Pre-Class Problems 11 for Wednesday, February 28

These are the type of problems that you will be working on in class.

You can go to the solution for each problem by clicking on the problem number or letter.

Discussion of zeros (roots) of polynomials and their multiplicity.

1. Find the zeros (roots) and their multiplicities. Discuss the implication of the multiplicity on the graph of the polynomial.

a. $f(x) = 2x^3 - 72x$

b. $g(x) = 6x^3 - 7x^2$

c. $h(x) = 48 + 32x - 3x^2 - 2x^3$

d. $p(x) = 5x^3 + 20x$

e. $q(x) = (6 - x)(x - 3)^2$

f. $q(t) = t^3 + 16t^2 + 64t$

g. $f(x) = x^3 - 2x^2 + 9x - 18$

h. $g(x) = 3x^4 + 5x^3 - 12x^2$

i. $h(x) = x^4 - 29x^2 + 100$

j. $f(x) = -16x^4 + 56x^2 - 49$

k. $p(x) = 15 - 2x^2 - x^4$

l. $q(x) = 8x^3 - 14x^4$

m. $f(x) = 54 + 27x - 2x^3 - x^4$

n. $g(t) = 3t^4 - 24t^3 + 48t^2$

o. $h(x) = (x - 1)(x + 2)(4x - 3)(5x + 2)$

p. $g(x) = 5x^4(8 - x)(2x + 7)^3$

Theorems on relative (local) extremum points (turning points).

2. Find the zeros (roots) and their multiplicities. Discuss the implication of the multiplicity on the graph of the polynomial. Determine the sign of the infinity that the polynomial values approaches as x or t approaches positive infinity and negative infinity. Determine whether the polynomial is even, odd, or neither in order to make use of symmetry if possible. Use this information to determine the number of relative (local) extremum points (turning points) that the graph of the polynomial has. Sketch a graph of the polynomial.

a. $f(x) = 2x^3 - 72x$

b. $g(x) = 6x^3 - 7x^2$

c. $h(x) = 48 + 32x - 3x^2 - 2x^3$

d. $p(x) = 5x^3 + 20x$

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m. $f(x) = 54 + 27x - 2x^3 - x^4$

n. $g(t) = 3t^4 - 24t^3 + 48t^2$

Problems available in the textbook: Page 311 … 21 – 40, 59 – 80 and Examples 2 – 7 starting on page 304.

SOLUTIONS:

1a. $f(x) = 2x^3 - 72x$ Back to Problem 1.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow 2x^3 - 72x = 0 \Rightarrow$

$2x(x^2 - 36) = 0 \Rightarrow 2x(x + 6)(x - 6) = 0 \Rightarrow x = 0, \; x = -6,$

$x = 6$

Since the factor x produces the zero (root) of 0, its multiplicity is one. Since the factor $x + 6$ produces the zero (root) of -6, its multiplicity is one. Since the factor $x - 6$ produces the zero (root) of 6, its multiplicity is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
-6	1	Crosses the *x*-axis at $(-6, 0)$
0	1	Crosses the *x*-axis at $(0, 0)$
6	1	Crosses the *x*-axis at $(6, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1b. $g(x) = 6x^3 - 7x^2$ Back to Problem 1.

Zeros (Roots) of g: $g(x) = 0 \Rightarrow 6x^3 - 7x^2 = 0 \Rightarrow$

$$x^2(6x - 7) = 0 \Rightarrow x = 0, \; x = \frac{7}{6}$$

Since the factor x^2 produces the zero (root) of 0, its multiplicity is two. Since the factor $6x - 7$ produces the zero (root) of $\frac{7}{6}$, its multiplicity is one.

Zero (Root)	Multiplicity	**Implication for the Graph of the Polynomial**
0	2	Touches the *x*-axis at $(0, 0)$
$\frac{7}{6}$	1	Crosses the *x*-axis at $\left(\frac{7}{6}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1c. $h(x) = 48 + 32x - 3x^2 - 2x^3$ Back to Problem 1.

To solve for the zeros (roots) of this polynomial, you we need to recall a technique of factoring called "factor by grouping."

Zeros (Roots) of h: $h(x) = 0 \Rightarrow 48 + 32x - 3x^2 - 2x^3 = 0 \Rightarrow$

$16(3 + 2x) - x^2(3 + 2x) = 0 \Rightarrow (3 + 2x)(16 - x^2) = 0 \Rightarrow$

$(3 + 2x)(4 + x)(4 - x) = 0 \Rightarrow x = -\dfrac{3}{2}, \; x = -4, \; x = 4$

Since the factor $3 + 2x$ produces the zero (root) of $-\frac{3}{2}$, its multiplicity is one. Since the factor $4 + x$ produces the zero (root) of -4, its multiplicity is one. Since the factor $4 - x$ produces the zero (root) of 4, its multiplicity is one.

Zero (Root)	Multiplicity	**Implication for the Graph of the Polynomial**
-4	1	Crosses the *x*-axis at $(-4, 0)$
$-\frac{3}{2}$	1	Crosses the *x*-axis at $\left(-\frac{3}{2}, 0\right)$

4	1	Crosses the *x*-axis at $(4, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1d. $p(x) = 5x^3 + 20x$ Back to Problem 1.

Zeros (Roots) of p: $p(x) = 0 \Rightarrow 5x^3 + 20x = 0 \Rightarrow$

$5x(x^2 + 4) = 0 \Rightarrow x = 0, \; x = \pm 2i$

NOTE: $x^2 + 4 = 0 \Rightarrow x^2 = -4 \Rightarrow x = \pm\sqrt{-4} = \pm 2i$

Since the factor x produces the zero (root) of 0, its multiplicity is one. Since the factor $x^2 + 4$ produces the zeros (roots) of $\pm 2i$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	1	Crosses the *x*-axis at $(0, 0)$
$-2i$	1	No implication
$2i$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1e. $q(x) = (6 - x)(x - 3)^2$ Back to Problem 1.

Zeros (Roots) of q: $q(x) = 0 \Rightarrow 54 - 45x + 12x^2 - x^3 = 0 \Rightarrow$

$(6 - x)(x - 3)^2 = 0 \Rightarrow x = 6, \ x = 3$

Since the factor $6 - x$ produces the zero (root) of 6, its multiplicity is one. Since the factor $(x - 3)^2$ produces the zero (root) of 3, its multiplicity is two.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
3	2	Touches the *x*-axis at $(3, 0)$
6	1	Crosses the *x*-axis at $(6, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1f. $q(t) = t^3 + 16t^2 + 64t$

Back to Problem 1.

For this problem, it will be helpful to recall the following special factoring formula:

$$a^2 + 2ab + b^2 = (a + b)^2$$

Of course, we also have that $a^2 - 2ab + b^2 = (a - b)^2$

Zeros (Roots) of q: $q(t) = 0 \Rightarrow t^3 + 16t^2 + 64t = 0 \Rightarrow$

$t(t^2 + 16t + 64) = 0 \Rightarrow t(t + 8)^2 = 0 \Rightarrow t = 0, \ t = -8$

Since the factor t produces the zero (root) of 0, its multiplicity is one. Since the factor $(t + 8)^2$ produces the zero (root) of -8, its multiplicity is two.

Zero (Root)	Multiplicity	Implication for the Graph of the Polynomial
0	1	Crosses the t-axis at $(0, 0)$
-8	2	Touches the t-axis at $(-8, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the t-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the t-axis.

1g. $f(x) = x^3 - 2x^2 + 9x - 18$ Back to Problem 1.

NOTE: The expression $x^3 - 2x^2 + 9x - 18$ can be factored by grouping.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow x^3 - 2x^2 + 9x - 18 = 0 \Rightarrow$ $x^2(x - 2) + 9(x - 2) = 0 \Rightarrow (x - 2)(x^2 + 9) = 0 \Rightarrow$

$x = 2, \; x = \pm 3i$

Since the factor $x - 2$ produces the zero (root) of 2, its multiplicity is one. Since the factor $x^2 + 9$ produces the zeros (roots) of $\pm 3i$, the multiplicity of each zero (root) is one.

Zero (Root)	Multiplicity	Implication for the Graph of the Polynomial
2	1	Crosses the x-axis at $(2, 0)$
$-3i$	1	No implication
$3i$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1h. $g(x) = 3x^4 + 5x^3 - 12x^2$ Back to Problem 1.

Zeros (Roots) of g: $g(x) = 0 \Rightarrow 3x^4 + 5x^3 - 12x^2 = 0 \Rightarrow$

$x^2(3x^2 + 5x - 12) = 0 \Rightarrow x^2(x + 3)(3x - 4) = 0 \Rightarrow$

$x = 0, \ x = -3 \ \ x = \dfrac{4}{3}$

Since the factor x^2 produces the zero (root) of 0, its multiplicity is two. Since the factor $x + 3$ produces the zero (root) of -3, its multiplicity is one. Since the factor $3x - 4$ produces the zero (root) of $\frac{4}{3}$, its multiplicity is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
-3	1	Crosses the *x*-axis at $(-3, 0)$
0	2	Touches the *x*-axis at $(0, 0)$
$\frac{4}{3}$	1	Crosses the *x*-axis at $\left(\frac{4}{3}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1i. $h(x) = x^4 - 29x^2 + 100$ Back to Problem 2.

NOTE: The expression $x^4 - 29x^2 + 100$ is quadratic in x^2. Thus, it factors like $a^2 - 29a + 100$, where $a = x^2$. Since $a^2 - 29a + 100 = (a - 4)(a - 25)$, then $x^4 - 29x^2 + 100 = (x^2 - 4)(x^2 - 25)$.

Zeros (Roots) of h: $h(x) = 0 \Rightarrow x^4 - 29x^2 + 100 = 0 \Rightarrow$

$(x^2 - 4)(x^2 - 25) = 0 \Rightarrow x = \pm 2,\; x = \pm 5$

Since the factor $x^2 - 4$ produces the zeros (roots) of -2 and 2, the multiplicity of each zero (root) is one. Since the factor $x^2 - 25$ produces the zeros (roots) of -5 and 5, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
-5	1	Crosses the *x*-axis at $(-5, 0)$
-2	1	Crosses the *x*-axis at $(-2, 0)$
2	1	Crosses the *x*-axis at $(2, 0)$
5	1	Crosses the *x*-axis at $(5, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1j. $f(x) = -16x^4 + 56x^2 - 49$ Back to Problem 1.

The expression $-16x^4 + 56x^2 - 49$ is quadratic in x^2. Thus, it factors like $-16u^2 + 56u - 49$, where $u = x^2$. Since $-16u^2 + 56u - 49 = -(16u^2 - 56u + 49) = -(4u - 7)^2$, then $-16x^4 + 56x^2 - 49 = -(4x^2 - 7)^2$

NOTE: We used the special factoring formula $a^2 - 2ab + b^2 = (a - b)^2$ to factor $16u^2 - 56u + 49$ since $16u^2 - 56u + 49 =$ $(4u)^2 - 2(28u) + 7^2$.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow -16x^4 + 56x^2 - 49 = 0 \Rightarrow$

$$-(4x^2 - 7)^2 = 0 \Rightarrow 4x^2 - 7 = 0 \Rightarrow x = \pm\frac{\sqrt{7}}{2}$$

Since the factor $(4x^2 - 7)^2$ produces the zeros (roots) of $-\frac{\sqrt{7}}{2}$ and $\frac{\sqrt{7}}{2}$, the multiplicity of each zero (root) is two.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
$-\frac{\sqrt{7}}{2}$	2	Touches the x-axis at $\left(-\frac{\sqrt{7}}{2}, 0\right)$
$\frac{\sqrt{7}}{2}$	2	Touches the x-axis at $\left(\frac{\sqrt{7}}{2}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the x-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the x-axis.

1k. $p(x) = 15 - 2x^2 - x^4$ Back to Problem 1.

NOTE: The expression $15 - 2x^2 - x^4$ is quadratic in x^2. Thus, it factors like $15 - 2a - a^2$, where $a = x^2$. Since $15 - 2a - a^2 =$ $(5 + a)(3 - a)$, then $15 - 2x^2 - x^4 = (5 + x^2)(3 - x^2)$.

Zeros (Roots) of h: $p(x) = 0 \Rightarrow 15 - 2x^2 - x^4 = 0 \Rightarrow$

$$(5 + x^2)(3 - x^2) = 0 \Rightarrow x = \pm i\sqrt{5},\ x = \pm\sqrt{3}$$

Since the factor $5 + x^2$ produces the zeros (roots) of $-i\sqrt{5}$ and $i\sqrt{5}$, the multiplicity of each zero (root) is one. Since the factor $3 - x^2$ produces the zeros (roots) of $-\sqrt{3}$ and $\sqrt{3}$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
$-\sqrt{3}$	1	Crosses the *x*-axis at $(-\sqrt{3}, 0)$
$\sqrt{3}$	1	Crosses the *x*-axis at $(\sqrt{3}, 0)$
$-i\sqrt{5}$	1	No implication
$i\sqrt{5}$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1l. $q(x) = 8x^3 - 14x^4$ Back to Problem 1.

Zeros (Roots) of q: $q(x) = 0 \Rightarrow 8x^3 - 14x^4 = 0 \Rightarrow$

$$2x^3(4 - 7x) = 0 \Rightarrow x = 0, \; x = \frac{4}{7}$$

Since the factor x^3 produces the zero (root) of 0, its multiplicity is three. Since the factor $4 - 7x$ produces the zero (root) of $\frac{4}{7}$, its multiplicity is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	3	Crosses the *x*-axis at $(0, 0)$
$\frac{4}{7}$	1	Crosses the *x*-axis at $\left(\frac{4}{7}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1m. $f(x) = 54 + 27x - 2x^3 - x^4$ Back to Problem 1.

NOTE: The expression $54 + 27x - 2x^3 - x^4$ can be factored by grouping.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow 54 + 27x - 2x^3 - x^4 = 0 \Rightarrow$

$27(2 + x) - x^3(2 + x) = 0 \Rightarrow (x + 2)(27 - x^3) = 0$

Recalling the difference of cubes factoring formula:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Thus, $27 - x^3 = (3 - x)(9 + 3x + x^2) = (3 - x)(x^2 + 3x + 9)$

Thus, $(x + 2)(27 - x^3) = 0 \Rightarrow (x + 2)(3 - x)(x^2 + 3x + 9) = 0 \Rightarrow$

$x = -2, \; x = 3, \; x^2 + 3x + 9 = 0$

Using the Quadratic Formula to solve $x^2 + 3x + 9 = 0$, we have that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 4(1)9}}{2} = \frac{-3 \pm \sqrt{9[1 - 4(1)]}}{2}$$

$$= \frac{-3 \pm \sqrt{9(1 - 4)}}{2} = \frac{-3 \pm 3\sqrt{-3}}{2} = \frac{-3 \pm 3i\sqrt{3}}{2}$$

Since the factor $x + 2$ produces the zero (root) of -2, its multiplicity is one. Since the factor $3 - x$ produces the zero (root) of 3, its multiplicity is one. Since the factor $x^2 + 3x + 9$ produces the zeros (roots) of $\frac{-3 - 3i\sqrt{3}}{2}$ and $\frac{-3 + 3i\sqrt{3}}{2}$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
-2	1	Crosses the *x*-axis at $(-2, 0)$
3	1	Crosses the *x*-axis at $(3, 0)$
$\frac{-3 - 3i\sqrt{3}}{2}$	1	No implication
$\frac{-3 + 3i\sqrt{3}}{2}$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1n. $g(t) = 3t^4 - 24t^3 + 48t^2$ Back to Problem 1.

Zeros (Roots) of g: $g(t) = 0 \Rightarrow 3t^4 - 24t^3 + 48t^2 = 0 \Rightarrow$

$$3t^2(t^2 - 8t + 16) = 0 \Rightarrow 3t^2(t - 4)^2 = 0 \Rightarrow t = 0, \ t = 4$$

Since the factor t^2 produces the zero (root) of 0, its multiplicity is two. Since the factor $(t - 4)^2$ produces the zero (root) of 4, its multiplicity is two.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	2	Touches the *t*-axis at $(0, 0)$
4	2	Touches the *t*-axis at $(4, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1o. $h(x) = (x - 1)(x + 2)(4x - 3)(5x + 2)$ Back to Problem 1.

Zeros (Roots) of h: $h(x) = 0 \Rightarrow$
$(x - 1)(x + 2)(4x - 3)(5x + 2) = 0 \Rightarrow$

$$x = 1, \ x = -2, \ x = \frac{3}{4}, \ x = -\frac{2}{5}$$

Since the factor $x - 1$ produces the zero (root) of 1, its multiplicity is one. Since the factor $x + 2$ produces the zero (root) of -2, its multiplicity is one. Since the factor $4x - 3$ produces the zero (root) of $\frac{3}{4}$, its multiplicity is one. Since the factor $5x + 2$ produces the zero (root) of $-\frac{2}{5}$, its multiplicity is one.

Zero (Root)	Multiplicity	Implication for the Graph of the Polynomial
-2	1	Crosses the *x*-axis at $(-2, 0)$
$-\dfrac{2}{5}$	1	Crosses the *x*-axis at $\left(-\dfrac{2}{5}, 0\right)$
$\dfrac{3}{4}$	1	Crosses the *x*-axis at $\left(\dfrac{3}{4}, 0\right)$
1	1	Crosses the *x*-axis at $(1, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

1p. $g(x) = 5x^4(8 - x)(2x + 7)^3$ Back to Problem 1.

Zeros (Roots) of g: $g(x) = 0 \Rightarrow 5x^4(8 - x)(2x + 7)^3 = 0 \Rightarrow$

$x = 0, \; x = 8 \; x = -\dfrac{7}{2}$

Since the factor x^4 produces the zero (root) of 0, its multiplicity is four. Since the factor $8 - x$ produces the zero (root) of 8, its multiplicity is one. Since the factor $(2x + 7)^3$ produces the zero (root) of $-\dfrac{7}{2}$, its multiplicity is three.

Zero (Root)	Multiplicity	Implication for the Graph of the Polynomial
$-\dfrac{7}{2}$	3	Crosses the *x*-axis at $\left(-\dfrac{7}{2}, 0\right)$

0	4	Touches the *x*-axis at $(0, 0)$
8	1	Crosses the *x*-axis at $(8, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

2a. $f(x) = 2x^3 - 72x$ Back to Problem 2.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow 2x^3 - 72x = 0 \Rightarrow$

$2x(x^2 - 36) = 0 \Rightarrow 2x(x + 6)(x - 6) = 0 \Rightarrow x = 0, \; x = -6,$

$x = 6$

Since the factor x produces the zero (root) of 0, its multiplicity is one. Since the factor $x + 6$ produces the zero (root) of -6, its multiplicity is one. Since the factor $x - 6$ produces the zero (root) of 6, its multiplicity is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
– 6	1	Crosses the *x*-axis at $(-6, 0)$
0	1	Crosses the *x*-axis at $(0, 0)$
6	1	Crosses the *x*-axis at $(6, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

For infinitely large values of *x*, $f(x) \approx 2x^3$

As $x \to \infty$, $x^3 \to \infty$. Thus, since 2 > 0, then $f(x) \approx 2x^3 \to \infty$.

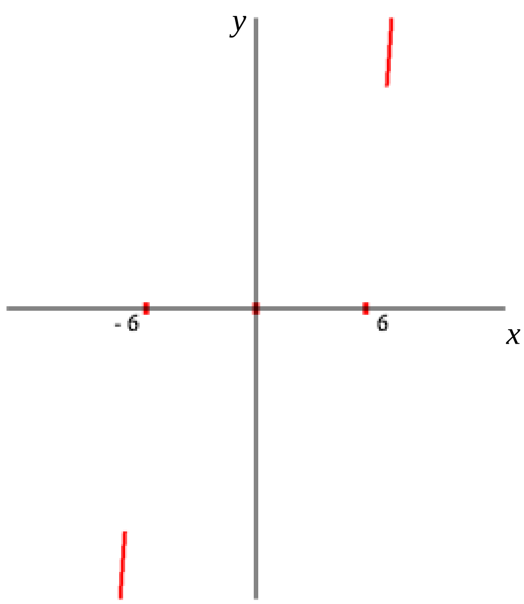
As $x \to -\infty$, $x^3 \to -\infty$. Thus, since 2 > 0, then $f(x) \approx 2x^3 \to -\infty$.

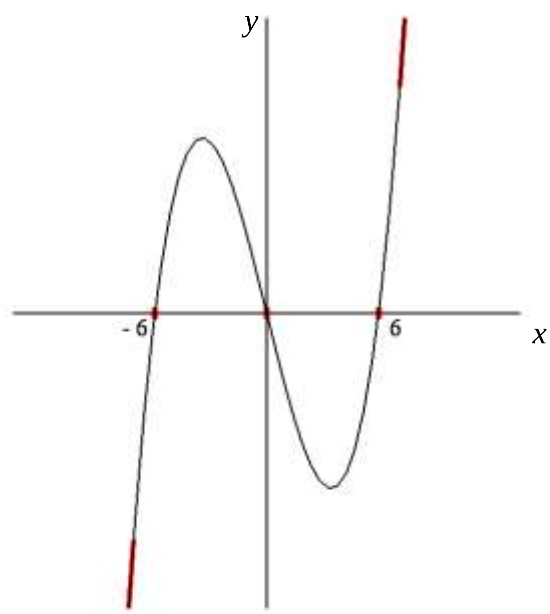
The polynomial is odd: $f(x) = 2x^3 - 72x \Rightarrow$

$$f(-x) = 2(-x)^3 - 72(-x) = -2x^3 + 72x = -(2x^3 - 72x) = -f(x)$$

Thus, the graph of f is symmetric through the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the point $(-6, 0)$. Then the graph must cross the *x*-axis at the origin. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -6 and 0. Then the graph must cross the *x*-axis at the point $(6, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between 0 and 6. We would need calculus in order to obtain information about the *x*-coordinate of these two local extremum points.

Thus, the graph of the polynomial has two local extremum points (turning points).

2b. $g(x) = 6x^3 - 7x^2$ Back to Problem 2.

Zeros (Roots) of g: $g(x) = 0 \Rightarrow 6x^3 - 7x^2 = 0 \Rightarrow$

$$x^2(6x - 7) = 0 \Rightarrow x = 0, \ x = \frac{7}{6}$$

Since the factor x^2 produces the zero (root) of 0, its multiplicity is two. Since the factor $6x - 7$ produces the zero (root) of $\frac{7}{6}$, its multiplicity is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	2	Touches the *x*-axis at $(0, 0)$
$\frac{7}{6}$	1	Crosses the *x*-axis at $\left(\frac{7}{6}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

For infinitely large values of *x*, $g(x) \approx 6x^3$

As $x \to \infty$, $x^3 \to \infty$. Thus, since 6 > 0, then $g(x) \approx 6x^3 \to \infty$.

As $x \to -\infty$, $x^3 \to -\infty$. Thus, since 6 > 0, then $g(x) \approx 6x^3 \to -\infty$.

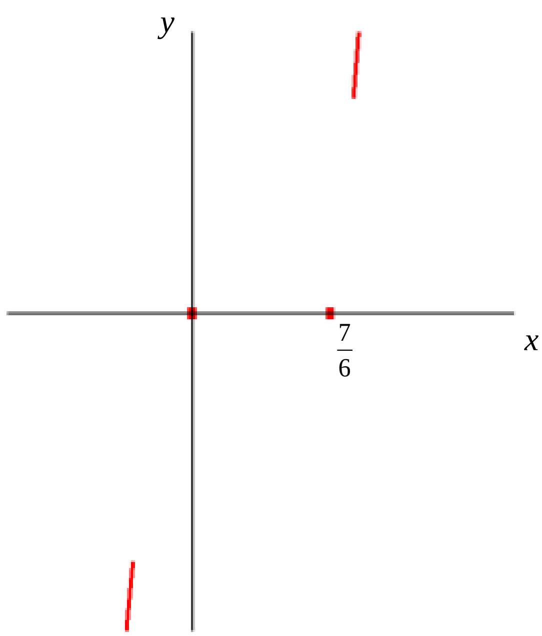
The polynomial is neither even nor odd: $g(x) = 6x^3 - 7x^2 \Rightarrow$

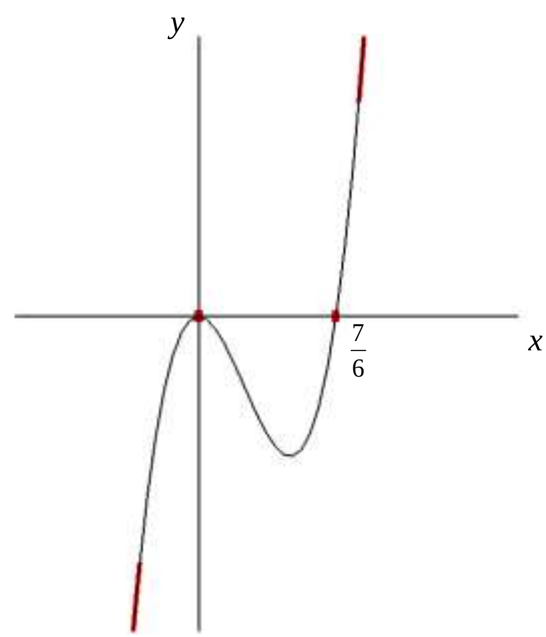
$g(-x) = 6(-x)^3 - 7(-x)^2 = -6x^3 - 7x^2$

Thus, $g(-x) \neq g(x)$ and $g(-x) \neq -g(x)$.

Thus, the graph of the polynomial is not symmetric with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

Since the graph touches at the origin, then there is a local extremum point (turning point) at the origin. Since the graph of the continuous polynomial must cross the x-axis at the point $\left(\frac{7}{6}, 0\right)$, then there is another local extremum point (turning point) whose x-coordinate is between 0 and $\frac{7}{6}$. We would need calculus in order to obtain information about this x-coordinate.

Thus, the graph of the polynomial has two local extremum points (turning points).

2c. $h(x) = 48 + 32x - 3x^2 - 2x^3$ Back to Problem 2.

To solve for the zeros (roots) of this polynomial, you we need to recall a technique of factoring called "factor by grouping."

Zeros (Roots) of h: $h(x) = 0 \Rightarrow 48 + 32x - 3x^2 - 2x^3 = 0 \Rightarrow$

$16(3 + 2x) - x^2(3 + 2x) = 0 \Rightarrow (3 + 2x)(16 - x^2) = 0 \Rightarrow$

$(3 + 2x)(4 + x)(4 - x) = 0 \Rightarrow x = -\frac{3}{2}, \ x = -4, \ x = 4$

Since the factor $3 + 2x$ produces the zero (root) of $-\frac{3}{2}$, its multiplicity is one. Since the factor $4 + x$ produces the zero (root) of -4, its multiplicity is one. Since the factor $4 - x$ produces the zero (root) of 4, its multiplicity is one.

Zero (Root)	Multiplicity	Implication for the Graph of the Polynomial
-4	1	Crosses the *x*-axis at $(-4, 0)$
$-\frac{3}{2}$	1	Crosses the *x*-axis at $\left(-\frac{3}{2}, 0\right)$
4	1	Crosses the *x*-axis at $(4, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

For infinitely large values of *x*, $h(x) \approx -2x^3$

As $x \to \infty$, $x^3 \to \infty$. Thus, since $-2 < 0$, then $h(x) \approx -2x^3 \to -\infty$.

As $x \to -\infty$, $x^3 \to -\infty$. Thus, since $-2 < 0$, then $h(x) \approx -2x^3 \to \infty$.

The polynomial is neither even nor odd:

$h(x) = 48 + 32x - 3x^2 - 2x^3 \Rightarrow$

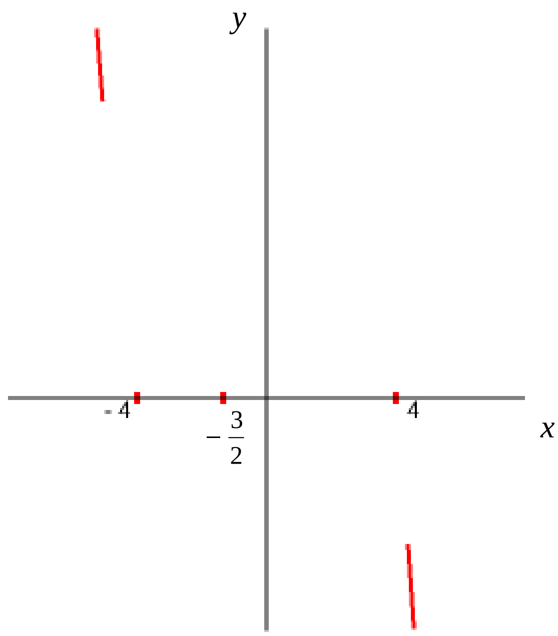
$h(-x) = 48 + 32(-x) - 3(-x)^2 - 2(-x)^3 =$

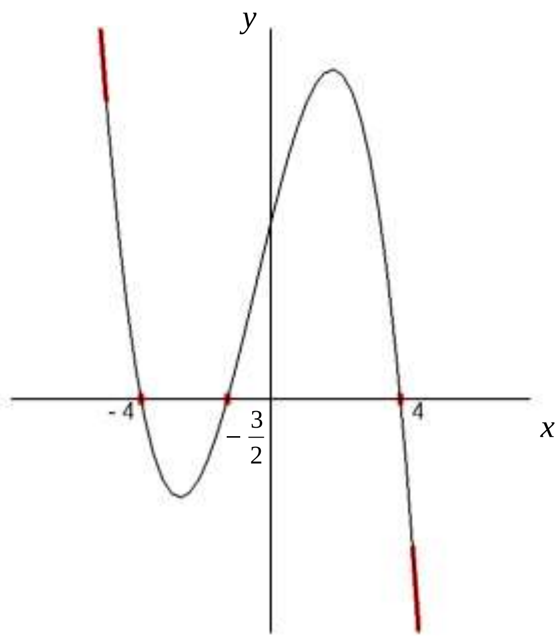
$48 - 32x - 3x^2 + 2x^3$

Thus, $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$.

Thus, the graph of the polynomial is not symmetric with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the point $(-4, 0)$. Then the graph must cross the *x*-axis at the point $\left(-\frac{3}{2}, 0\right)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -4 and $-\frac{3}{2}$. Then the graph must cross the *x*-axis at the point $(4, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between $-\frac{3}{2}$ and 4. We would need calculus in order to obtain information about the *x*-coordinate of these two local extremum points.

Thus, the graph of the polynomial has two local extremum points (turning points).

2d. $p(x) = 5x^3 + 20x$ Back to Problem 2.

Zeros (Roots) of p: $p(x) = 0 \Rightarrow 5x^3 + 20x = 0 \Rightarrow$

$5x(x^2 + 4) = 0 \Rightarrow \quad x = 0, \; x = \pm 2i$

NOTE: $x^2 + 4 = 0 \Rightarrow x^2 = -4 \Rightarrow x = \pm\sqrt{-4} = \pm 2i$

Since the factor x produces the zero (root) of 0, its multiplicity is one. Since the factor $x^2 + 4$ produces the zeros (roots) of $\pm 2i$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	1	Crosses the *x*-axis at $(0, 0)$
$-2i$	1	No implication
$2i$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

For infinitely large values of *x*, $p(x) \approx 5x^3$

As $x \to \infty$, $x^3 \to \infty$. Thus, since 5 > 0, then $p(x) \approx 5x^3 \to \infty$.

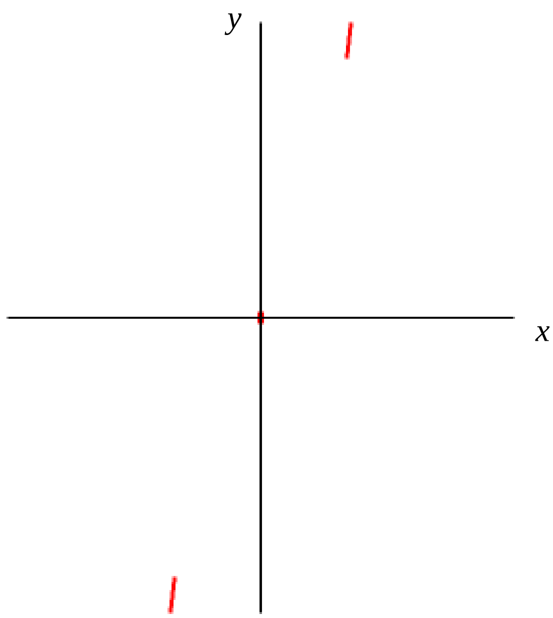
As $x \to -\infty$, $x^3 \to -\infty$. Thus, since 5 > 0, then $p(x) \approx 5x^3 \to -\infty$.

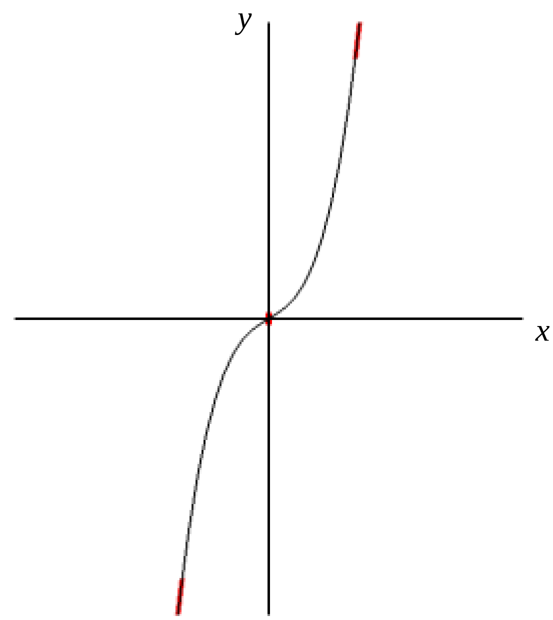
The polynomial is odd: $p(x) = 5x^3 + 20x \Rightarrow$

$$p(-x) = 5(-x)^3 + 20(-x) = -5x^3 - 20x = -(5x^3 - 20x) = -p(x)$$

Thus, the graph of p is symmetric through the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the origin. If the graph had a local extremum point (turning point) for some $x < 0$, then it would have to have a second one in order to pass through the origin. Because the graph is symmetric through the origin, the graph would have to have two local extremum points (turning points) for $x > 0$. Thus, the polynomial would have four local extremum points (turning points).

Thus, the graph of the polynomial has no local extremum points (turning points).

2e. $q(x) = (6 - x)(x - 3)^2$ Back to Problem 2.

Zeros (Roots) of q: $q(x) = 0 \Rightarrow (6 - x)(x - 3)^2 = 0 \Rightarrow$
$(6 - x)(x - 3)^2 = 0 \Rightarrow x = 6, \; x = 3$

Since the factor $6 - x$ produces the zero (root) of 6, its multiplicity is one. Since the factor $(x - 3)^2$ produces the zero (root) of 3, its multiplicity is two.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
3	2	Touches the *x*-axis at $(3, 0)$
6	1	Crosses the *x*-axis at $(6, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

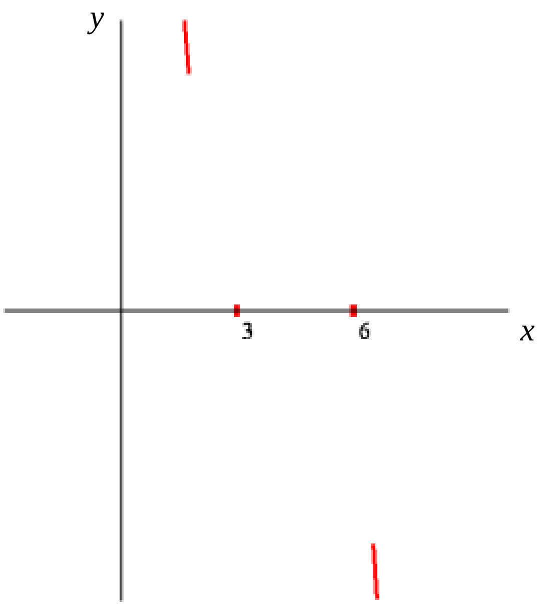
For infinitely large values of *x*, $q(x) \approx -x(x^2) = -x^3$

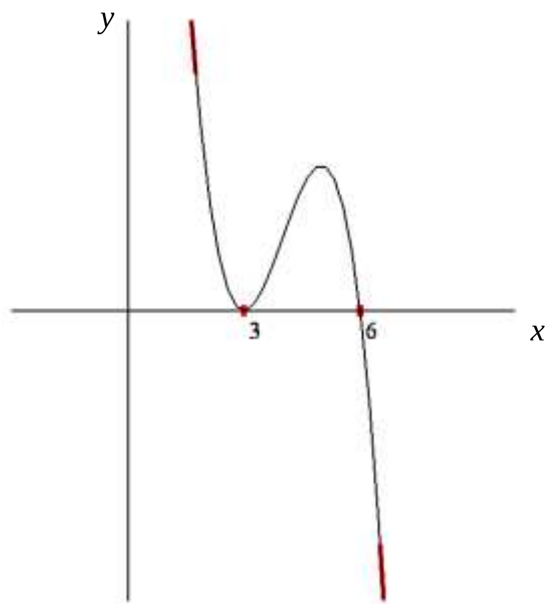
As $x \to \infty$, $x^3 \to \infty$. Thus, since $-1 < 0$, then $q(x) \approx -x^3 \to -\infty$.

As $x \to -\infty$, $x^3 \to -\infty$. Thus, since $-1 < 0$, then $f(x) \approx -x^3 \to \infty$.

The polynomial is neither even nor odd. Thus, the graph of the polynomial is not symmetric with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

Since the graph touches at the point $(3, 0)$, then there is a local extremum point (turning point) at this point. Since the graph of the continuous polynomial must cross the *x*-axis at the point $(6, 0)$, then there is another local extremum point (turning point) whose *x*-coordinate is between 3 and 6. We would need calculus in order to obtain information about this *x*-coordinate.

Thus, the graph of the polynomial has two local extremum points (turning points).

2f. $q(t) = t^3 + 16t^2 + 64t$ Back to Problem 2.

For this problem, it will be helpful to recall the following special factoring formula:

$$a^2 + 2ab + b^2 = (a + b)^2$$

Of course, we also have that $a^2 - 2ab + b^2 = (a - b)^2$

Zeros (Roots) of q: $q(t) = 0 \Rightarrow t^3 + 16t^2 + 64t = 0 \Rightarrow$

$t(t^2 + 16t + 64) = 0 \Rightarrow t(t + 8)^2 = 0 \Rightarrow t = 0, \; t = -8$

Since the factor t produces the zero (root) of 0, its multiplicity is one. Since the factor $(t + 8)^2$ produces the zero (root) of -8, its multiplicity is two.

Zero (Root)	Multiplicity	Implication for the Graph of the Polynomial
0	1	Crosses the *t*-axis at $(0, 0)$
-8	2	Touches the *t*-axis at $(-8, 0)$

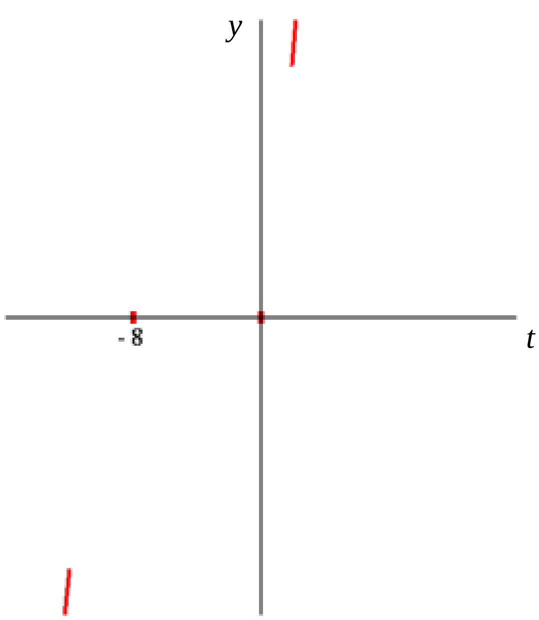
Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *t*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *t*-axis.

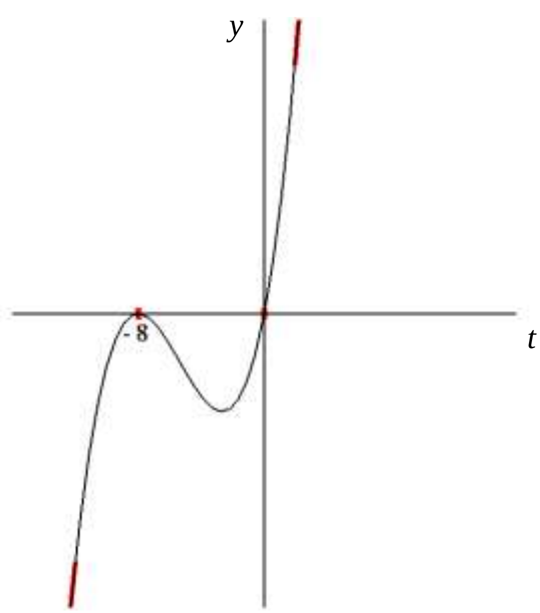
For infinitely large values of *t*, $q(t) \approx t^3$

As $t \to \infty$, $t^3 \to \infty$. Thus, $q(t) \approx t^3 \to \infty$.

As $t \to -\infty$, $t^3 \to -\infty$. Thus, $q(t) \approx t^3 \to -\infty$.

The polynomial is neither even nor odd. Thus, the graph of the polynomial is not symmetric with respect to the *y*-axis nor the origin.
Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $t \to \infty$ and as $t \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

Since the graph touches at the point $(-8, 0)$, then there is a local extremum point (turning point) at this point. Since the graph of the continuous polynomial must cross the *t*-axis at the origin, then there is another local extremum point (turning point) whose *t*-coordinate is between -8 and 0. We would need calculus in order to obtain information about this *t*-coordinate.

Thus, the graph of the polynomial has two local extremum points (turning points).

2g. $f(x) = x^3 - 2x^2 + 9x - 18$ Back to Problem 2.

NOTE: The expression $x^3 - 2x^2 + 9x - 18$ can be factored by grouping.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow x^3 - 2x^2 + 9x - 18 = 0 \Rightarrow$
$x^2(x - 2) + 9(x - 2) = 0 \Rightarrow (x - 2)(x^2 + 9) = 0 \Rightarrow$

$x = 2, \; x = \pm 3i$

Since the factor $x - 2$ produces the zero (root) of 2, its multiplicity is one. Since the factor $x^2 + 9$ produces the zeros (roots) of $\pm 3i$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
2	1	Crosses the *x*-axis at $(2, 0)$
$-3i$	1	No implication

$3i$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

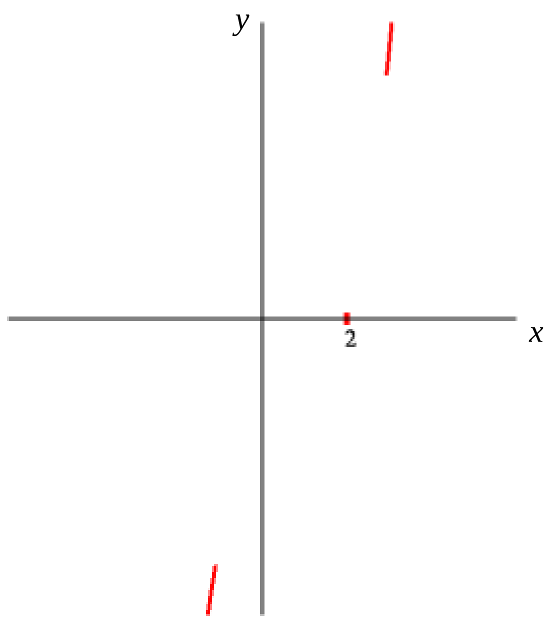
For infinitely large values of *x*, $f(x) \approx x^3$

As $x \to \infty$, $x^3 \to \infty$. Thus, $f(x) \approx x^3 \to \infty$.

As $x \to -\infty$, $x^3 \to -\infty$. Thus, $f(x) \approx x^3 \to -\infty$.

The polynomial is neither even nor odd. Thus, the graph of the polynomial is not symmetric with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.



Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

The graph of the continuous polynomial could have no local extremum points (turning points) or it could have two local extremum points (turning points). There is not enough information from the zeros (roots) and the graph is not symmetric to the *y*-axis nor the origin. We need calculus in order to get information about the local extremum point(s).

2h. $g(x) = 3x^4 + 5x^3 - 12x^2$ Back to Problem 2.

Zeros (Roots) of g: $g(x) = 0 \Rightarrow 3x^4 + 5x^3 - 12x^2 = 0 \Rightarrow$

$x^2(3x^2 + 5x - 12) = 0 \Rightarrow x^2(x + 3)(3x - 4) = 0 \Rightarrow$

$x = 0,\ x = -3\ x = \frac{4}{3}$

Since the factor x^2 produces the zero (root) of 0, its multiplicity is two. Since the factor $x + 3$ produces the zero (root) of -3, its multiplicity is one. Since the factor $3x - 4$ produces the zero (root) of $\frac{4}{3}$, its multiplicity is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
-3	1	Crosses the *x*-axis at $(-3, 0)$
0	2	Touches the *x*-axis at $(0, 0)$
$\frac{4}{3}$	1	Crosses the *x*-axis at $\left(\frac{4}{3}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

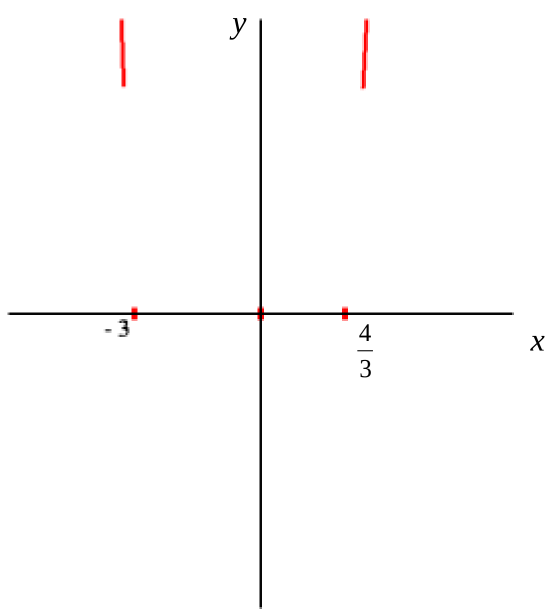
For infinitely large values of x, $g(x) \approx 3x^4$

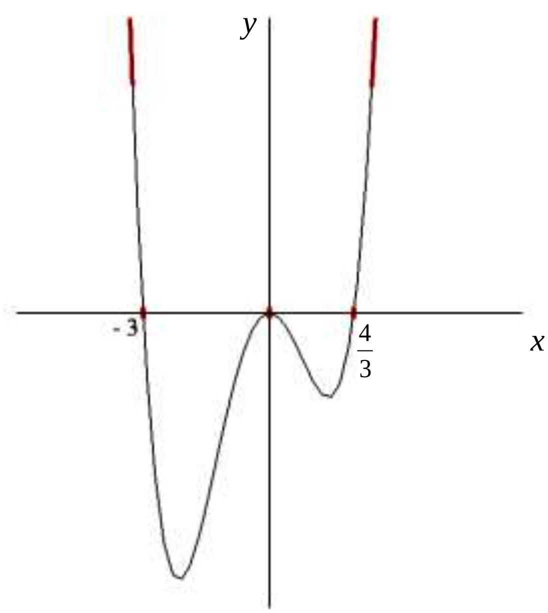
As $x \to \infty$, $x^4 \to \infty$. Thus, since $3 > 0$, then $g(x) \approx 3x^4 \to \infty$.

As $x \to -\infty$, $x^4 \to \infty$. Thus, since $3 > 0$, then $g(x) \approx 3x^4 \to \infty$.

The polynomial is neither even nor odd. Thus, the graph of the polynomial is not symmetric with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the point $(-3, 0)$. Then the graph must touch the *x*-axis at the origin. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -3 and 0. Since the graph touches at the origin, then there is a local extremum point (turning point) at the origin. Then the graph of the polynomial must cross the *x*-axis at the point $\left(\frac{4}{3}, 0\right)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between 0 and $\frac{4}{3}$. We would need calculus in order to obtain information about the *x*-coordinate of two of these local extremum points (turning points).

Thus, the graph of the polynomial has three local extremum points (turning points).

2i. $h(x) = x^4 - 29x^2 + 100$ Back to Problem 2.

NOTE: The expression $x^4 - 29x^2 + 100$ is quadratic in x^2. Thus, it factors like $a^2 - 29a + 100$, where $a = x^2$. Since $a^2 - 29a + 100 = (a - 4)(a - 25)$, then $x^4 - 29x^2 + 100 = (x^2 - 4)(x^2 - 25)$.

Zeros (Roots) of h: $h(x) = 0 \Rightarrow x^4 - 29x^2 + 100 = 0 \Rightarrow$

$(x^2 - 4)(x^2 - 25) = 0 \Rightarrow x = \pm 2, \ x = \pm 5$

Since the factor $x^2 - 4$ produces the zeros (roots) of -2 and 2, the multiplicity of each zero (root) is one. Since the factor $x^2 - 25$ produces the zeros (roots) of -5 and 5, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
-5	1	Crosses the *x*-axis at $(-5, 0)$
-2	1	Crosses the *x*-axis at $(-2, 0)$
2	1	Crosses the *x*-axis at $(2, 0)$

5	1	Crosses the *x*-axis at $(5, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

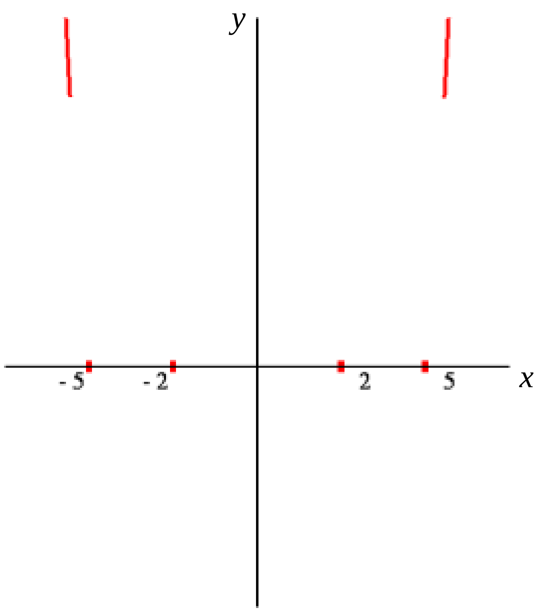
For infinitely large values of *x*, $h(x) \approx x^4$

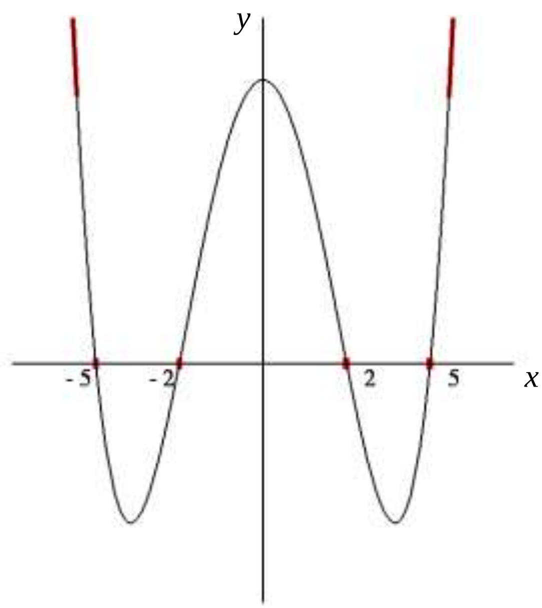
As $x \to \infty$, $x^4 \to \infty$. Thus, $h(x) \approx x^4 \to \infty$.

As $x \to -\infty$, $x^4 \to \infty$. Thus, $h(x) \approx x^4 \to \infty$.

The polynomial is even. Thus, the graph of h is symmetric about the *y*-axis.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the point $(-5, 0)$. Then the graph must cross the *x*-axis at the point $(-2, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -5 and -2. Then the graph must cross the *x*-axis at the point $(2, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -2 and 2. Then the graph must cross the *x*-axis at the point $(5, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between 2 and 5. We would need calculus in order to obtain information about the *x*-coordinate of these three local extremum points.

Thus, the graph of the polynomial has three local extremum points (turning points).

2j. $f(x) = -16x^4 + 56x^2 - 49$ Back to Problem 2.

The expression $-16x^4 + 56x^2 - 49$ is quadratic in x^2. Thus, it factors like $-16u^2 + 56u - 49$, where $u = x^2$. Since $-16u^2 + 56u - 49 = -(16u^2 - 56u + 49) = -(4u - 7)^2$, then $-16x^4 + 56x^2 - 49 = -(4x^2 - 7)^2$

NOTE: We used the special factoring formula $a^2 - 2ab + b^2 = (a - b)^2$ to factor $16u^2 - 56u + 49$ since $16u^2 - 56u + 49 = (4u)^2 - 2(28u) + 7^2$.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow -16x^4 + 56x^2 - 49 = 0 \Rightarrow$

$$-(4x^2 - 7)^2 = 0 \Rightarrow 4x^2 - 7 = 0 \Rightarrow x = \pm\frac{\sqrt{7}}{2}$$

Since the factor $(4x^2 - 7)^2$ produces the zeros (roots) of $-\frac{\sqrt{7}}{2}$ and $\frac{\sqrt{7}}{2}$, the multiplicity of each zero (root) is two.

Zero (Root)	Multiplicity	Implication for the Graph of the Polynomial
$-\frac{\sqrt{7}}{2}$	2	Touches the *x*-axis at $\left(-\frac{\sqrt{7}}{2}, 0\right)$
$\frac{\sqrt{7}}{2}$	2	Touches the *x*-axis at $\left(\frac{\sqrt{7}}{2}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

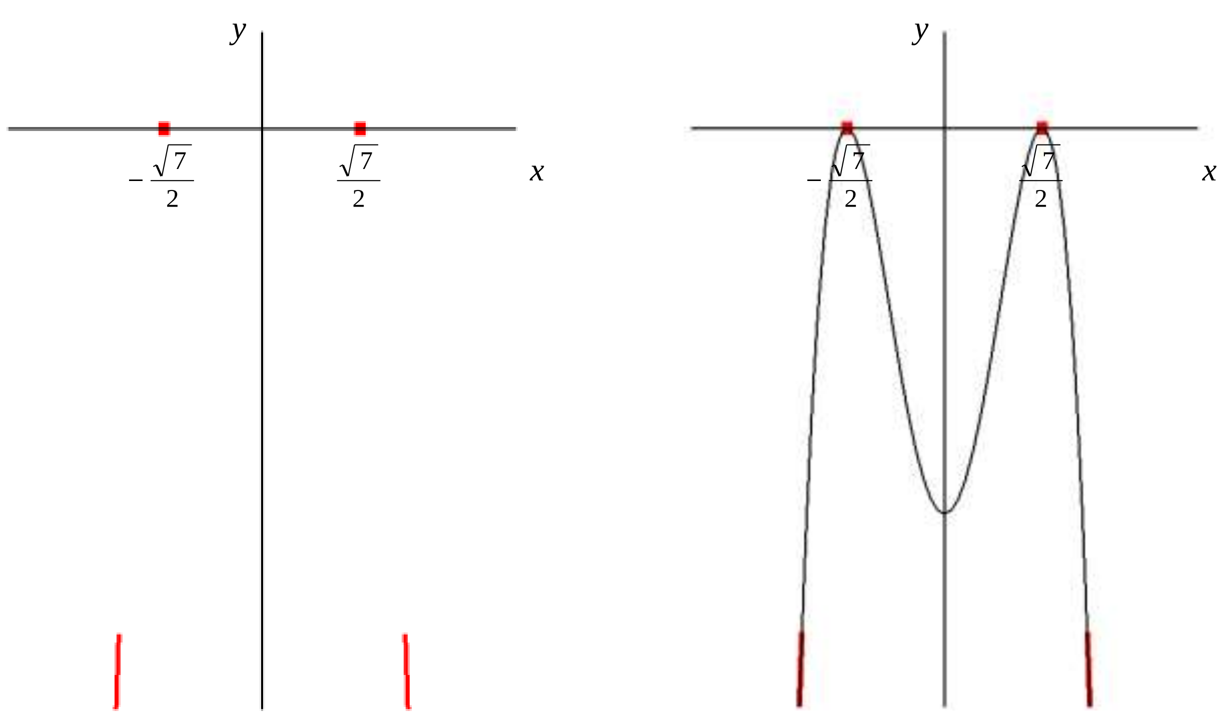
For infinitely large values of *x*, $f(x) \approx -16x^4$

As $x \to \infty$, $x^4 \to \infty$. Thus, since $-16 < 0$, then $f(x) \approx -16x^4 \to -\infty$.

As $x \to -\infty$, $x^4 \to \infty$. Thus, since $-16 < 0$, then $f(x) \approx -16x^4 \to -\infty$.

The polynomial is even. Thus, the graph of f is symmetric about the *y*-axis.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.



The Drawing of this Sketch

Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

Since the graph of the continuous polynomial touches the x-axis at the point $\left(-\frac{\sqrt{7}}{2}, 0 \right)$, then there is a local extremum point (turning point) at this point. Then the graph must touch the x-axis at the point $\left(\frac{\sqrt{7}}{2}, 0 \right)$. In order for this to happen, there must be a local extremum point (turning point) whose x-coordinate is between $-\frac{\sqrt{7}}{2}$ and $\frac{\sqrt{7}}{2}$. Since the graph touches at the point $\left(\frac{\sqrt{7}}{2}, 0 \right)$, then there is a local extremum point (turning point) at this point. We would need calculus in order to obtain information about the x-coordinate of the local extremum points (turning points) whose x-coordinate is between $-\frac{\sqrt{7}}{2}$ and $\frac{\sqrt{7}}{2}$.

Thus, the graph of the polynomial has three local extremum points (turning points).

2k. $p(x) = 15 - 2x^2 - x^4$ Back to Problem 2.

NOTE: The expression $15 - 2x^2 - x^4$ is quadratic in x^2. Thus, it factors like $15 - 2a - a^2$, where $a = x^2$. Since $15 - 2a - a^2 =$ $(5 + a)(3 - a)$, then $15 - 2x^2 - x^4 = (5 + x^2)(3 - x^2)$.

Zeros (Roots) of h: $p(x) = 0 \Rightarrow 15 - 2x^2 - x^4 = 0 \Rightarrow$

$(5 + x^2)(3 - x^2) = 0 \Rightarrow x = \pm i\sqrt{5},\ x = \pm\sqrt{3}$

Since the factor $5 + x^2$ produces the zeros (roots) of $-i\sqrt{5}$ and $i\sqrt{5}$, the multiplicity of each zero (root) is one. Since the factor $3 - x^2$

produces the zeros (roots) of $-\sqrt{3}$ and $\sqrt{3}$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
$-\sqrt{3}$	1	Crosses the *x*-axis at $(-\sqrt{3}, 0)$
$\sqrt{3}$	1	Crosses the *x*-axis at $(\sqrt{3}, 0)$
$-i\sqrt{5}$	1	No implication
$i\sqrt{5}$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

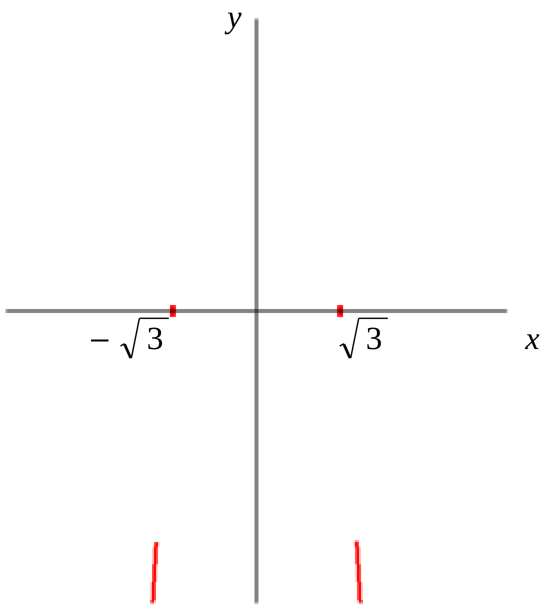
For infinitely large values of *x*, $p(x) \approx -x^4$

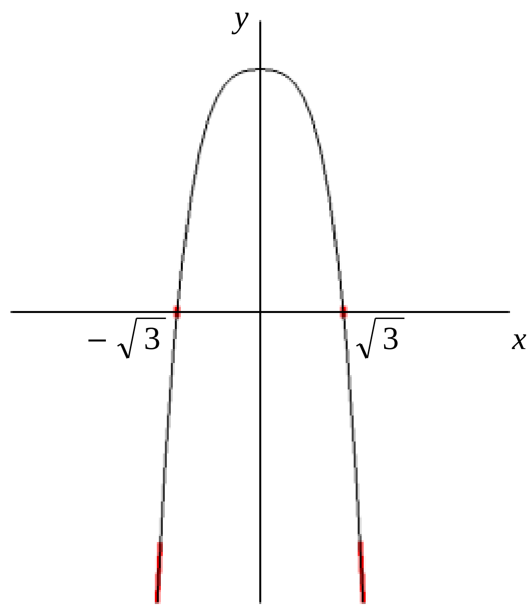
As $x \to \infty$, $x^4 \to \infty$. Thus, since $-1 < 0$, then $p(x) \approx -x^4 \to -\infty$.

As $x \to -\infty$, $x^4 \to \infty$. Thus, since $-1 < 0$, then $p(x) \approx -x^4 \to -\infty$.

The polynomial is even. Thus, the graph of p is symmetric about the *y*-axis.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

The graph of the continuous polynomial must cross the x-axis at the point $(-\sqrt{3}, 0)$. Then the graph must cross the x-axis at the point $(\sqrt{3}, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose x-coordinate is between $-\sqrt{3}$ and $\sqrt{3}$. Since the graph of the polynomial is symmetric about the y-axis, the x-coordinate of this local extremum point is 0. If the graph had a local extremum point (turning point) for some $x < -\sqrt{3}$, then it would have to have another one in order to pass through the point $(-\sqrt{3}, 0)$. Because the graph is symmetric about the y-axis, the graph would have to have two more local extremum points (turning points) for $x > \sqrt{3}$. Thus, the polynomial would have five local extremum points (turning points). So, there are no local extremum points (turning points) for $x < -\sqrt{3}$ nor for $x > \sqrt{3}$. The same argument shows that there are no local extremum points (turning points) for $-\sqrt{3} < x < 0$ nor for $0 < x < \sqrt{3}$.

Thus, the graph of the polynomial has one local extremum points (turning points).

2l. $q(x) = 8x^3 - 14x^4$

Back to Problem 2.

Zeros (Roots) of q: $q(x) = 0 \Rightarrow 8x^3 - 14x^4 = 0 \Rightarrow$

$2x^3(4 - 7x) = 0 \Rightarrow x = 0, \ x = \frac{4}{7}$

Since the factor x^3 produces the zero (root) of 0, its multiplicity is three. Since the factor $4 - 7x$ produces the zero (root) of $\frac{4}{7}$, its multiplicity is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	3	Crosses the *x*-axis at $(0, 0)$
$\frac{4}{7}$	1	Crosses the *x*-axis at $\left(\frac{4}{7}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

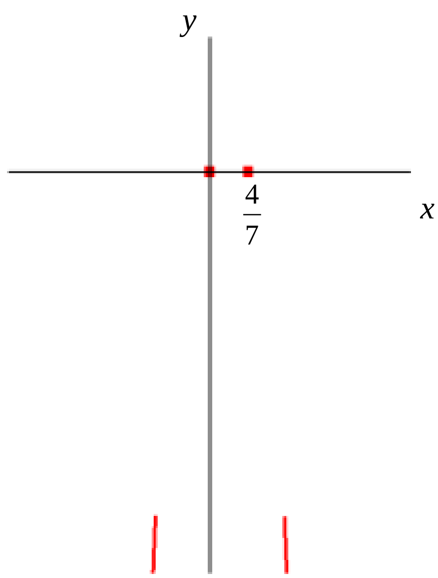
For infinitely large values of *x*, $q(x) \approx -14x^4$

As $x \to \infty$, $x^4 \to \infty$. Thus, since $-14 < 0$, then $q(x) \approx -14x^4 \to -\infty$.

As $x \to -\infty$, $x^4 \to \infty$. Thus, since $-14 < 0$, then $q(x) \approx -14x^4 \to -\infty$.

The polynomial is neither even nor odd. Thus, the graph of the polynomial is not symmetric with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.



Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the origin. Then the graph must cross the *x*-axis at the point $\left(\frac{4}{7}, 0\right)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between 0 and $\frac{4}{7}$. If the graph had a local extremum point (turning point) for some $x < 0$, then it would have to have another local extremum point (turning point) in order to pass through the origin. Since there is no symmetry in graph, this is possible and the graph would have three local extremum points (turning points). The same argument would be true for $0 < x < \frac{4}{7}$ and for $x > \frac{4}{7}$. We need calculus in order to get information about the local extremum point(s).

Thus, the graph of the polynomial has one or three local extremum points (turning points).

2m. $f(x) = 54 + 27x - 2x^3 - x^4$ Back to Problem 2.

NOTE: The expression $54 + 27x - 2x^3 - x^4$ can be factored by grouping.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow 54 + 27x - 2x^3 - x^4 = 0 \Rightarrow$

$27(2 + x) - x^3(2 + x) = 0 \Rightarrow (x + 2)(27 - x^3) = 0$

Recalling the difference of cubes factoring formula:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Thus, $27 - x^3 = (3 - x)(9 + 3x + x^2) = (3 - x)(x^2 + 3x + 9)$

Thus, $(x + 2)(27 - x^3) = 0 \Rightarrow (x + 2)(3 - x)(x^2 + 3x + 9) = 0 \Rightarrow$

$x = -2,\ x = 3,\ x^2 + 3x + 9 = 0$

Using the Quadratic Formula to solve $x^2 + 3x + 9 = 0$, we have that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 4(1)9}}{2} = \frac{-3 \pm \sqrt{9[1 - 4(1)]}}{2}$$

$$= \frac{-3 \pm \sqrt{9(1 - 4)}}{2} = \frac{-3 \pm 3\sqrt{-3}}{2} = \frac{-3 \pm 3i\sqrt{3}}{2}$$

Since the factor $x + 2$ produces the zero (root) of -2, its multiplicity is one. Since the factor $3 - x$ produces the zero (root) of 3, its multiplicity is one. Since the factor $x^2 + 3x + 9$ produces the zeros (roots) of

$\dfrac{-3 - 3i\sqrt{3}}{2}$ and $\dfrac{-3 + 3i\sqrt{3}}{2}$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
-2	1	Crosses the *x*-axis at $(-2, 0)$
3	1	Crosses the *x*-axis at $(3, 0)$
$\dfrac{-3 - 3i\sqrt{3}}{2}$	1	No implication
$\dfrac{-3 + 3i\sqrt{3}}{2}$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

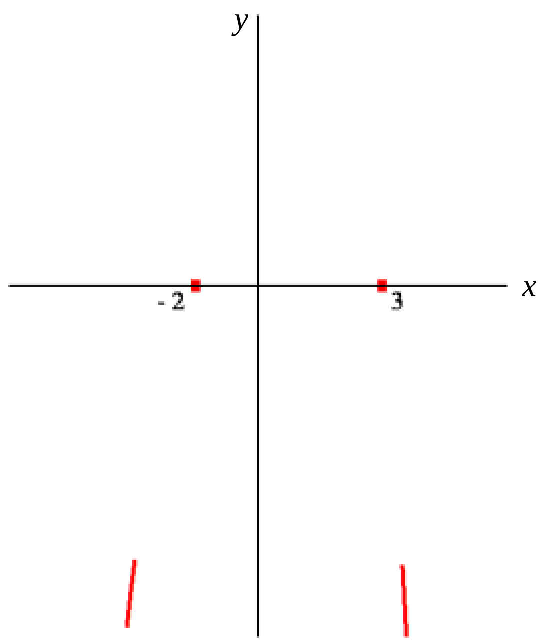
For infinitely large values of *x*, $f(x) \approx -x^4$

As $x \to \infty$, $x^4 \to \infty$. Thus, since $-1 < 0$, then $f(x) \approx -x^4 \to -\infty$.

As $x \to -\infty$, $x^4 \to \infty$. Thus, since $-1 < 0$, then $f(x) \approx -x^4 \to -\infty$.

The polynomial is neither even nor odd.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.



Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the point $(-2, 0)$. Then the graph must cross the *x*-axis at the point $(3, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -2 and 3. If the graph had a local extremum point (turning point) for some $x < -2$, then it would have to have another local extremum point (turning point) in order to pass through the point $(-2, 0)$. Since there is no symmetry in graph, this is possible and the graph would have three local extremum points (turning points). The same argument would be true for $-2 < x < 3$ and for $x > 3$. We need calculus in order to get information about the local extremum point(s).

Thus, the graph of the polynomial has one or three local extremum points (turning points).

2n. $g(t) = 3t^4 - 24t^3 + 48t^2$ Back to Problem 2.

Zeros (Roots) of g: $g(t) = 0 \Rightarrow 3t^4 - 24t^3 + 48t^2 = 0 \Rightarrow$

$$3t^2(t^2 - 8t + 16) = 0 \Rightarrow 3t^2(t - 4)^2 = 0 \Rightarrow t = 0, \ t = 4$$

Since the factor t^2 produces the zero (root) of 0, its multiplicity is two. Since the factor $(t - 4)^2$ produces the zero (root) of 4, its multiplicity is two.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	2	Touches the *t*-axis at $(0, 0)$
4	2	Touches the *t*-axis at $(4, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

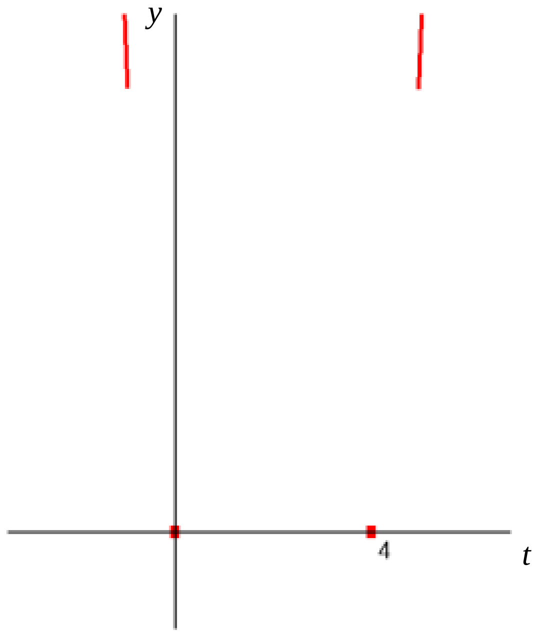
For infinitely large values of *t*, $g(t) \approx 3t^4$

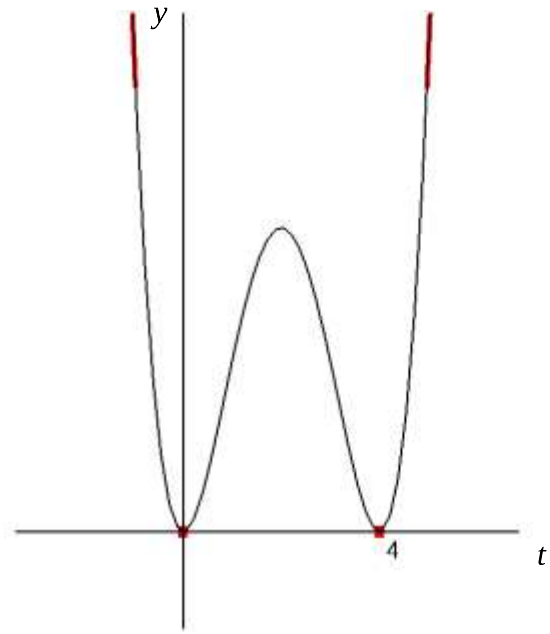
As $t \to \infty$, $t^4 \to \infty$. Thus, since 3 > 0, then $g(t) \approx 3t^4 \to \infty$.

As $t \to -\infty$, $t^4 \to \infty$. Thus, since 3 > 0, then $g(t) \approx 3t^4 \to \infty$.

The polynomial is neither even nor odd.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $t \to \infty$ and as $t \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

Since the graph of the continuous polynomial touches the *t*-axis at the origin, then there is a local extremum point (turning point) at the origin. Then the graph must touch the *t*-axis at the point $(4, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *t*-coordinate is between 0 and 4. We would need calculus in order to obtain information about the *t*-coordinate of this local extremum point (turning point). Since the graph touches at the point $(4, 0)$, then there is a local extremum point (turning point) at this point.

Thus, the graph of the polynomial has three local extremum points (turning points).

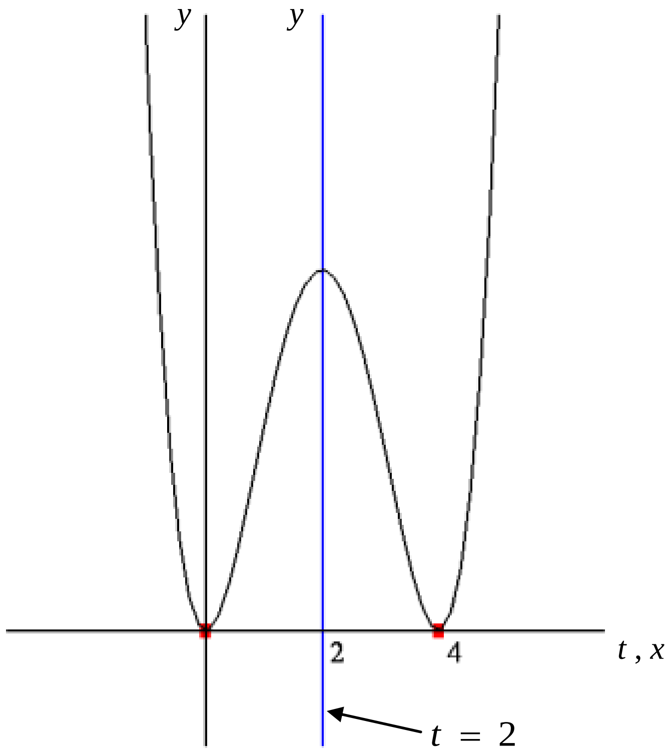
COMMENT: It appears that the graph of the polynomial g is symmetric about the vertical line $t = 2$. If we introduced a new xy coordinate system, whose origin is at the point $(2, 0)$ in the ty coordinate system, we would have that the point $(2, 0) = (t, y)$ in the original ty coordinate plane is the point $(0, 0) = (x, y)$ in the new xy coordinate plane. Thus, we have that $x = t - 2$, or $t = x + 2$.

Thus, $g(t) = 3t^4 - 24t^3 + 48t^2 = 3t^2(t-4)^2 \Rightarrow$
$h(x) = g(x+2) = 3(x+2)^2(x+2-4)^2 = 3(x+2)^2(x-2)^2 =$

$3[(x+2)(x-2)]^2 = 3(x^2-4)^2$

The polynomial h is an even function since $h(-x) = 3[(-x)^2 - 4]^2 =$

$3(x^2-4)^2 = h(x)$ Thus, the graph of the polynomial h is symmetric about the *y*-axis in the *xy*-plane.



Definition A real or complex number z is called a zero or a root of the polynomial p if and only if $p(z) = 0$.

Theorem If $z = a + bi$ is a zero (root) of a polynomial with real coefficients, then the conjugate $\overline{z} = a - bi$ is also a zero (root) of the polynomial.

Proof Recall the following properties for conjugates.

1. $\overline{z+w} = \overline{z} + \overline{w}$

2. By mathematical induction,

$$\overline{z_1 + z_2 + z_3 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \overline{z_3} + \cdots + \overline{z_n}$$

3. $\overline{zw} = \overline{z}\,\overline{w}$

4. By mathematical induction, $\overline{z_1 z_2 z_3 \cdots z_n} = \overline{z_1}\,\overline{z_2}\,\overline{z_3} \cdots \overline{z_n}$

5. $\overline{z^n} = \overline{z}^{\,n}$

6. If a is a real number, then $\overline{a} = a$.

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$.

Since z is a zero (root) of the polynomial, then $p(z) = 0$. We want to show that $p(\overline{z}) = 0$. $p(z) = 0 \Rightarrow$

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0 = 0 \Rightarrow$$

$$\overline{a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0} = \overline{0} \Rightarrow$$

$$\overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \overline{a_{n-2} z^{n-2}} + \cdots + \overline{a_2 z^2} + \overline{a_1 z} + \overline{a_0} = 0 \Rightarrow$$

$$\overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \overline{a_{n-2}}\,\overline{z^{n-2}} + \cdots + \overline{a_2}\,\overline{z^2} + \overline{a_1}\,\overline{z} + \overline{a_0} = 0 \Rightarrow$$

$$a_n \overline{z}^{\,n} + a_{n-1} \overline{z}^{\,n-1} + a_{n-2} \overline{z}^{\,n-2} + \cdots + a_2 \overline{z}^{\,2} + a_1 \overline{z} + a_0 = 0 \Rightarrow$$

$p(\overline{z}) = 0$. Thus, $\overline{z}$ is a zero (root) of the polynomial.

Note, since the coefficients of the polynomial are real numbers, then $\overline{a_i} = a_i$ for $i = 0, 1, 2, 3, \ldots, n$.

NOTE: This proof is easier to read (and type) if we make use of summation notation

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 =$$

$$\sum_{i=0}^{n} a_i x^i . \text{ Then } p(z) = 0 \Rightarrow \sum_{i=0}^{n} a_i z^i = 0 \Rightarrow \overline{\sum_{i=0}^{n} a_i z^i} = \overline{0} \Rightarrow$$

$$\sum_{i=0}^{n} \overline{a_i z^i} = 0 \Rightarrow \sum_{i=0}^{n} \overline{a_i}\, \overline{z^i} = 0 \Rightarrow \sum_{i=0}^{n} a_i \overline{z}^{\,i} = 0 \Rightarrow p(\overline{z}) = 0 .$$

<u>Theorem</u> Let a be a real number. Let p be any polynomial. Then the following statements are equivalent.

1. $x = a$ is a zero (root) of the polynomial p

2. $p(a) = 0$

3. $x - a$ is a factor of $p(x)$

4. $(a, 0)$ is an *x*-intercept of the graph of $y = p(x)$

<u>Definition</u> Let p be any polynomial. If $(x - z)^m$, where m is positive integer, is a factor of $p(x)$, then $x = z$ is called a zero (root) of multiplicity m.

<u>Theorem</u> Let p be any polynomial. Let a be a real number. If $(x - a)^m$ is a factor of $p(x)$, then

1. if m is odd, then the graph of $y = p(x)$ crosses the *x*-axis at the *x*-intercept $(a, 0)$,

2. if m is even, then the graph of $y = p(x)$ touches the x-axis at the x-intercept $(a, 0)$ but does not cross the x-axis.

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Theorem (The Fundamental Theorem of Algebra) An nth degree polynomial has at most n zeros (roots). If you count the multiplicity of the zero (root), then an nth degree polynomial has exactly n zeros (roots).

Theorem An nth degree polynomial has at most $n - 1$ relative (local) extremum points (turning points).

Information about local extremum points can be obtained using calculus.

Theorem If a and b are real zeros (roots) of a polynomial and $a \neq b$, then the polynomial has at least one relative (local) extremum point whose x-coordinate is between a and b.

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