Pre-Class Problems 12 for Monday, March 12

These are the type of problems that you will be working on in class.

You can go to the solution for each problem by clicking on the problem number or letter.

Discussion on Long Division and Synthetic Division.

1. Use synthetic division to divide the following polynomials.

   a. $(3x^2 - 14x + 21) \div (x - 4)$

   b. $(7x^2 + 6x - 18) \div (x + 3)$

   c. $(x^3 + 4x^2 - 9x - 16) \div (x - 2)$

   d. $(3x^3 + 4x^2 - 48x - 64) \div (x + 4)$

   e. $(2x^3 - 65x - 77) \div (x - 6)$

   f. $(2x^3 - 65x - 77) \div (x + 6)$

   g. $\dfrac{24 - 18x^2 - x^4}{x + 2}$ h. $\dfrac{98 - 15x + 50x^2 - 2x^4}{x - 5}$

   i. $\dfrac{x^3 - 27}{x - 3}$ j. $\dfrac{x^3 - a^3}{x - a}$ k. $\dfrac{x^3 + 64}{x + 4}$

Discussion of the Remainder Theorem.

2. If $f(x) = 2x^3 - 5x^2 + 8x - 17$, then use the Remainder Theorem to find the following polynomial values.

a. $f(-6)$ b. $f(8)$ c. $f(\sqrt{3})$ d. $f\left(-\frac{1}{2}\right)$

3. If $g(x) = x^4 + 6x^3 - 15x + 12$, then use the Remainder Theorem to find the following polynomial values.

a. $g(-9)$ b. $g(7)$ c. $g(-\sqrt{2})$ d. $g\left(\frac{1}{2}\right)$

4. If $p(x) = x^4 - 6x^3 - 3x^2 + 16x + 12$, then use the Remainder Theorem to determine if the given number is a zero (root) of the polynomial.

a. 3 b. -1 c. 6 d. -6

5. If $h(t) = t^4 + 8t^3 + 11t^2 - 40t - 80$, then use the Remainder Theorem to determine if the given number is a zero (root) of the polynomial.

a. 4 b. -4 c. $-\sqrt{5}$ d. $3i$

Discussion of the Factor Theorem.

6. If $f(x) = 9x^4 + 18x^3 - 43x^2 - 32x + 48$, then use the Factor Theorem to determine if the given binomial is a factor of the polynomial.

a. $x - 2$ b. $x + 3$ c. $x - \frac{2}{3}$ d. $x + \frac{4}{3}$

Discussion of Rational Zeros (Roots) of Polynomials

7. Find the zeros (roots) of the following polynomials. Also, give a factorization for the polynomial.

a. $f(x) = x^3 - 5x^2 - 2x + 24$

b. $g(x) = 3x^3 - 23x^2 + 57x - 45$

c. $h(t) = 4t^3 - 4t^2 - 9t + 30$

d. $p(x) = x^4 - 6x^3 - 3x^2 + 16x + 12$

e. $f(z) = 6z^4 - 11z^3 - 53z^2 + 108z - 36$

f. $g(x) = 9x^4 + 18x^3 - 43x^2 - 32x + 48$

g. $h(x) = x^4 + 8x^3 + 11x^2 - 40x - 80$

h. $p(t) = t^4 - 12t^3 + 54t^2 - 108t + 81$

Problems available in the textbook: Page 325 … 23 – 70 and Examples 4 – 9 starting on page 303. Page 341 … 17 – 28, 39 – 48, 59 – 64 and Examples 1 – 4, 6, 9, 10 starting on page 330.

SOLUTIONS:

1a. $(3x^2 - 14x + 21) \div (x - 4)$ Back to Problem 1.

$$\begin{array}{rrr|l} 3 & -14 & 21 & 4 \\ & 12 & -8 & \\ \hline 3 & -2 & 13 & \end{array}$$

Answer: $\dfrac{3x^2 - 14x + 21}{x - 4} = 3x - 2 + \dfrac{13}{x - 4}$

1b. $(7x^2 + 6x - 18) \div (x + 3)$ Back to Problem 1.

$$\begin{array}{rrr|r} 7 & 6 & -18 & -3 \\ & -21 & 45 & \\ \hline 7 & -15 & 27 & \end{array}$$

Answer: $\dfrac{7x^2 + 6x - 18}{x + 3} = 7x - 15 + \dfrac{27}{x + 3}$

1c. $(x^3 + 5x^2 - 9x - 16) \div (x - 2)$ Back to Problem 1.

$$\begin{array}{rrrr|r} 1 & 5 & -9 & -16 & 2 \\ & 2 & 14 & 10 & \\ \hline 1 & 7 & 5 & -6 & \end{array}$$

Answer: $\dfrac{x^3 + 4x^2 - 9x - 16}{x - 2} = x^2 + 7x + 5 - \dfrac{6}{x - 2}$

1d. $(3x^3 + 4x^2 - 48x - 64) \div (x + 4)$ Back to Problem 1.

$$\begin{array}{rrrr|r} 3 & 4 & -48 & -64 & -4 \\ & -12 & 32 & 64 & \\ \hline 3 & -8 & -16 & 0 & \end{array}$$

Answer: $\dfrac{3x^3 + 4x^2 - 48x - 64}{x + 4} = 3x^2 - 8x - 16$

1e. $(2x^3 - 65x - 77) \div (x - 6)$ Back to Problem 1.

$$\begin{array}{rrrr|l} 2 & 0 & -65 & -77 & 6 \\ & 12 & 72 & 42 & \\ \hline 2 & 12 & 7 & -35 & \end{array}$$

Answer: $\dfrac{2x^3 - 65x - 77}{x - 6} = 2x^2 + 12x + 7 - \dfrac{35}{x - 6}$

1f. $(2x^3 - 65x - 77) \div (x + 6)$ Back to Problem 1.

$$\begin{array}{rrrr|l} 2 & 0 & -65 & -77 & -6 \\ & -12 & 72 & -42 & \\ \hline 2 & -12 & 7 & -119 & \end{array}$$

Answer: $\dfrac{2x^3 - 65x - 77}{x + 6} = 2x^2 - 12x + 7 - \dfrac{119}{x + 6}$

1g. $(24 - 18x^2 - x^4) \div (x + 2)$ Back to Problem 1.

$(-x^4 - 18x^2 + 24) \div (x + 2)$

$$\begin{array}{rrrrr|r}
-1 & 0 & -18 & 0 & 24 & -2 \\
 & 2 & -4 & 44 & -88 & \\
\hline
-1 & 2 & -22 & 44 & -64 &
\end{array}$$

Answer: $\dfrac{24 - 18x^2 - x^4}{x + 2} = -x^3 + 2x^2 - 22x + 44 - \dfrac{64}{x + 2}$

1h. $\dfrac{98 - 15x + 50x^2 - 2x^4}{x - 5}$ Back to Problem 1.

$$\frac{-2x^4 + 50x^2 - 15x + 98}{x - 5}$$

$$\begin{array}{rrrrr|r}
-2 & 0 & 50 & -15 & 98 & 5 \\
 & -10 & -50 & 0 & -75 & \\
\hline
-2 & -10 & 0 & -15 & 23 &
\end{array}$$

Answer: $\dfrac{98 - 15x + 50x^2 - 2x^4}{x - 5} = -2x^3 - 10x^2 - 15 + \dfrac{23}{x - 5}$

1i. $\dfrac{x^3 - 27}{x - 3}$ Back to Problem 1.

$$\begin{array}{rrrr|r}
1 & 0 & 0 & -27 & 3 \\
 & 3 & 9 & 27 & \\
\hline
1 & 3 & 9 & 0 &
\end{array}$$

Answer: $\dfrac{x^3 - 27}{x - 3} = x^2 + 3x + 9$

NOTE: $\dfrac{x^3 - 27}{x - 3} = x^2 + 3x + 9 \Rightarrow x^3 - 27 = (x - 3)(x^2 + 3x + 9)$

This is the factorization of the difference of cubes $x^3 - 27$. If you can remember that $x - 3$ is one of the factors of $x^3 - 27$, then you can get the other factor of $x^2 + 3x + 9$ by synthetic division.

1j. $\dfrac{x^3 - a^3}{x - a}$ Back to Problem 1.

$$\begin{array}{cccc|c} 1 & 0 & 0 & -a^3 & a \\ & a & a^2 & a^3 & \\ \hline 1 & a & a^2 & 0 & \end{array}$$

Answer: $\dfrac{x^3 - a^3}{x - a} = x^2 + ax + a^2$

1k. $\dfrac{x^3 + 64}{x + 4}$ Back to Problem 1.

$$\begin{array}{cccc|c} 1 & 0 & 0 & 64 & -4 \\ & -4 & 16 & -64 & \\ \hline 1 & -4 & 16 & 0 & \end{array}$$

Answer: $\dfrac{x^3 + 64}{x + 4} = x^2 - 4x + 16$

NOTE: $\dfrac{x^3 + 64}{x + 4} = x^2 - 4x + 16 \Rightarrow x^3 + 64 = (x + 4)(x^2 - 4x + 16)$

This is the factorization of the sum of cubes $x^3 + 64$. If you can remember that $x + 4$ is one of the factors of $x^3 + 64$, then you can get the other factor of $x^2 - 4x + 16$ by synthetic division.

2a. $f(x) = 2x^3 - 5x^2 + 8x - 17$ Back to Problem 2.

Find $f(-6)$.

$$\begin{array}{rrrr|r}
2 & -5 & 8 & -17 & -6 \\
 & -12 & 102 & -660 & \\
\hline
2 & -17 & 110 & -677 &
\end{array}$$

Answer: -667

2b. $f(x) = 2x^3 - 5x^2 + 8x - 17$ Back to Problem 2.

Find $f(8)$.

$$\begin{array}{rrrr|r}
2 & -5 & 8 & -17 & 8 \\
 & 16 & 88 & 768 & \\
\hline
2 & 11 & 96 & 751 &
\end{array}$$

Answer: 751

2c. $f(x) = 2x^3 - 5x^2 + 8x - 17$ Back to Problem 2.

Find $f(\sqrt{3})$.

$$
\begin{array}{rrrr|l}
2 & -5 & 8 & -17 & \sqrt{3} \\
 & 2\sqrt{3} & 6 - 5\sqrt{3} & 14\sqrt{3} - 15 & \\
\hline
2 & 2\sqrt{3} - 5 & 14 - 5\sqrt{3} & 14\sqrt{3} - 32 &
\end{array}
$$

NOTE: In order to find $f(\sqrt{3})$, I think it easier to do the evaluation of the function:

$f(x) = 2x^3 - 5x^2 + 8x - 17 \Rightarrow f(\sqrt{3}) = 6\sqrt{3} - 15 + 8\sqrt{3} - 17 =$

$14\sqrt{3} - 32$

NOTE: $(\sqrt{3})^3 = \sqrt{3}\sqrt{3}\sqrt{3} = 3\sqrt{3}$ Thus, $2(\sqrt{3})^3 = 6\sqrt{3}$

Answer: $14\sqrt{3} - 32$

2d. $f(x) = 2x^3 - 5x^2 + 8x - 17$ Back to Problem 2.

Find $f\left(-\dfrac{1}{2}\right)$.

$$\begin{array}{rrrr|l} 2 & -5 & 8 & -17 = -\frac{34}{2} & -\frac{1}{2} \\ & -1 & 3 & -\frac{11}{2} & \\ \hline 2 & -6 & 11 & -\frac{45}{2} & \end{array}$$

Answer: $-\frac{45}{2}$

3a. $g(x) = x^4 + 6x^3 - 15x + 12$

Back to Problem 3.

Find $g(-9)$.

$$\begin{array}{rrrrr|l} 1 & 6 & 0 & -15 & 12 & -9 \\ & -9 & 27 & -243 & 2322 & \\ \hline 1 & -3 & 27 & -258 & 2345 & \end{array}$$

Answer: 2345

3b. $g(x) = x^4 + 6x^3 - 15x + 12$

Back to Problem 3.

Find $g(7)$.

$$\begin{array}{rrrrr|l} 1 & 6 & 0 & -15 & 12 & 7 \\ & 7 & 91 & 637 & 4354 & \\ \hline 1 & 13 & 91 & 622 & 4366 & \end{array}$$

Answer: 4366

3c. $g(x) = x^4 + 6x^3 - 15x + 12$ Back to Problem 3.

Find $g(-\sqrt{2})$.

$$
\begin{array}{cccccc|l}
1 & 6 & 0 & -15 & 12 & & -\sqrt{2} \\
 & -\sqrt{2} & -6\sqrt{2}+2 & -2\sqrt{2}+12 & 4+3\sqrt{2} & & \\
\hline
1 & 6-\sqrt{2} & 2-6\sqrt{2} & -2\sqrt{2}-3 & 3\sqrt{2}+16 & &
\end{array}
$$

NOTE: In order to find $g(-\sqrt{2})$, I think it easier to do the evaluation of the function:

$g(x) = x^4 + 6x^3 - 15x + 12 \Rightarrow g(-\sqrt{2}) = 4 - 12\sqrt{2} + 15\sqrt{2} + 12 =$

$3\sqrt{2} + 16$

NOTE: $(-\sqrt{2})^4 = +\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2} = 2 \cdot 2 = 4$

NOTE: $(-\sqrt{2})^3 = -\sqrt{2}\sqrt{2}\sqrt{2} = -2\sqrt{2}$ Thus, $6(-\sqrt{2})^3 = -12\sqrt{2}$

Answer: $3\sqrt{2} + 16$

3d. $g(x) = x^4 + 6x^3 - 15x + 12$ Back to Problem 3.

Find $g\left(\frac{1}{2}\right)$.

$$\begin{array}{cccccc|c} 1 & 6 & 0 & -15 = -\frac{120}{8} & 12 = \frac{192}{16} & & \frac{1}{2} \\ & \frac{1}{2} & \frac{13}{4} & \frac{13}{8} & -\frac{107}{16} & & \\ \hline 1 & \frac{13}{2} & \frac{13}{4} & -\frac{107}{8} & \frac{85}{16} & & \end{array}$$

Answer: $\frac{85}{16}$

4a. $p(x) = x^4 - 6x^3 - 3x^2 + 16x + 12$ Back to Problem 4.

$$\begin{array}{ccccc|c} 1 & -6 & -3 & 16 & 12 & 3 \\ & 3 & -9 & -36 & -60 & \\ \hline 1 & -3 & -12 & -20 & -48 & \end{array}$$

$p(3) = -48$

Answer: 3 is not a zero (root) of the polynomial

4b. $p(x) = x^4 - 6x^3 - 3x^2 + 16x + 12$ Back to Problem 4.

$$\begin{array}{ccccc|c} 1 & -6 & -3 & 16 & 12 & -1 \\ & -1 & 7 & -4 & -12 & \\ \hline 1 & -7 & 4 & 12 & 0 & \end{array}$$

$p(-1) = 0$

Answer: -1 is a zero (root) of the polynomial

4c. $p(x) = x^4 - 6x^3 - 3x^2 + 16x + 12$ Back to Problem 4.

$$\begin{array}{rrrrr|l} 1 & -6 & -3 & 16 & 12 & \underline{6} \\ & 6 & 0 & -18 & -12 & \\ \hline 1 & 0 & -3 & -2 & 0 & \end{array}$$

$p(6) = 0$

Answer: 6 is a zero (root) of the polynomial

4d. $p(x) = x^4 - 6x^3 - 3x^2 + 16x + 12$ Back to Problem 4.

$$\begin{array}{rrrrr|l} 1 & -6 & -3 & 16 & 12 & \underline{-6} \\ & -6 & 72 & -414 & 2388 & \\ \hline 1 & -12 & 69 & -398 & 2400 & \end{array}$$

$p(-6) = 2400$

Answer: -6 is not a zero (root) of the polynomial

5a. $h(t) = t^4 + 8t^3 + 11t^2 - 40t - 80$ Back to Problem 5.

$$\begin{array}{rrrrr|l} 1 & 8 & 11 & -40 & -80 & 4 \\ & 4 & 48 & 236 & 784 & \\ \hline 1 & 12 & 59 & 196 & 704 & \end{array}$$

$h(4) = 704$

Answer: 4 is not a zero (root) of the polynomial

5b. $h(t) = t^4 + 8t^3 + 11t^2 - 40t - 80$ Back to Problem 5.

$$\begin{array}{rrrrr|l} 1 & 8 & 11 & -40 & -80 & -4 \\ & -4 & -16 & 20 & 80 & \\ \hline 1 & 4 & -5 & -20 & 0 & \end{array}$$

$h(-4) = 0$

Answer: -4 is a zero (root) of the polynomial

5c. $h(t) = t^4 + 8t^3 + 11t^2 - 40t - 80$ Back to Problem 5.

$$\begin{array}{rrrrr|l} 1 & 8 & 11 & -40 & -80 & -\sqrt{5} \\ & -\sqrt{5} & -8\sqrt{5} + 5 & -16\sqrt{5} + 40 & 80 & \\ \hline 1 & 8 - \sqrt{5} & 16 - 8\sqrt{5} & -16\sqrt{5} & 0 & \end{array}$$

$h(-\sqrt{5}\,) = 0$

NOTE: In order to find $h(-\sqrt{5}\,)$, I think it easier to do the evaluation of the function:

$h(t) = t^4 + 8t^3 + 11t^2 - 40t - 80 \Rightarrow$

$h(-\sqrt{5}\,) = 25 - 40\sqrt{5} + 55 + 40\sqrt{5} - 80 = 0$

Answer: $-\sqrt{5}$ is a zero (root) of the polynomial

5d. $h(t) = t^4 + 8t^3 + 11t^2 - 40t - 80$ Back to Problem 5.

$$
\begin{array}{ccccc|c}
1 & 8 & 11 & -40 & -80 & 3i \\
 & 3i & 24i - 9 & 6i - 72 & -336i - 18 & \\
\hline
1 & 8 + 3i & 2 + 24i & -112 + 6i & -98 - 336i &
\end{array}
$$

$h(3i) = -98 - 336i$

Answer: $3i$ is not a zero (root) of the polynomial

6a. $f(x) = 9x^4 + 18x^3 - 43x^2 - 32x + 48$ Back to Problem 6.

$x - 2$ is a factor of the polynomial f if and only if $f(2) = 0$.

9	18	– 43	– 32	48	2
	18	72	58	52	
9	36	29	26	100	

$f(2) = 100$

Answer: $x - 2$ is not a factor of the polynomial

6b. $f(x) = 9x^4 + 18x^3 - 43x^2 - 32x + 48$ Back to Problem 6.

$x + 3$ is a factor of the polynomial f if and only if $f(-3) = 0$.

9	18	– 43	– 32	48	– 3
	– 27	27	48	– 48	
9	– 9	– 16	16	0	

$f(-3) = 0$

Answer: $x + 3$ is a factor of the polynomial

6c. $f(x) = 9x^4 + 18x^3 - 43x^2 - 32x + 48$ Back to Problem 6.

$x - \dfrac{2}{3}$ is a factor of the polynomial f if and only if $f\left(\dfrac{2}{3}\right) = 0$.

$$
\begin{array}{cccccc|c}
9 & 18 & -43 & -32 & 48 = \frac{144}{3} & & \frac{2}{3} \\
 & 6 & 16 & -18 & -\frac{100}{3} & & \\
\hline
9 & 24 & -27 & -50 & \frac{44}{3} & &
\end{array}
$$

$$f\left(\frac{2}{3}\right) = \frac{44}{3}$$

Answer: $x - \frac{2}{3}$ is not a factor of the polynomial

6d. $f(x) = 9x^4 + 18x^3 - 43x^2 - 32x + 48$ Back to Problem 6.

$x + \frac{4}{3}$ is a factor of the polynomial f if and only if $f\left(-\frac{4}{3}\right) = 0$.

$$
\begin{array}{ccccc|c}
9 & 18 & -43 & -32 & 48 & -\frac{4}{3} \\
 & -12 & -8 & 68 & -48 & \\
\hline
9 & 6 & -51 & 36 & 0 &
\end{array}
$$

$$f\left(-\frac{4}{3}\right) = 0$$

Answer: $x + \frac{4}{3}$ is a factor of the polynomial

7a. $f(x) = x^3 - 5x^2 - 2x + 24$ Back to Problem 7.

To find the zeros (roots) of f, we want to solve the equation $f(x) = 0 \Rightarrow$

$x^3 - 5x^2 - 2x + 24 = 0$. The expression $x^3 - 5x^2 - 2x + 24$ can

not be factored by grouping.

We need to find one rational zero (root) for the polynomial f. This will produce a linear factor for the polynomial and the other factor will be quadratic.

Factors of 24: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$

Factors of 1: 1

Possible rational zeros (roots): $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$

Trying 1:

$$\begin{array}{cccc|c} \multicolumn{4}{c}{\overbrace{}^{Coeff\ of\ x^3 - 5x^2 - 2x + 24}} & \\ 1 & -5 & -2 & 24 & 1 \\ & 1 & -4 & -6 & \\ \hline 1 & -4 & -6 & 18 & \end{array}$$

Thus, $f(1) = 18 \neq 0 \Rightarrow x - 1$ is not a factor of f and 1 is not a zero (root) of f.

Trying -1:

$$\begin{array}{cccc|c} \multicolumn{4}{c}{\overbrace{}^{Coeff\ of\ x^3 - 5x^2 - 2x + 24}} & \\ 1 & -5 & -2 & 24 & -1 \\ & -1 & 6 & -4 & \\ \hline 1 & -6 & 4 & 20 & \end{array}$$

Thus, $f(-1) = 20 \neq 0 \Rightarrow x + 1$ is not a factor of f and -1 is not a zero (root) of f.

Trying 2:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hspace{8em}}^{Coeff\ of\ x^3 - 5x^2 - 2x + 24}} & \\ 1 & -5 & -2 & 24 & 2 \\ & 2 & -6 & -16 & \\ \hline 1 & -3 & -8 & 8 & \end{array}$$

Thus, $f(2) = 8 \neq 0 \Rightarrow x - 2$ is not a factor of f and 2 is not a zero (root) of f.

Trying -2:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hspace{8em}}^{Coeff\ of\ x^3 - 5x^2 - 2x + 24}} & \\ 1 & -5 & -2 & 24 & -2 \\ & -2 & 14 & -24 & \\ \hline 1 & -7 & 12 & 0 & \end{array}$$

Thus, $f(-2) = 0 \Rightarrow x + 2$ is a factor of f and -2 is a zero (root) of f.

NOTE: By the Bound Theorem, -2 is a lower bound for the negative zeros (roots) of f since we alternate from **positive** 1 to **negative** 7 to **positive** 12 to **negative** 0 in the third row of the synthetic division.

The third row in the synthetic division gives us the coefficients of the other factor starting with x^2. Thus, the other factor is $x^2 - 7x + 12$.

Thus, we have that $x^3 - 5x^2 - 2x + 24 = (x + 2)(x^2 - 7x + 12)$.

Now, we can try to find a factorization for the expression $x^2 - 7x + 12$:
$x^2 - 7x + 12 = (x - 3)(x - 4)$

Thus, we have that $x^3 - 5x^2 - 2x + 24 = (x + 2)(x^2 - 7x + 12) = (x + 2)(x - 3)(x - 4)$

Thus, $x^3 - 5x^2 - 2x + 24 = 0 \Rightarrow (x + 2)(x - 3)(x - 4) = 0 \Rightarrow$

$x = -2, \ x = 3, \ x = 4$

Answer: Zeros (Roots): -2, 3, 4

Factorization: $x^3 - 5x^2 - 2x + 24 = (x + 2)(x - 3)(x - 4)$

7b. $g(x) = 3x^3 - 23x^2 + 57x - 45$ Back to Problem 7.

To find the zeros (roots) of g, we want to solve the equation $g(x) = 0 \Rightarrow$

$3x^3 - 23x^2 + 57x - 45 = 0$. The expression $3x^3 - 23x^2 + 57x - 45$

can't be factored by grouping.

We need to find one rational zero (root) for the polynomial g. This will produce a linear factor for the polynomial and the other factor will be quadratic.

Factors of -45: $\pm 1,\ \pm 3,\ \pm 5,\ \pm 9,\ \pm 15,\ \pm 45$

Factors of 3: 1, 3

The rational numbers obtained using the factors of -45 for the numerator and the 1 as the factor of 3 for the denominator:

$$\pm 1,\ \pm 3,\ \pm 5,\ \pm 9,\ \pm 15,\ \pm 45$$

The rational numbers obtained using the factors of -45 for the numerator and the 3 as the factor of 3 for the denominator:

$$\pm \frac{1}{3},\ \pm 1,\ \pm \frac{5}{3},\ \pm 3,\ \pm 5,\ \pm 15$$

NOTE: $\pm \dfrac{3}{3} = \pm 1$, $\pm \dfrac{9}{3} = \pm 3$, $\pm \dfrac{15}{3} = \pm 5$, and $\pm \dfrac{45}{3} = \pm 15$

Possible rational zeros (roots): $\pm\frac{1}{3}, \pm 1, \pm\frac{5}{3}, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45$

Trying 1:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{}^{Coeff\ of\ 3x^3 - 23x^2 + 57x - 45}} & \\ 3 & -23 & 57 & -45 & \underline{1} \\ & 3 & -20 & 37 & \\ \hline 3 & -20 & 37 & -8 & \end{array}$$

Thus, $g(1) = -8 \neq 0 \Rightarrow x - 1$ is not a factor of g and 1 is not a zero (root) of g.

Trying -1:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{}^{Coeff\ of\ 3x^3 - 23x^2 + 57x - 45}} & \\ 3 & -23 & 57 & -45 & \underline{-1} \\ & -3 & 26 & -83 & \\ \hline 3 & -26 & 83 & -128 & \end{array}$$

Thus, $g(-1) = -128 \neq 0 \Rightarrow x + 1$ is not a factor of g and -1 is not a zero (root) of g.

NOTE: By the Bound Theorem, -1 is a lower bound for the negative zeros (roots) of g since we alternate from **positive** 3 to **negative** 26 to **positive** 83 to **negative** 128 in the third row of the synthetic division. Thus, $-\frac{5}{3}$, -3, -5, -9, -15, and -45 can't be rational zeros (roots) of g.

Trying 2:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{}^{Coeff\ of\ 3x^3 - 23x^2 + 57x - 45}} & \\ 3 & -23 & 57 & -45 & \underline{2} \\ & 6 & -34 & 46 & \\ \hline 3 & -17 & 23 & 1 & \end{array}$$

Thus, $g(2) = 1 \neq 0 \Rightarrow x - 2$ is not a factor of g and 2 is not a zero (root) of g.

Trying 3:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{}^{Coeff\ of\ 3x^3 - 23x^2 + 57x - 45}} & \\ 3 & -23 & 57 & -45 & 3 \\ & 9 & -42 & 45 & \\ \hline 3 & -14 & 15 & 0 & \end{array}$$

Thus, $g(3) = 0 \Rightarrow x - 3$ is a factor of g and 3 is a zero (root) of g.

The third row in the synthetic division gives us the coefficients of the other factor starting with x^2. Thus, the other factor is $3x^2 - 14x + 15$.

Thus, we have that $3x^3 - 23x^2 + 57x - 45 = (x - 3)(3x^2 - 14x + 15)$.

Now, we can try to find a factorization for the expression $3x^2 - 14x + 15$:
$3x^2 - 14x + 15 = (x - 3)(3x - 5)$

Thus, we have that $3x^3 - 23x^2 + 57x - 45 = (x - 3)(3x^2 - 14x + 15)$
$= (x - 3)(x - 3)(3x - 5) = (x - 3)^2(3x - 5)$

Thus, $3x^3 - 23x^2 + 57x - 45 = 0 \Rightarrow (x - 3)^2(3x - 5) = 0 \Rightarrow$

$x = 3, \ x = \dfrac{5}{3}$

Answer: Zeros (Roots): $\dfrac{5}{3}$, 3 (multiplicity 2)

Factorization: $3x^3 - 23x^2 + 57x - 45 = (x - 3)^2(3x - 5)$

7c. $h(t) = 4t^3 - 4t^2 - 9t + 30$ Back to Problem 7.

To find the zeros (roots) of h, we want to solve the equation $h(t) = 0 \Rightarrow$

$4t^3 - 4t^2 - 9t + 30 = 0$. The expression $4t^3 - 4t^2 - 9t + 30$ can not be factored by grouping.

We need to find one rational zero (root) for the polynomial h. This will produce a linear factor for the polynomial and the other factor will be quadratic.

Factors of 30: $\pm 1, \ \pm 2, \ \pm 3, \ \pm 5, \ \pm 6, \ \pm 10, \ \pm 15, \ \pm 30$

Factors of 4: 1, 2, 4

The rational numbers obtained using the factors of -30 for the numerator and the 1 as the factor of 4 for the denominator:

$$\pm 1, \ \pm 2, \ \pm 3, \ \pm 5, \ \pm 6, \ \pm 10, \ \pm 15, \ \pm 30$$

The rational numbers obtained using the factors of -30 for the numerator and the 2 as the factor of 4 for the denominator:

$$\pm \frac{1}{2}, \ \pm 1, \ \pm \frac{3}{2}, \ \pm \frac{5}{2}, \ \pm 3, \ \pm 5, \ \pm \frac{15}{2}, \ \pm 15$$

Eliminating the ones that are already listed above, we have

$$\pm \frac{1}{2}, \ \pm \frac{3}{2}, \ \pm \frac{5}{2}, \ \pm \frac{15}{2}$$

The rational numbers obtained using the factors of -30 for the numerator and the 4 as the factor of 4 for the denominator:

$$\pm \frac{1}{4}, \ \pm \frac{1}{2}, \ \pm \frac{3}{4}, \ \pm \frac{5}{4}, \ \pm \frac{3}{2}, \ \pm \frac{5}{2}, \ \pm \frac{15}{4}, \ \pm \frac{15}{2}$$

Eliminating the ones that are already listed above, we have

$$\pm\frac{1}{4},\ \pm\frac{3}{4},\ \pm\frac{5}{4},\ \pm\frac{15}{4}$$

Possible rational zeros (roots): $\pm\frac{1}{4},\ \pm\frac{1}{2},\ \pm\frac{3}{4},\ \pm 1,\ \pm\frac{5}{4},\ \pm\frac{3}{2},\ \pm 2,\ \pm\frac{5}{2},\ \pm 3,\ \pm\frac{15}{4},\ \pm 5,\ \pm 6,\ \pm\frac{15}{2},\ \pm 10,\ \pm 15,\ \pm 30$

Trying 1:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{4 \quad -4 \quad -9 \quad 30}}^{Coeff\ of\ 4t^3\ -\ 4t^2\ -\ 9t\ +\ 30}} & \\ 4 & -4 & -9 & 30 & \ 1 \\ & 4 & 0 & -9 & \\ \hline 4 & 0 & -9 & 21 & \end{array}$$

Thus, $h(1) = 21 \neq 0 \Rightarrow t - 1$ is not a factor of h and 1 is not a zero (root) of h.

Trying 2:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{4 \quad -4 \quad -9 \quad 30}}^{Coeff\ of\ 4t^3\ -\ 4t^2\ -\ 9t\ +\ 30}} & \\ 4 & -4 & -9 & 30 & \ 2 \\ & 8 & 8 & -2 & \\ \hline 4 & 4 & -1 & 28 & \end{array}$$

Thus, $h(2) = 28 \neq 0 \Rightarrow t - 2$ is not a factor of h and 2 is not a zero (root) of h.

Trying 3:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{4 \quad -4 \quad -9 \quad 30}}^{Coeff\ of\ 4t^3\ -\ 4t^2\ -\ 9t\ +\ 30}} & \\ 4 & -4 & -9 & 30 & \ 3 \\ & 12 & 24 & 45 & \\ \hline 4 & 8 & 15 & 75 & \end{array}$$

Thus, $h(3) = 75 \neq 0 \Rightarrow t - 3$ is not a factor of h and 3 is not a zero (root) of h.

NOTE: By the Bound Theorem, 3 is an upper bound for the positive zeros (roots) of h since all the numbers are positive in the third row of the synthetic division. Thus, $\frac{15}{4}$, 5, 6, $\frac{15}{2}$, 10, 15, and 30 can't be rational zeros (roots) of h.

Trying -1:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{4 \quad -4 \quad -9 \quad 30}}^{Coeff\ of\ 4t^3 - 4t^2 - 9t + 30}} & \\ 4 & -4 & -9 & 30 & -1 \\ \cline{5-5} & -4 & 8 & 1 & \\ \hline 4 & -8 & -1 & 31 & \end{array}$$

Thus, $h(-1) = 31 \neq 0 \Rightarrow t + 1$ is not a factor of h and -1 is not a zero (root) of h.

Trying -2:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{4 \quad -4 \quad -9 \quad 30}}^{Coeff\ of\ 4t^3 - 4t^2 - 9t + 30}} & \\ 4 & -4 & -9 & 30 & -2 \\ \cline{5-5} & -8 & 24 & -30 & \\ \hline 4 & -12 & 15 & 0 & \end{array}$$

Thus, $h(-2) = 0 \Rightarrow t + 2$ is factor of h and -2 is a zero (root) of h.

The third row in the synthetic division gives us the coefficients of the other factor starting with t^2. Thus, the other factor is $4t^2 - 12t + 15$.

Thus, we have that $4t^3 - 4t^2 - 9t + 30 = (t + 2)(4t^2 - 12t + 15)$.

Now, we can try to find a factorization for the expression $4t^2 - 12t + 15$. However, it does not factor.

Thus, we have that $4t^3 - 4t^2 - 9t + 30 = (t + 2)(4t^2 - 12t + 15)$

Thus, $4t^3 - 4t^2 - 9t + 30 = 0 \Rightarrow (t + 2)(4t^2 - 12t + 15) = 0 \Rightarrow$

$t = -2, \; 4t^2 - 12t + 15 = 0$

We will need to use the Quadratic Formula to solve $4t^2 - 12t + 15 = 0$.

Thus, $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{12 \pm \sqrt{12 \cdot 12 - 4(4)15}}{8} =$

$\dfrac{12 \pm \sqrt{4 \cdot 4\,[3 \cdot 3 - 1(1)15]}}{8} = \dfrac{12 \pm 4\sqrt{9 - 15}}{8} = \dfrac{12 \pm 4\sqrt{-6}}{8} =$

$\dfrac{12 \pm 4i\sqrt{6}}{8} = \dfrac{3 \pm i\sqrt{6}}{2}$

Answer: Zeros (Roots): $-2, \; \dfrac{3 + i\sqrt{6}}{2}, \; \dfrac{3 - i\sqrt{6}}{2}$

Factorization: $4t^3 - 4t^2 - 9t + 30 = (t + 2)(4t^2 - 12t + 15)$

7d. $p(x) = x^4 - 6x^3 - 3x^2 + 16x + 12$ Back to Problem 7.

To find the zeros (roots) of p, we want to solve the equation $p(x) = 0 \Rightarrow$

$x^4 - 6x^3 - 3x^2 + 16x + 12 = 0$.

We need to find two rational zeros (roots) for the polynomial p. This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of 12: $\pm 1, \; \pm 2, \; \pm 3, \; \pm 4, \; \pm 6, \; \pm 12$

Factors of 1: 1

Possible rational zeros (roots): $\pm 1, \ \pm 2, \ \pm 3, \ \pm 4, \ \pm 6, \ \pm 12$

Trying 1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad -6 \quad -3 \quad 16 \quad 12}}^{Coeff\ of\ x^4 - 6x^3 - 3x^2 + 16x + 12}} & \\ 1 & -6 & -3 & 16 & 12 & \ 1 \\ & 1 & -5 & -8 & 8 & \\ \hline 1 & -5 & -8 & 8 & 20 & \end{array}$$

Thus, $p(1) = 20 \neq 0 \Rightarrow x - 1$ is not a factor of p and 1 is not a zero (root) of p.

Trying -1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad -6 \quad -3 \quad 16 \quad 12}}^{Coeff\ of\ x^4 - 6x^3 - 3x^2 + 16x + 12}} & \\ 1 & -6 & -3 & 16 & 12 & \ -1 \\ & -1 & 7 & -4 & -12 & \\ \hline 1 & -7 & 4 & 12 & 0 & \end{array}$$

Thus, $p(-1) = 0 \Rightarrow x + 1$ is a factor of p and -1 is a zero (root) of p.

The third row in the synthetic division gives us the coefficients of the other factor starting with x^3. Thus, the other factor is $x^3 - 7x^2 + 4x + 12$.

Thus, we have that $x^4 - 6x^3 - 3x^2 + 16x + 12 =$ $(x + 1)(x^3 - 7x^2 + 4x + 12)$.

Note that the remaining zeros of the polynomial p must also be zeros (roots) of the quotient polynomial $q(x) = x^3 - 7x^2 + 4x + 12$. We will use this polynomial to find the remaining zeros (roots) of p, including another zero (root) of -1.

Trying -1 again:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{1 \quad -7 \quad 4 \quad 12}}^{Coeff\ of\ x^3 - 7x^2 + 4x + 12}} & \\ 1 & -7 & 4 & 12 & \ -1 \\ & -1 & 8 & -12 & \\ \hline 1 & -8 & 12 & 0 & \end{array}$$

The remainder is 0. Thus, $x+1$ is a factor of the quotient polynomial $q(x) = x^3 - 7x^2 + 4x + 12$ and -1 is a zero (root) of multiplicity of the polynomial *p*.

Thus, we have that $x^3 - 7x^2 + 4x + 12 = (x+1)(x^2 - 8x + 12)$.

Thus, we have that $x^4 - 6x^3 - 3x^2 + 16x + 12 =$
$(x+1)(x^3 - 7x^2 + 4x + 12) = (x+1)(x+1)(x^2 - 8x + 12) =$
$(x+1)^2(x^2 - 8x + 12)$.

Now, we can try to find a factorization for the expression $x^2 - 8x + 12$:
$x^2 - 8x + 12 = (x-2)(x-6)$

Thus, we have that $x^4 - 6x^3 - 3x^2 + 16x + 12 =$
$(x+1)^2(x^2 - 8x + 12) = (x+1)^2(x-2)(x-6)$

Thus, $x^4 - 6x^3 - 3x^2 + 16x + 12 = 0 \Rightarrow$

$(x+1)^2(x-2)(x-6) = 0 \Rightarrow x = -1, \; x = 2, \; x = 6$

Answer: Zeros (Roots): -1 (multiplicity 2), 2, 6

Factorization: $x^4 - 6x^3 - 3x^2 + 16x + 12 = (x+1)^2(x-2)(x-6)$

7e. $f(z) = 6z^4 - 11z^3 - 53z^2 + 108z - 36$ Back to Problem 7.

To find the zeros (roots) of *f*, we want to solve the equation $f(z) = 0 \Rightarrow$

$6z^4 - 11z^3 - 53z^2 + 108z - 36 = 0$.

We need to find two rational zeros (roots) for the polynomial f. This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of -36: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 9,\ \pm 12,\ \pm 18,\ \pm 36$

Factors of 6: $1,\ 2,\ 3,\ 6$

The rational numbers obtained using the factors of -36 for the numerator and the 1 as the factor of 6 for the denominator:

$$\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 9,\ \pm 12,\ \pm 18,\ \pm 36$$

The rational numbers obtained using the factors of -36 for the numerator and the 2 as the factor of 6 for the denominator:

$$\pm \frac{1}{2},\ \pm 1,\ \pm \frac{3}{2},\ \pm 2,\ \pm 3,\ \pm \frac{9}{2},\ \pm 6,\ \pm 9,\ \pm 18$$

Eliminating the ones that are already listed above, we have

$$\pm \frac{1}{2},\ \pm \frac{3}{2},\ \pm \frac{9}{2}$$

The rational numbers obtained using the factors of -36 for the numerator and the 3 as the factor of 6 for the denominator:

$$\pm \frac{1}{3},\ \pm \frac{2}{3},\ \pm 1,\ \pm \frac{4}{3},\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 12$$

Eliminating the ones that are already listed above, we have

$$\pm \frac{1}{3},\ \pm \frac{2}{3},\ \pm \frac{4}{3}$$

The rational numbers obtained using the factors of -36 for the numerator and the 6 as the factor of 6 for the denominator:

$$\pm\frac{1}{6},\ \pm\frac{1}{3},\ \pm\frac{1}{2},\ \pm\frac{2}{3},\ \pm 1,\ \pm\frac{3}{2},\ \pm 2,\ \pm 3,\ \pm 6$$

Eliminating the ones that are already listed above, we have $\pm\frac{1}{6}$

Possible rational zeros (roots): $\pm\frac{1}{6},\ \pm\frac{1}{3},\ \pm\frac{1}{2},\ \pm\frac{2}{3},\ \pm 1,\ \pm\frac{4}{3},\ \pm\frac{3}{2},$
$\pm 2,\ \pm 3,\ \pm 4,\ \pm\frac{9}{2},\ \pm 6,\ \pm 9,\ \pm 12,\ \pm 18,\ \pm 36$

Trying 1:

$$\begin{array}{ccccc|c} \multicolumn{5}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{Coeff\ \ of\ \ 6z^4\ -\ 11z^3\ -\ 53z^2\ +\ 108z\ -\ 36}} & \\ 6 & -11 & -53 & 108 & -36 & 1 \\ & 6 & -5 & -58 & 50 & \\ \hline 6 & -5 & -58 & 50 & 14 & \end{array}$$

Thus, $f(1) = 14 \neq 0 \Rightarrow z - 1$ is not a factor of f and 1 is not a zero (root) of f.

Trying -1:

$$\begin{array}{ccccc|c} \multicolumn{5}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{Coeff\ \ of\ \ 6z^4\ -\ 11z^3\ -\ 53z^2\ +\ 108z\ -\ 36}} & \\ 6 & -11 & -53 & 108 & -36 & -1 \\ & -6 & 17 & 36 & -144 & \\ \hline 6 & -17 & -36 & 144 & -180 & \end{array}$$

Thus, $f(-1) = -180 \Rightarrow z + 1$ is not a factor of p and -1 is not a zero (root) of f.

Trying 2:

$$\begin{array}{ccccc|c} \multicolumn{5}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{Coeff\ \ of\ \ 6z^4\ -\ 11z^3\ -\ 53z^2\ +\ 108z\ -\ 36}} & \\ 6 & -11 & -53 & 108 & -36 & 2 \\ & 12 & 2 & -102 & 12 & \\ \hline 6 & 1 & -51 & 6 & -24 & \end{array}$$

Thus, $f(2) = -24 \neq 0 \Rightarrow z - 2$ is not a factor of f and 2 is not a zero (root) of f.

Trying -2:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{6 \quad -11 \quad -53 \quad 108 \quad -36}}^{Coeff\ of\ 6z^4 - 11z^3 - 53z^2 + 108z - 36}} & \\ 6 & -11 & -53 & 108 & -36 & -2 \\ & -12 & 46 & 14 & -244 & \\ \hline 6 & -23 & -7 & 122 & -280 & \end{array}$$

Thus, $f(-2) = -280 \Rightarrow z + 2$ is not a factor of p and -2 is not a zero (root) of f.

Trying 3:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{6 \quad -11 \quad -53 \quad 108 \quad -36}}^{Coeff\ of\ 6z^4 - 11z^3 - 53z^2 + 108z - 36}} & \\ 6 & -11 & -53 & 108 & -36 & 3 \\ & 18 & 21 & -96 & 36 & \\ \hline 6 & 7 & -32 & 12 & 0 & \end{array}$$

Thus, $f(3) = 0 \Rightarrow z - 3$ is a factor of f and 3 is a zero (root) of f.

The third row in the synthetic division gives us the coefficients of the other factor starting with z^3. Thus, the other factor is $6z^3 + 7z^2 - 32z + 12$.

Thus, we have that $6z^4 - 11z^3 - 53z^2 + 108z - 36 =$
$(z - 3)(6z^3 + 7z^2 - 32z + 12)$.

Note that the remaining zeros of the polynomial f must also be zeros (roots) of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$. We will use this polynomial to find the remaining zeros (roots) of f, including another zero (root) of 3.

Trying 3 again:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hspace{12em}}^{Coeff\ of\ 6z^3 + 7z^2 - 32z + 12}} & \\ 6 & 7 & -32 & 12 & \underline{3} \\ & 18 & 75 & 129 & \\ \hline 6 & 25 & 43 & 141 & \end{array}$$

The remainder is 141 and not 0. Thus, $z - 3$ is not a factor of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ and 3 is not a zero (root) of q. Thus, the multiplicity of the zero (root) of 3 is one.

NOTE: By the Bound Theorem, 3 is an upper bound for the positive zeros (roots) of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ since all the numbers are positive in the third row of the synthetic division. Thus, 4, $\frac{9}{2}$, 6, 9, 12, 18, and 36 can't be rational zeros (roots) of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ nor of the polynomial $f(z) = 6z^4 - 11z^3 - 53z^2 + 108z - 36 =$ $(z - 3)(6z^3 + 7z^2 - 32z + 12)$.

Trying -3:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hspace{12em}}^{Coeff\ of\ 6z^3 + 7z^2 - 32z + 12}} & \\ 6 & 7 & -32 & 12 & \underline{-3} \\ & -18 & 33 & -3 & \\ \hline 6 & -11 & 1 & 9 & \end{array}$$

The remainder is 9 and not 0. Thus, $z + 3$ is not a factor of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ and -3 is not a zero (root) of q.

Trying -4:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hspace{12em}}^{Coeff\ of\ 6z^3 + 7z^2 - 32z + 12}} & \\ 6 & 7 & -32 & 12 & \underline{-4} \\ & -24 & 68 & -144 & \\ \hline 6 & -17 & 36 & -132 & \end{array}$$

The remainder is -132 and not 0. Thus, $z + 4$ is not a factor of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ and -4 is not a zero (root) of q.

NOTE: By the Bound Theorem, -4 is a lower bound for the negative zeros (roots) of the quotient polynomial q since we alternate from **positive** 6 to **negative** 17 to **positive** 36 to **negative** 132 in the third row of the synthetic division. Thus, $-\frac{9}{2}$, -6, -9, -12, -18, and -36 can't be rational zeros (roots) of q.

Thus, the only possible rational zeros (roots) which are left to be checked are $\pm\frac{1}{6}$, $\pm\frac{1}{3}$, $\pm\frac{1}{2}$, $\pm\frac{2}{3}$, $\pm\frac{4}{3}$, and $\pm\frac{3}{2}$.

Trying $\frac{3}{2}$:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{6 \quad 7 \quad -32 \quad 12}}^{Coeff\ of\ 6z^3 + 7z^2 - 32z + 12}} & \\ 6 & 7 & -32 & 12 & \frac{3}{2} \\ & 9 & 24 & -12 & \\ \hline 6 & 16 & -8 & 0 & \end{array}$$

The remainder is 0. Thus, $z - \frac{3}{2}$ is a factor of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ and $\frac{3}{2}$ is a zero (root) of q.

Thus, we have that $6z^3 + 7z^2 - 32z + 12 = \left(z - \frac{3}{2}\right)(6z^2 + 16z - 8) = \left(z - \frac{3}{2}\right)2(3z^2 + 8z - 4) = (2z - 3)(3z^2 + 8z - 4)$.

Thus, we have that $6z^4 - 11z^3 - 53z^2 + 108z - 36 =$
$(z - 3)(6z^3 + 7z^2 - 32z + 12) = (z - 3)(2z - 3)(3z^2 + 8z - 4)$.

Now, we can try to find a factorization for the expression $3z^2 + 8z - 4$. However, it does not factor.

Thus, we have that $6z^4 - 11z^3 - 53z^2 + 108z - 36 = (z - 3)(2z - 3)(3z^2 + 8z - 4)$.

Thus, $6z^4 - 11z^3 - 53z^2 + 108z - 36 = 0 \Rightarrow$

$(z - 3)(2z - 3)(3z^2 + 8z - 4) = 0 \Rightarrow z = 3, \; z = \dfrac{3}{2},$

$3z^2 + 8z - 4 = 0$

We will need to use the Quadratic Formula to solve $3z^2 + 8z - 4 = 0$.

Thus, $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-8 \pm \sqrt{64 - 4(3)(-4)}}{6} =$

$\dfrac{-8 \pm \sqrt{16(4 + 3)}}{6} = \dfrac{-8 \pm 4\sqrt{7}}{6} = \dfrac{-4 \pm 2\sqrt{7}}{3}$

Answer: Zeros (Roots): $\dfrac{-4 - 2\sqrt{7}}{3}, \; \dfrac{-4 + 2\sqrt{7}}{3}, \; \dfrac{3}{2}, \; 3$

Factorization: $6z^4 - 11z^3 - 53z^2 + 108z - 36 =$
$(z - 3)(2z - 3)(3z^2 + 8z - 4)$

7f. $g(x) = 9x^4 + 18x^3 - 43x^2 - 32x + 48$ Back to Problem 7.

To find the zeros (roots) of g, we want to solve the equation $g(x) = 0 \Rightarrow$

$9x^4 + 18x^3 - 43x^2 - 32x + 48 = 0$.

We need to find two rational zeros (roots) for the polynomial f. This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of 48: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 8,\ \pm 12,\ \pm 16, \pm 24,\ \pm 48$

Factors of 9: 1, 3, 9

The rational numbers obtained using the factors of 48 for the numerator and the 1 as the factor of 9 for the denominator:

$$\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 8,\ \pm 12,\ \pm 16, \pm 24,\ \pm 48$$

The rational numbers obtained using the factors of 48 for the numerator and the 3 as the factor of 9 for the denominator:

$$\pm \frac{1}{3},\ \pm \frac{2}{3},\ \pm 1,\ \pm \frac{4}{3},\ \pm 2,\ \pm \frac{8}{3},\ \pm 4,\ \pm \frac{16}{3}, \pm 8,\ \pm 16$$

Eliminating the ones that are already listed above, we have

$$\pm \frac{1}{3},\ \pm \frac{2}{3},\ \pm \frac{4}{3},\ \pm \frac{8}{3},\ \pm \frac{16}{3}$$

The rational numbers obtained using the factors of 48 for the numerator and the 9 as the factor of 9 for the denominator:

$$\pm \frac{1}{9},\ \pm \frac{2}{9},\ \pm \frac{1}{3},\ \pm \frac{4}{9},\ \pm \frac{2}{3},\ \pm \frac{8}{9},\ \pm \frac{4}{3},\ \pm \frac{16}{9}, \pm \frac{8}{3},\ \pm \frac{16}{3}$$

Eliminating the ones that are already listed above, we have

$$\pm \frac{1}{9},\ \pm \frac{2}{9},\ \pm \frac{4}{9},\ \pm \frac{8}{9},\ \pm \frac{16}{9}$$

Possible rational zeros (roots): $\pm\frac{1}{9}$, $\pm\frac{2}{9}$, $\pm\frac{1}{3}$, $\pm\frac{4}{9}$, $\pm\frac{2}{3}$, $\pm\frac{8}{9}$, ± 1, $\pm\frac{4}{3}$, $\pm\frac{16}{9}$, ± 2, $\pm\frac{8}{3}$, ± 3, ± 4, $\pm\frac{16}{3}$, ± 6, ± 8, ± 12, ± 16, ± 24, ± 48

Trying 1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\textit{Coeff of } 9x^4 + 18x^3 - 43x^2 - 32x + 48}} & \\ 9 & 18 & -43 & -32 & 48 & \underline{1} \\ & 9 & 27 & -16 & -48 & \\ \hline 9 & 27 & -16 & -48 & 0 & \end{array}$$

Thus, $g(1) = 0 \Rightarrow x - 1$ is a factor of g and 1 is a zero (root) of g.

The third row in the synthetic division gives us the coefficients of the other factor starting with x^3. Thus, the other factor is $9x^3 + 27x^2 - 16x - 48$.

Thus, we have that $9x^4 + 18x^3 - 43x^2 - 32x + 48 = (x - 1)(9x^3 + 27x^2 - 16x - 48)$.

Note that the remaining zeros of the polynomial g must also be zeros (roots) of the quotient polynomial $q(x) = 9x^3 + 27x^2 - 16x - 48$. We will use this polynomial to find the remaining zeros (roots) of g, including another zero (root) of 1.

NOTE: The expression $9x^3 + 27x^2 - 16x - 48$ can be factored by grouping:

$$9x^3 + 27x^2 - 16x - 48 = 9x^2(x + 3) - 16(x + 3) =$$

$$(x + 3)(9x^2 - 16) = (x + 3)(3x + 4)(3x - 4)$$

Thus, we have that $9x^4 + 18x^3 - 43x^2 - 32x + 48 = (x - 1)(9x^3 + 27x^2 - 16x - 48) = (x - 1)(x + 3)(3x + 4)(3x - 4)$.

Thus, $9x^4 + 18x^3 - 43x^2 - 32x + 48 = 0 \Rightarrow$

$(x - 1)(x + 3)(3x + 4)(3x - 4) = 0 \Rightarrow x = 1,\ x = -3,\ x = -\frac{4}{3},$

$x = \frac{4}{3}$

Answer: Zeros (Roots): $-3,\ -\frac{4}{3},\ 1,\ \frac{4}{3}$

Factorization: $9x^4 + 18x^3 - 43x^2 - 32x + 48 =$
$(x - 1)(x + 3)(3x + 4)(3x - 4)$

7g. $h(x) = x^4 + 8x^3 + 11x^2 - 40x - 80$ Back to Problem 7.

To find the zeros (roots) of h, we want to solve the equation $h(x) = 0 \Rightarrow$
$x^4 + 8x^3 + 11x^2 - 40x - 80 = 0$.

We need to find two rational zeros (roots) for the polynomial h. This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of -80: $\pm 1,\ \pm 2,\ \pm 4,\ \pm 5,\ \pm 8,\ \pm 10\ \ \pm 16,\ \pm 20,\ \pm 40,$
± 80

Factors of 1: 1

Possible rational zeros (roots): $\pm 1,\ \pm 2,\ \pm 4,\ \pm 5,\ \pm 8,\ \pm 10\ \ \pm 16,$
$\pm 20,\ \pm 40,\ \pm 80$

Trying 1:

$$\begin{array}{rrrrrl}
\multicolumn{5}{c}{\overbrace{\hspace{5cm}}^{Coeff\ \ of\ \ x^4\ +\ 8x^3\ +\ 11x^2\ -\ 40x\ -\ 80}} & \\
1 & 8 & 11 & -40 & -80 & \underline{|\ 1} \\
 & 1 & 9 & 20 & -20 & \\ \cline{1-5}
1 & 9 & 20 & -20 & -100 &
\end{array}$$

Thus, $h(1) = -100 \neq 0 \Rightarrow x - 1$ is not a factor of h and 1 is not a zero (root) of h.

Trying -1:

$$\begin{array}{rrrrrl}
\multicolumn{5}{c}{\overbrace{\hspace{5cm}}^{Coeff\ \ of\ \ x^4\ +\ 8x^3\ +\ 11x^2\ -\ 40x\ -\ 80}} & \\
1 & 8 & 11 & -40 & -80 & \underline{|\ -1} \\
 & -1 & -7 & -4 & 44 & \\ \cline{1-5}
1 & 7 & 4 & -44 & -36 &
\end{array}$$

Thus, $h(-1) = -36 \neq 0 \Rightarrow x + 1$ is not a factor of h and -1 is not a zero (root) of h.

Trying 2:

$$\begin{array}{rrrrrl}
\multicolumn{5}{c}{\overbrace{\hspace{5cm}}^{Coeff\ \ of\ \ x^4\ +\ 8x^3\ +\ 11x^2\ -\ 40x\ -\ 80}} & \\
1 & 8 & 11 & -40 & -80 & \underline{|\ 2} \\
 & 2 & 20 & 62 & 44 & \\ \cline{1-5}
1 & 10 & 31 & 22 & -36 &
\end{array}$$

Thus, $h(2) = -36 \neq 0 \Rightarrow x - 2$ is not a factor of h and 2 is not a zero (root) of h.

Trying -2:

$$\begin{array}{rrrrrl}
\multicolumn{5}{c}{\overbrace{\hspace{5cm}}^{Coeff\ \ of\ \ x^4\ +\ 8x^3\ +\ 11x^2\ -\ 40x\ -\ 80}} & \\
1 & 8 & 11 & -40 & -80 & \underline{|\ -2} \\
 & -2 & -12 & 2 & 76 & \\ \cline{1-5}
1 & 6 & -1 & -38 & -4 &
\end{array}$$

Thus, $h(-2) = -4 \neq 0 \Rightarrow x + 2$ is not a factor of h and -2 is not a zero (root) of h.

Trying 3:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{Coeff\ \ of\ \ x^4 + 8x^3 + 11x^2 - 40x - 80}} & \\ 1 & 8 & 11 & -40 & -80 & \ 3 \\ \cline{6-6} & 3 & 33 & 132 & 276 & \\ \hline 1 & 11 & 44 & 92 & 196 & \end{array}$$

Thus, $h(3) = 196 \neq 0 \Rightarrow x - 3$ is not a factor of h and 3 is not a zero (root) of h.

NOTE: By the Bound Theorem, 3 is an upper bound for the positive zeros (roots) of h since all the numbers are positive in the third row of the synthetic division. Thus, 4, 5, 8, 10 16, 20, 40, and 80 can't be rational zeros (roots) of h.

Trying -3:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{Coeff\ \ of\ \ x^4 + 8x^3 + 11x^2 - 40x - 80}} & \\ 1 & 8 & 11 & -40 & -80 & \ -3 \\ \cline{6-6} & -3 & -15 & 12 & 84 & \\ \hline 1 & 5 & -4 & -28 & 4 & \end{array}$$

Thus, $h(-3) = 4 \neq 0 \Rightarrow x + 3$ is not a factor of h and -3 is not a zero (root) of h.

Trying -4:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{Coeff\ \ of\ \ x^4 + 8x^3 + 11x^2 - 40x - 80}} & \\ 1 & 8 & 11 & -40 & -80 & \ -4 \\ \cline{6-6} & -4 & -16 & 20 & 80 & \\ \hline 1 & 4 & -5 & -20 & 0 & \end{array}$$

Thus, $h(-4) = 0 \Rightarrow x + 4$ is a factor of h and -4 is a zero (root) of h.

The third row in the synthetic division gives us the coefficients of the other factor starting with x^3. Thus, the other factor is $x^3 + 4x^2 - 5x - 20$.

Thus, we have that $x^4 + 8x^3 + 11x^2 - 40x - 80 =$
$(x + 4)(x^3 + 4x^2 - 5x - 20)$.

Note that the remaining zeros of the polynomial g must also be zeros (roots) of the quotient polynomial $q(x) = x^3 + 4x^2 - 5x - 20$. We will use this polynomial to find the remaining zeros (roots) of g, including another zero (root) of -4.

Trying -4 again:

$$\begin{array}{cccc|c} \multicolumn{4}{c}{\overbrace{\textit{Coeff of } x^3 + 4x^2 - 5x - 20}} & \\ 1 & 4 & -5 & -20 & -4 \\ & -4 & 0 & 20 & \\ \hline 1 & 0 & -5 & 0 & \end{array}$$

The remainder is 0. Thus, $x + 4$ is a factor of the quotient polynomial $q(x) = x^3 + 4x^2 - 5x - 20$ and -4 is a zero (root) of multiplicity of the polynomial h.

Thus, we have that $x^3 + 4x^2 - 5x - 20 = (x + 4)(x^2 - 5)$.

Thus, we have that $x^4 + 8x^3 + 11x^2 - 40x - 80 =$
$(x + 4)(x^3 + 4x^2 - 5x - 20) = (x + 4)(x + 4)(x^2 - 5) = (x + 4)^2(x^2 - 5)$.

Thus, $x^4 + 8x^3 + 11x^2 - 40x - 80 = 0 \Rightarrow$

$(x + 4)^2(x^2 - 5) = 0 \Rightarrow x = -4,\ x^2 - 5 = 0$

$x^2 - 5 = 0 \Rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5}$

Answer: Zeros (Roots): -4 (multiplicity 2), $-\sqrt{5}$, $\sqrt{5}$

Factorization: $x^4 + 8x^3 + 11x^2 - 40x - 80 = (x + 4)^2(x^2 - 5)$

7h. $p(t) = t^4 - 12t^3 + 54t^2 - 108t + 81$ Back to Problem 7.

To find the zeros (roots) of p, we want to solve the equation $p(t) = 0 \Rightarrow$ $t^4 - 12t^3 + 54t^2 - 108t + 81 = 0$.

We need to find two rational zeros (roots) for the polynomial p. This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of 81: $\pm 1, \pm 3, \pm 9, \pm 27, \pm 81$

Factors of 1: 1

Possible rational zeros (roots): $\pm 1, \pm 3, \pm 9, \pm 27, \pm 81$

Trying 1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad -12 \quad 54 \quad -108 \quad 81}}^{Coeff\ of\ t^4 - 12t^3 + 54t^2 - 108t + 81}} & \\ 1 & -12 & 54 & -108 & 81 & \underline{1} \\ & 1 & -11 & 43 & -65 & \\ \hline 1 & -11 & 43 & -65 & 16 & \end{array}$$

Thus, $p(1) = 16 \neq 0 \Rightarrow t - 1$ is not a factor of p and 1 is not a zero (root) of p.

Trying -1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad -12 \quad 54 \quad -108 \quad 81}}^{Coeff\ of\ t^4 - 12t^3 + 54t^2 - 108t + 81}} & \\ 1 & -12 & 54 & -108 & 81 & \underline{-1} \\ & -1 & 13 & -67 & 175 & \\ \hline 1 & -13 & 67 & -175 & 256 & \end{array}$$

Thus, $p(-1) = 256 \neq 0 \Rightarrow t + 1$ is not a factor of p and -1 is not a zero (root) of p.

NOTE: By the Bound Theorem, -1 is a lower bound for the negative zeros (roots) of p since we alternate from **positive** 1 to **negative** 13 to **positive** 67 to **negative** 175 to **positive** 256 in the third row of the synthetic division. Thus, -3, -9, -27, and -81 can't be rational zeros (roots) of p.

Trying 3:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad -12 \quad 54 \quad -108 \quad 81}}^{Coeff\ of\ t^4 - 12t^3 + 54t^2 - 108t + 81}} & \\ 1 & -12 & 54 & -108 & 81 & \underline{\;3\;} \\ & 3 & -27 & 81 & -81 & \\ \hline 1 & -9 & 27 & -27 & 0 & \end{array}$$

Thus, $p(3) = 0 \Rightarrow t - 3$ is a factor of p and 3 is a zero (root) of p.

The third row in the synthetic division gives us the coefficients of the other factor starting with t^3. Thus, the other factor is $t^3 - 9t^2 + 27t - 27$.

Thus, we have that $t^4 - 12t^3 + 54t^2 - 108t + 81 =$
$(t - 3)(t^3 - 9t^2 + 27t - 27)$.

Note that the remaining zeros of the polynomial p must also be zeros (roots) of the quotient polynomial $q(t) = t^3 - 9t^2 + 27t - 27$. We will use this polynomial to find the remaining zeros (roots) of p, including another zero (root) of 3.

Trying 3 again:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{1 \quad -9 \quad 27 \quad -27}}^{Coeff\ of\ t^3 - 9t^2 + 27t - 27}} & \\ 1 & -9 & 27 & -27 & \underline{\;3\;} \\ & 3 & -18 & 27 & \\ \hline 1 & -6 & 9 & 0 & \end{array}$$

The remainder is 0. Thus, $t - 3$ is a factor of the quotient polynomial $q(t) = t^3 - 9t^2 + 27t - 27$ and 3 is a zero (root) of multiplicity of the polynomial p.

Thus, we have that $t^3 - 9t^2 + 27t - 27 = (t - 3)(t^2 - 6t + 9)$.

Thus, we have that $t^4 - 12t^3 + 54t^2 - 108t + 81 =$
$(t - 3)(t^3 - 9t^2 + 27t - 27) = (t - 3)(t - 3)(t^2 - 6t + 9) =$
$(t - 3)^2(t^2 - 6t + 9)$.

Since $t^2 - 6t + 9 = (t - 3)^2$, then we have that
$t^4 - 12t^3 + 54t^2 - 108t + 81 = (t - 3)^2(t^2 - 6t + 9) =$

$(t-3)^2(t-3)^2 = (t-3)^4$

Thus, $t^4 - 12t^3 + 54t^2 - 108t + 81 = 0 \Rightarrow (t-3)^4 = 0 \Rightarrow t = 3$

Answer: Zeros (Roots): 3 (multiplicity 4)

Factorization: $t^4 - 12t^3 + 54t^2 - 108t + 81 = (t-3)^4$

Example Find $(2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20) \div (x^2 - 4x + 3)$.

$$
\begin{array}{r|l}
 & 2x^3 + 5x^2 - 3 \\
\cline{2-2}
x^2 - 4x + 3 & 2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20 \\
 & \underline{2x^5 - 8x^4 + 6x^3} \\
 & 5x^4 - 20x^3 + 12x^2 \\
 & \underline{5x^4 - 20x^3 + 15x^2} \\
 & -3x^2 + 16x - 20 \\
 & \underline{-3x^2 + 12x - 9} \\
 & 4x - 11
\end{array}
$$

NOTE: $2x^3(x^2 - 4x + 3) = 2x^5 - 8x^4 + 6x^3$

$5x^2(x^2 - 4x + 3) = 5x^4 - 20x^3 + 15x^2$

$-3(x^2 - 4x + 3) = -3x^2 + 12x - 9$

The expression $x^2 - 4x + 3$ is called the divisor in the division. The function $b(x) = x^2 - 4x + 3$ is called the divisor function.

The expression $2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20$ is called the dividend in the division. The function $a(x) = 2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20$ is called the dividend function.

The expression $2x^3 + 5x^2 - 3$ is called the quotient in the division. The function $q(x) = 2x^3 + 5x^2 - 3$ is called the quotient function.

The expression $4x - 11$ is called the remainder in the division. The function $r(x) = 4x - 11$ is called the remainder function.

We have that

$$\frac{2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20}{x^2 - 4x + 3} = 2x^3 + 5x^2 - 3 + \frac{4x - 11}{x^2 - 4x + 3}$$

Multiplying both sides of this equation by $x^2 - 4x + 3$, we have that

$$2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20 =$$

$$(x^2 - 4x + 3)(2x^3 + 5x^2 - 3) + (4x - 11)$$

Let a and b be polynomials. Then $\dfrac{a(x)}{b(x)} = q(x) + \dfrac{r(x)}{b(x)}$. The degree of the remainder polynomial r is less than the degree of divisor polynomial b, written $\deg r < \deg b$.

Multiplying both sides of the equation $\dfrac{a(x)}{b(x)} = q(x) + \dfrac{r(x)}{b(x)}$ by $r(x)$, we have that $a(x) = b(x)q(x) + r(x)$.

Example Find $(12x^4 - 17x^3 - 21x^2 + 7) \div (3x - 2)$.

$$
\begin{array}{r}
4x^3 - 3x^2 - 9x - 6 \\
3x - 2 \,\big)\overline{\,12x^4 - 17x^3 - 21x^2 + 0x + 7} \\
\underline{12x^4 - 8x^3} \\
-9x^3 - 21x^2 \\
\underline{-9x^3 + 6x^2} \\
-27x^2 + 0x \\
\underline{-27x^2 + 18x} \\
-18x + 7 \\
\underline{-18x + 12} \\
-5
\end{array}
$$

NOTE: $4x^3(3x - 2) = 12x^4 - 8x^3$

$-3x^2(3x - 2) = -9x^3 + 6x^2$

$-9x(3x - 2) = -27x^2 + 18x$

$-6(3x - 2) = -18x + 12$

The quotient function is $q(x) = 4x^3 - 3x^2 - 9x - 6$ and the remainder function is $r(x) = -5$. We have that

$$12x^4 - 17x^3 - 21x^2 + 7 = (3x - 2)(4x^3 - 3x^2 - 9x - 6) + (-5).$$

Example Find $(12x^4 - 17x^3 - 21x^2 + 7) \div \left(x - \dfrac{2}{3} \right)$.

$$
\begin{array}{r}
12x^3 - 9x^2 - 27x - 18 \\
x - \frac{2}{3}\overline{\big)\,12x^4 - 17x^3 - 21x^2 + 0x + 7} \\
\underline{12x^4 - 8x^3} \qquad\qquad\qquad\quad \\
-9x^3 - 21x^2 \qquad\qquad \\
\underline{-9x^3 + 6x^2} \qquad\qquad \\
-27x^2 + 0x \qquad \\
\underline{-27x^2 + 18x} \qquad \\
-18x + 7 \\
\underline{-18x + 12} \\
-5
\end{array}
$$

The quotient function is $q(x) = 12x^3 - 9x^2 - 27x - 18$ and the remainder function is $r(x) = -5$. We have that

$$12x^4 - 17x^3 - 21x^2 + 7 = \left(x - \frac{2}{3} \right)(12x^3 - 9x^2 - 27x - 18) + (-5).$$

NOTE: In the example above, we had that
$12x^4 - 17x^3 - 21x^2 + 7 = (3x - 2)(4x^3 - 3x^2 - 9x - 6) + (-5)$. Thus,

$$12x^4 - 17x^3 - 21x^2 + 7 = (3x - 2)(4x^3 - 3x^2 - 9x - 6) - 5 =$$

$$3\left(x - \frac{2}{3} \right)(4x^3 - 3x^2 - 9x - 6) - 5 =$$

$$\left(x - \frac{2}{3} \right)(12x^3 - 9x^2 - 27x - 18) - 5$$

<u>Example</u> Find $(x^5 + 4x^4 + 72x^2 - 8x - 20) \div (x + 6)$.

$$
\begin{array}{rl}
 & \phantom{x^5 + 4x^4 + {}} x^4 - 2x^3 + 12x^2 - 8 \\
x + 6 \,\big) & \overline{x^5 + 4x^4 + 0x^3 + 72x^2 - 8x - 20} \\
 & \underline{x^5 + 6x^4} \\
 & - 2x^4 + 0x^3 \\
 & \underline{- 2x^4 - 12x^3} \\
 & 12x^3 + 72x^2 \\
 & \underline{12x^3 + 72x^2} \\
 & - 8x - 20 \\
 & \underline{- 8x - 48} \\
 & \phantom{x^5 + 4x^4 + 0x^3 + 72x^2 - 8x - {}} 28
\end{array}
$$

The quotient function is $q(x) = x^4 - 2x^3 + 12x^2 - 8$ and the remainder function is $r(x) = 28$. We have that

$$x^5 + 4x^4 + 72x^2 - 8x - 20 = (x + 6)(x^4 - 2x^3 + 12x^2 - 8) + 28.$$

Consider the following.

$$
\begin{array}{cccccc|l}
\multicolumn{6}{c}{\overbrace{\hspace{9cm}}^{\text{Coefficients of } x^5 + 4x^4 + 72x^2 - 8x - 20}} & -6 \\
1 & 4 & 0 & 72 & -8 & -20 & \\
 & -6 & 12 & -72 & 0 & 48 & \\ \hline
1 & -2 & 12 & 0 & -8 & 28 &
\end{array}
$$

What do the numbers in the third row represent?

$$
\begin{array}{cccccc|l}
1 & 4 & 0 & 72 & -8 & -20 & -6 \\
 & -6 & 12 & -72 & 0 & 48 & \\ \hline
\multicolumn{5}{c}{\underbrace{1 \qquad -2 \qquad 12 \qquad 0 \qquad -8}_{\text{Coefficients of the quotient function starting with } x^4}} & \underbrace{28}_{\text{Remainder}} &
\end{array}
$$

Thus, the quotient function is $q(x) = x^4 - 2x^3 + 12x^2 - 8$ and the remainder function is $r(x) = 28$. These are the same answers that we obtained above using long division.

This process is called synthetic division. Synthetic division can only be used to divide a polynomial by another polynomial of degree one with a leading coefficient of one. Thus, you can't use synthetic division to find $(12x^4 - 17x^3 - 21x^2 + 7) \div (3x - 2)$. However, we can do the following division.

Example Use synthetic division to find $(12x^4 - 17x^3 - 21x^2 + 7) \div \left(x - \frac{2}{3}\right)$.

$$\begin{array}{rrrrr|l}
\multicolumn{5}{c}{\overbrace{\textit{Coefficients of } 12x^4 - 17x^3 - 21x^2 + 7}} & \\
12 & -17 & -21 & 0 & 7 & \frac{2}{3} \\
 & 8 & -6 & -18 & -12 & \\ \hline
12 & -9 & -27 & -18 & -5 & \\
\multicolumn{4}{c}{\underbrace{\textit{Coeff of quotient function starting with } x^3}} & &
\end{array}$$

Thus, the quotient function is $q(x) = 12x^3 - 9x^2 - 27x - 18$ and the remainder function is $r(x) = -5$. These are the same answers that we obtained above using long division.

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Example If $f(x) = x^5 + 4x^4 + 72x^2 - 8x - 20$, then find $f(-6)$.

$$f(-6) = (-6)^5 + 4(-6)^4 + 72(-6)^2 - 8(-6) - 20 =$$

$$-7776 + 4(1296) + 72(36) - 8(-6) - 20 =$$

$$-7776 + 5184 + 2592 + 48 - 20 = 28$$

This calculation would have been faster (and easier) using the fact that

$$x^5 + 4x^4 + 72x^2 - 8x - 20 = (x + 6)(x^4 - 2x^3 + 12x^2 - 8) + 28$$

that we obtained in the example above. Thus, $f(x) = (x + 6)q(x) + 28$, where $q(x) = x^4 - 2x^3 + 12x^2 - 8$.

Thus, $f(-6) = (-6 + 6)q(-6) + 28 = 0 \cdot q(-6) + 28 = 0 + 28 = 28$.

This result can be explained by the following theorem.

Theorem (The Remainder Theorem) Let p be a polynomial. If $p(x)$ is divided by $x - a$, then the remainder is $p(a)$.

Proof If $p(x)$ is divided by $x - a$, then $p(x) = (x - a)q(x) + r$. Thus, $p(a) = (a - a)q(a) + r = 0 \cdot q(a) + r = 0 + r = r$.

Example If $g(x) = 12x^4 - 17x^3 - 21x^2 + 7$, then find $g\left(\frac{2}{3}\right)$.

Using synthetic division to find $(12x^4 - 17x^3 - 21x^2 + 7) \div \left(x - \frac{2}{3}\right)$, we have that

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{12 \quad -17 \quad -21 \quad 0 \quad 7}}^{Coefficients\ of\ 12x^4 - 17x^3 - 21x^2 + 7}} & \\ 12 & -17 & -21 & 0 & 7 & \frac{2}{3} \\ & 8 & -6 & -18 & -12 & \\ \hline 12 & -9 & -27 & -18 & -5 & \end{array}$$

Thus, the remainder is -5. Thus, $g\left(\frac{2}{3}\right) = -5$.

Example If $h(x) = x^5 - 8x^4 + 12x^3 + 23x^2 - 16x - 37$, then find $h(3)$.

Using synthetic division to find $(x^5 - 8x^4 + 12x^3 + 23x^2 - 16x - 37) \div (x - 3)$, we have that

$$\begin{array}{rrrrrr|l}
\multicolumn{6}{c}{\overbrace{\textit{Coefficients of } x^5 - 8x^4 + 12x^3 + 23x^2 - 16x - 37}} & \\
1 & -8 & 12 & 23 & -16 & -37 & \ 3 \\
 & 3 & -15 & -9 & 42 & 78 & \\
\hline
1 & -5 & -3 & 14 & 26 & 41 &
\end{array}$$

Thus, the remainder is 41. Thus, $h(3) = 41$.

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Theorem (The Factor Theorem) Let p be a polynomial. The expression $x - a$ is a factor of $p(x)$ if and only if $p(a) = 0$.

Proof $(\Rightarrow)$ Suppose that $x - a$ is a factor of $p(x)$. Then the remainder upon division by $x - a$ must be zero. By the Remainder Theorem, $p(a) = 0$.

$(\Leftarrow)$ Suppose that $p(a) = 0$. By the Remainder Theorem, we have that $p(x) = (x - a)q(x) + p(a)$. Thus, $p(x) = (x - a)q(x)$. Thus, $x - a$ is a factor of $p(x)$.

Example Show that $x + 4$ is a factor of $p(x) = 5x^3 + 12x^2 - 20x + 48$.

We will use the Factor Theorem and show that $p(-4) = 0$. We will use the Remainder Theorem and synthetic division to find $p(-4)$.

$$\begin{array}{rrrr|l}
\multicolumn{4}{c}{\overbrace{\textit{Coeff of } 5x^3 + 12x^2 - 20x + 48}} & \\
5 & 12 & -20 & 48 & \ -4 \\
 & -20 & 32 & -48 & \\
\hline
5 & -8 & 12 & 0 &
\end{array}$$

Thus, $p(-4) = 0$. Thus, by the Factor Theorem, $x + 4$ is a factor of $p(x) = 5x^3 + 12x^2 - 20x + 48$.

NOTE: The third row in the synthetic division gives us the coefficients of the other factor starting with x^2. Thus, the other factor is $5x^2 - 8x + 12$.

Thus, we have that $5x^3 + 12x^2 - 20x + 48 = (x + 4)(5x^2 - 8x + 12)$.

Example Show that $t - 6$ is not a factor of $q(t) = 2t^4 - 7t^2 + 15$.

We will use the Factor Theorem and show that $q(6) \neq 0$. We will use the Remainder Theorem and synthetic division to find $q(6)$.

$$\begin{array}{ccccc|l}
\multicolumn{5}{c}{\overbrace{\hspace{12em}}^{\textit{Coefficients of } 2t^4 - 7t^2 + 15}} & \\
2 & 0 & -7 & 0 & 15 & \underline{6} \\
 & 12 & 72 & 390 & 2340 & \\
\hline
2 & 12 & 65 & 390 & 2355 &
\end{array}$$

Thus, $q(6) = 2355 \neq 0$. Thus, by the Factor Theorem, $t - 6$ is not a factor of $q(t) = 2t^4 - 7t^2 + 15$.

Example Find the value(s) of c so that $x + 3$ is a factor of $f(x) = 2x^4 - x^3 - 9x^2 + 22x + c$.

By the Factor Theorem, $x + 3$ is a factor of the polynomial f if and only if $f(-3) = 0$.

$$f(-3) = 162 + 27 - 81 - 66 + c = c + 42$$

Thus, $f(-3) = 0 \Rightarrow c + 42 = 0 \Rightarrow c = -42$.

Using the Remainder Theorem and synthetic division to find $f(-3)$, we have

$$\begin{array}{ccccc|l} \multicolumn{5}{c}{\overbrace{\hphantom{2 \quad -1 \quad -9 \quad 22 \quad c+42}}^{Coeff\ of\ 2x^4 - x^3 - 9x^2 + 22x + c}} & \\ 2 & -1 & -9 & 22 & c & -3 \\ & -6 & 21 & -36 & 42 & \\ \hline 2 & -7 & 12 & -14 & c+42 & \end{array}$$

By the Remainder Theorem, $f(-3) = c + 42$. Thus, $f(-3) = 0 \Rightarrow$

$c + 42 = 0 \Rightarrow c = -42$.

Answer: -42

<u>Example</u> Find the value(s) of c so that $t - 2$ is a factor of $g(t) = t^5 + 5t^3 - 6t^2 + ct - 64$.

By the Factor Theorem, $t - 2$ is a factor of the polynomial g if and only if $g(2) = 0$.

$$g(2) = 32 + 40 - 24 + 2c - 64 = 2c - 16$$

Thus, $g(2) = 0 \Rightarrow 2c - 16 = 0 \Rightarrow c = 8$.

Using the Remainder Theorem and synthetic division to find $g(2)$, we have

$$\begin{array}{cccccc|l} \multicolumn{6}{c}{\overbrace{\hphantom{1 \quad 0 \quad 5 \quad -6 \quad c+24 \quad 2c-16}}^{Coeff\ of\ t^5 + 5t^3 - 6t^2 + ct - 64}} & \\ 1 & 0 & 5 & -6 & c & -64 & 2 \\ & 2 & 4 & 18 & 24 & 2c+48 & \\ \hline 1 & 2 & 9 & 12 & c+24 & 2c-16 & \end{array}$$

By the Remainder Theorem, $g(2) = 2c - 16$. Thus, $g(2) = 0 \Rightarrow$

$2c - 16 = 0 \Rightarrow c = 8$.

Answer: 8

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Recall that a rational number is a quotient of integers. That is, a rational number is of the form $\frac{a}{b}$, where a and b are integers. A rational number $\frac{a}{b}$ is said to be in reduced form if the greatest common divisor (GCD) of a and b is one.

Examples $\frac{-3}{6}$, $\frac{4}{5}$, $\frac{17}{-9}$, and $\frac{8}{12}$ are rational numbers. The numbers $\frac{4}{5}$ and $\frac{17}{-9}$ are in reduced form. The numbers $\frac{-3}{6}$ and $\frac{8}{12}$ are not in reduced form. Of course, we can write $\frac{-3}{6}$ and $\frac{17}{-9}$ as $-\frac{3}{6}$ and $-\frac{17}{9}$ respectively. In reduced form, $-\frac{3}{6}$ is $-\frac{1}{2}$ and $\frac{8}{12}$ is $\frac{2}{3}$.

Examples $\frac{\sqrt{7}}{4}$ and $\frac{\pi}{6}$ are not rational numbers since $\sqrt{7}$ and π are not integers.

Theorem Let p be a polynomial with integer coefficients. If $\frac{c}{d}$ is a rational zero (root) in reduced form of

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdot \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0,$$

where the a_i's are integers for $i = 1, 2, 3, \ldots, n$ and $a_n \neq 0$ and $a_0 \neq 0$, then c is a factor of a_0 and d is a factor of a_n.

Theorem (Bounds for Real Zeros (Roots) of Polynomials) Let p be a polynomial with real coefficients and positive leading coefficient.

1. If $p(x)$ is synthetically divided by $x - a$, where $a > 0$, and all the numbers in the third row of the division process are either positive or zero, then a is an upper bound for the real solutions of the equation $p(x) = 0$.

2. If $p(x)$ is synthetically divided by $x + a$, where $a > 0$, and all the numbers in the third row of the division process are alternately positive and negative (and a 0 can be considered to be either positive or negative as needed), then $-a$ is a lower bound for the real solutions of the equation $p(x) = 0$.

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