned}$$

Back to Problem 1.

This system of equations can also be solved by the addition method.

$$\begin{aligned} 3x^2 - 4y^2 &= 25 \\ x^2 + 6y^2 &= 12 \end{aligned} \Rightarrow \begin{array}{r} 3x^2 - \ \ 4y^2 = \ \ \ 25 \\ \underline{-3x^2 - 18y^2 = -36} \\ -22y^2 = -11 \end{array}$$

$$-22y^2 = -11 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

$$x^2 + 6y^2 = 12 \text{ and } y = \pm\frac{1}{\sqrt{2}} \Rightarrow x^2 + 6\left(\frac{1}{2}\right) = 12 \Rightarrow x^2 + 3 = 12 \Rightarrow$$

$$x^2 = 9 \Rightarrow x = \pm 3$$

Answer: $\left(3, \frac{\sqrt{2}}{2}\right), \left(3, -\frac{\sqrt{2}}{2}\right), \left(-3, \frac{\sqrt{2}}{2}\right), \left(-3, -\frac{\sqrt{2}}{2}\right)$

2a. $$\begin{aligned} x - 3y - 2z &= -1 \\ 3x + y + 5z &= 32 \\ -4x + 6y - z &= -29 \end{aligned}$$

Back to Problem 2.

First, form the augmented matrix for this system of equations:

$$\left[\begin{array}{cccc} 1 & -3 & -2 & -1 \\ 3 & 1 & 5 & 32 \\ -4 & 6 & -1 & -29 \end{array}\right]$$

Multiply Row 1 by -3 and add it to Row 2 $(-3R_1 + R_2)$ and multiply Row 1 by 4 and add it to Row 3 $(4R_1 + R_3)$:

$$\left[\begin{array}{cccc} 1 & -3 & -2 & -1 \\ 3 & 1 & 5 & 32 \\ -4 & 6 & -1 & -29 \end{array}\right] \xrightarrow[4R_1 + R_3]{-3R_1 + R_2}$$

$$\begin{bmatrix} 1 & -3 & -2 & -1 \\ -3+3 & 9+1 & 6+5 & 3+32 \\ 4+(-4) & -12+6 & -8+(-1) & -4+(-29) \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -2 & -1 \\ 0 & 10 & 11 & 35 \\ 0 & -6 & -9 & -33 \end{bmatrix}$$

Divide Row 3 by -3 $\left(-\frac{1}{3}R_3\right)$:

$$\begin{bmatrix} 1 & -3 & -2 & -1 \\ 0 & 10 & 11 & 35 \\ 0 & -6 & -9 & -33 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_3} \begin{bmatrix} 1 & -3 & -2 & -1 \\ 0 & 10 & 11 & 35 \\ 0 & 2 & 3 & 11 \end{bmatrix}$$

Interchange Row 2 and Row 3 $(R_2 \leftrightarrow R_3)$:

$$\begin{bmatrix} 1 & -3 & -2 & -1 \\ 0 & 10 & 11 & 35 \\ 0 & 2 & 3 & 11 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -3 & -2 & -1 \\ 0 & 2 & 3 & 11 \\ 0 & 10 & 11 & 35 \end{bmatrix}$$

Multiply Row 2 by -5 and add to Row 3 $(-5R_2 + R_3)$:

$$\begin{bmatrix} 1 & -3 & -2 & -1 \\ 0 & 2 & 3 & 11 \\ 0 & 10 & 11 & 35 \end{bmatrix} \xrightarrow{-5R_2 + R_3} \begin{bmatrix} 1 & -3 & -2 & -1 \\ 0 & 2 & 3 & 11 \\ 0 & -10+10 & -15+11 & -55+35 \end{bmatrix}$$

$$\left[\begin{array}{cccc} 1 & -3 & -2 & -1 \\ 0 & 2 & 3 & 11 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

Divide Row 2 by 2 $\left(\frac{1}{2}R_2\right)$ and divide Row 3 by -4 $\left(-\frac{1}{4}R_3\right)$:

$$\left[\begin{array}{cccc} 1 & -3 & -2 & -1 \\ 0 & 2 & 3 & 11 \\ 0 & 0 & -4 & -20 \end{array}\right] \xrightarrow{\frac{1}{2}R_2,\ -\frac{1}{4}R_3} \left[\begin{array}{cccc} 1 & -3 & -2 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Row 3 reads $0x + 0y + z = 5 \Rightarrow z = 5$

Row 2 reads $0x + y + \frac{3}{2}z = \frac{11}{2} \Rightarrow y + \frac{3}{2}z = \frac{11}{2}$

Since $z = 5$, then $y + \frac{15}{2} = \frac{11}{2} \Rightarrow y = -\frac{4}{2} = -2$

Row 1 reads $x - 3y - 2z = -1$

Since $y = -2$ and $z = 5$, then $x + 6 - 10 = -1 \Rightarrow x - 4 = -1 \Rightarrow x = 3$

Answer: $(3, -2, 5)$

NOTE: Geometrically, if you graph the three planes in the system of equations, the planes will intersect at the point $(3, -2, 5)$.

2b.
$$\begin{aligned} 5x + 4y - 3z &= -36 \\ 3x - 2y + 7z &= -15 \\ -2x - 6y + 9z &= 21 \end{aligned}$$

Back to Problem 2.

First, form the augmented matrix for this system of equations:

$$\begin{bmatrix} 5 & 4 & -3 & -36 \\ 3 & -2 & 7 & -15 \\ -2 & -6 & 9 & 21 \end{bmatrix}$$

Multiply Row 3 by 2 and add to Row 1 $(2R_3 + R_1)$:

$$\begin{bmatrix} 5 & 4 & -3 & -36 \\ 3 & -2 & 7 & -15 \\ -2 & -6 & 9 & 21 \end{bmatrix} \xrightarrow{2R_3 + R_1}$$

$$\begin{bmatrix} -4+5 & -12+4 & 18+(-3) & 42+(-36) \\ 3 & -2 & 7 & -15 \\ -2 & -6 & 9 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -8 & 15 & 6 \\ 3 & -2 & 7 & -15 \\ -2 & -6 & 9 & 21 \end{bmatrix}$$

Multiply Row 1 by -3 and add it to Row 2 $(-3R_1 + R_2)$ and multiply Row 1 by 2 and add it to Row 3 $(2R_1 + R_3)$:

$$\begin{bmatrix} 1 & -8 & 15 & 6 \\ 3 & -2 & 7 & -15 \\ -2 & -6 & 9 & 21 \end{bmatrix} \xrightarrow[2R_1 + R_2]{-3R_1 + R_2}$$

$$\begin{bmatrix} 1 & -8 & 15 & 6 \\ -3 + 3 & 24 + (-2) & -45 + 7 & -18 + (-15) \\ 2 + (-2) & -16 + (-6) & 30 + 9 & 12 + 21 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -8 & 15 & 6 \\ 0 & 22 & -38 & -33 \\ 0 & -22 & 39 & 33 \end{bmatrix}$$

Add Row 2 to Row 3 $(R_2 + R_3)$:

$$\begin{bmatrix} 1 & -8 & 15 & 6 \\ 0 & 22 & -38 & -33 \\ 0 & -22 & 39 & 33 \end{bmatrix} \xrightarrow{R_2 + R_3}$$

$$\begin{bmatrix} 1 & -8 & 15 & 6 \\ 0 & 22 & -38 & -33 \\ 0 & 22 + (-22) & -38 + 39 & -33 + 33 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -8 & 15 & 6 \\ 0 & 22 & -38 & -33 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Divide Row 2 by 22 $\left(\frac{1}{22} R_2 \right)$:

$$\begin{bmatrix} 1 & -8 & 15 & 6 \\ 0 & 22 & -38 & -33 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{22}R_2} \begin{bmatrix} 1 & -8 & 15 & 6 \\ 0 & 1 & -\frac{19}{11} & -\frac{3}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Row 3 reads $0x + 0y + z = 0 \Rightarrow z = 0$

Row 2 reads $0x + y - \dfrac{19}{11}z = -\dfrac{3}{2} \Rightarrow y - \dfrac{19}{11}z = -\dfrac{3}{2}$

Since $z = 0$, then $y - \dfrac{19}{11}z = -\dfrac{3}{2} \Rightarrow y - 0 = -\dfrac{3}{2} \Rightarrow y = -\dfrac{3}{2}$

Row 1 reads $x - 8y + 15z = 6$

Since $y = -\dfrac{3}{2}$ and $z = 0$, then $x + 12 + 0 = 6 \Rightarrow x = -6$

Answer: $\left(-6, -\dfrac{3}{2}, 0\right)$

NOTE: Geometrically, if you graph the three planes in the system of equations, the planes will intersect at the point $\left(-6, -\dfrac{3}{2}, 0\right)$.

2c. $$\begin{aligned} 2x \qquad\quad - 7z &= 19 \\ 3y + z &= 9 \\ 4x - 5y \qquad &= -24 \end{aligned}$$

Back to Problem 2.

First, form the augmented matrix for this system of equations:

$$\begin{bmatrix} 2 & 0 & -7 & 19 \\ 0 & 3 & 1 & 9 \\ 4 & -5 & 0 & -24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -7 & 19 \\ 0 & 3 & 1 & 9 \\ 4 & -5 & 0 & -24 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \begin{bmatrix} 2 & 0 & -7 & 19 \\ 0 & 3 & 1 & 9 \\ 0 & -5 & 14 & -62 \end{bmatrix} \xrightarrow{2R_2 + R_3}$$

$$\begin{bmatrix} 2 & 0 & -7 & 19 \\ 0 & 3 & 1 & 9 \\ 0 & 1 & 16 & -44 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 0 & -7 & 19 \\ 0 & 1 & 16 & -44 \\ 0 & 3 & 1 & 9 \end{bmatrix} \xrightarrow{-3R_2 + R_3}$$

$$\begin{bmatrix} 2 & 0 & -7 & 19 \\ 0 & 1 & 16 & -44 \\ 0 & 0 & -47 & 141 \end{bmatrix} \xrightarrow{-\frac{1}{47}R_3} \begin{bmatrix} 2 & 0 & -7 & 19 \\ 0 & 1 & 16 & -44 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Row 3 reads $z = -3$

Row 2 reads $y + 16z = -44$

Since $z = -3$, then $y - 48 = -44 \Rightarrow y = 4$

Row 1 reads $2x - 7z = 19$

Since $z = -3$, then $2x + 21 = 19 \Rightarrow 2x = -2 \Rightarrow x = -1$

Answer: $(-1,\ 4,\ -3)$

NOTE: Geometrically, if you graph the three planes in the system of equations, the planes will intersect at the point $(-1,\ 4,\ -3)$.