# LESSON 9 EXPONENTIAL FUNCTIONS

**Definition** The exponential function $f$ with base $b$ is the function defined by $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

The domain of $f$ is the set of real numbers. That is, the domain of $f$ is $(-\infty, \infty)$.

The range of $f$ is the set of positive real numbers. That is, the range of $f$ is $(0, \infty)$.

**Definition** The natural exponential function is the exponential function whose base is the irrational number *e*. Thus, the natural exponential function is the function defined by $f(x) = e^x$, where $e = 2.718281828\ldots$.

Notation: Sometimes, $e^x$ is written $\exp(x)$.

Recall the following properties of exponents, where $m$ and $n$ be integers:

1. $(a^m)^n = a^{mn}$

2. $a^m a^n = a^{m+n}$

3. If $a \neq 0$, then $\dfrac{a^m}{a^n} = a^{m-n}$ if $m > n$ OR $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$ if $n > m$.

4. If $a \neq 0$, then $a^0 = 1$.

5. If $a \neq 0$ and $n$ is positive, then $a^{-n} = \dfrac{1}{a^n}$.

6. $(ab)^n = a^n b^n$

7. If $b \neq 0$, then $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.

8. If $a \neq 0$ and $b \neq 0$ and $n$ is positive, then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.

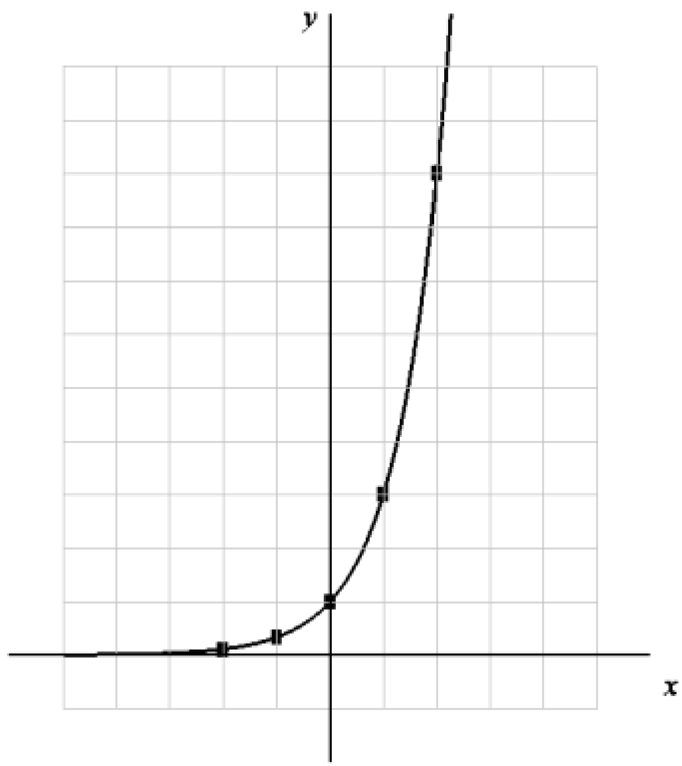
9. $a^{m/n} = \sqrt[n]{a^m}$

**Examples** Graph the following exponential functions.

1. $f(x) = 3^x$

In order to graph the function $f$ given by $f(x) = 3^x$, we set $f(x) = y$ and graph the equation $y = 3^x$.

| $x$ | $y$ |
|---|---|
| $-2$ | $\frac{1}{9}$ |
| $-1$ | $\frac{1}{3}$ |
| 0 | 1 |
| 1 | 3 |
| 2 | 9 |




The Drawing of this Graph

The $y$-intercept of the graph of the function is the point $(0, 1)$.

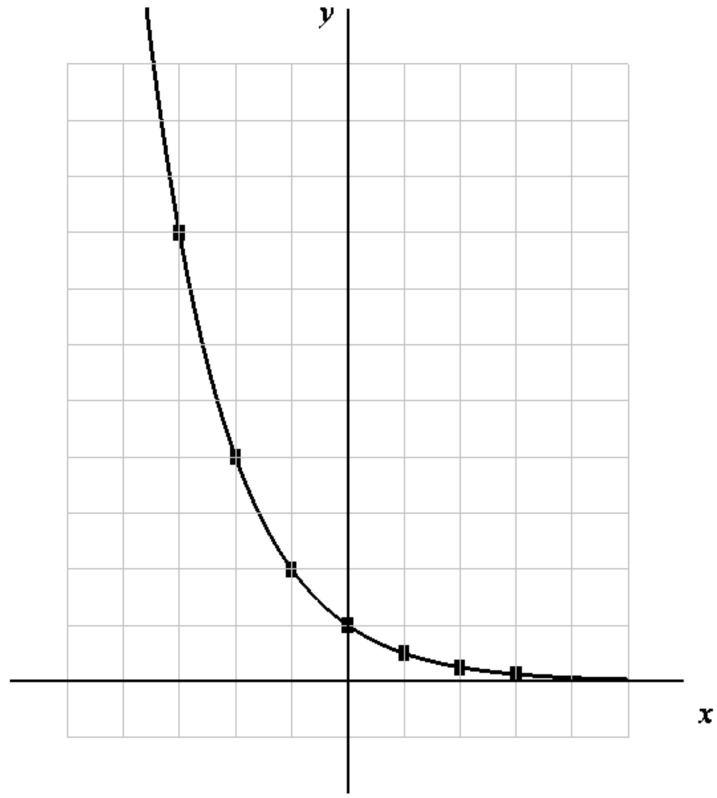
Note that as $x \to \infty$, $y = 3^x \to \infty$ and as $x \to -\infty$, $y = 3^x \to 0$.

Since as $x \to -\infty$, $y = 3^x \to 0$, then the horizontal line of $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

2. $g(x) = \left(\frac{1}{2}\right)^x$

In order to graph the function $g$ given by $g(x) = \left(\frac{1}{2}\right)^x$, we set $g(x) = y$ and graph the equation $y = \left(\frac{1}{2}\right)^x$.

| $x$ | $y$ |
|---|---|
| $-3$ | $8$ |
| $-2$ | $4$ |
| $-1$ | $2$ |
| $0$ | $1$ |
| $1$ | $\frac{1}{2}$ |
| $2$ | $\frac{1}{4}$ |
| $3$ | $\frac{1}{8}$ |




The Drawing of this Graph

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$$

The $y$-intercept of the graph of the function is the point $(0, 1)$.

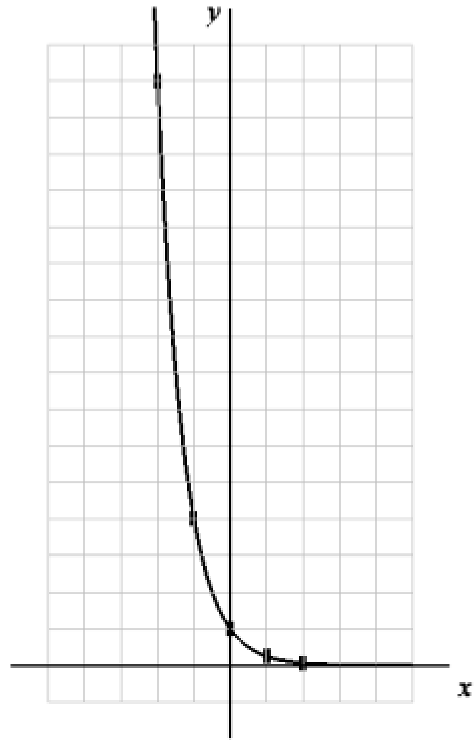
Note that as $x \to \infty$, $y = \left(\frac{1}{2}\right)^x \to 0$ and as $x \to -\infty$, $y = \left(\frac{1}{2}\right)^x \to \infty$.

Since as $x \to \infty$, $y = \left(\frac{1}{2}\right)^x \to 0$, then the horizontal line of $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

3. $h(x) = 4^{-x}$

In order to graph the function $h$ given by $h(x) = 4^{-x}$, we set $h(x) = y$ and graph the equation $y = 4^{-x}$.

| $x$ | $y$ |
|---|---|
| $-2$ | 16 |
| $-1$ | 4 |
| 0 | 1 |
| 1 | $\frac{1}{4}$ |
| 2 | $\frac{1}{16}$ |




The Drawing of this Graph

NOTE: $4^{-x} = (4^{-1})^{x} = \left(\dfrac{1}{4}\right)^{x}$

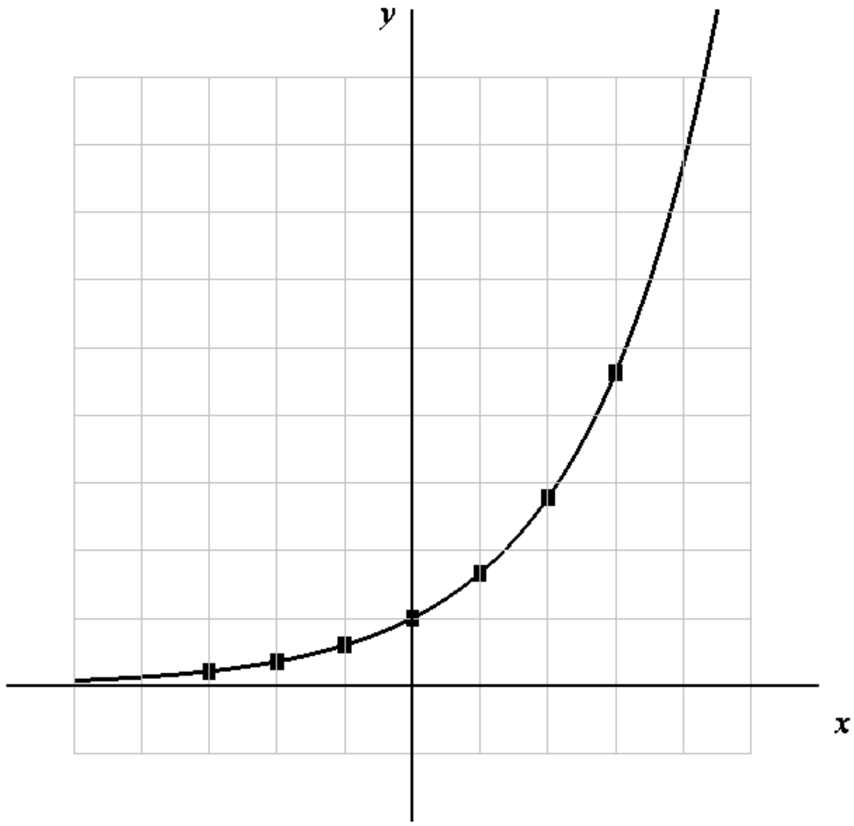
The $y$-intercept of the graph of the function is the point $(0, 1)$.

Note that as $x \to \infty$, $y = 4^{-x} \to 0$ and as $x \to -\infty$, $y = 4^{-x} \to \infty$.

Since as $x \to \infty$, $y = 4^{-x} \to 0$, then the horizontal line of $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

4. $y = \left(\dfrac{3}{5}\right)^{-x}$

| $x$ | $y$ |
|---|---|
| $-3$ | $\frac{27}{125}$ |
| $-2$ | $\frac{9}{25}$ |
| $-1$ | $\frac{3}{5}$ |
| $0$ | $1$ |
| $1$ | $\frac{5}{3}$ |
| $2$ | $\frac{25}{9}$ |
| $3$ | $\frac{125}{27}$ |




The Drawing of this Graph

NOTE: $\left(\frac{3}{5}\right)^{-x} = \left(\frac{5}{3}\right)^{x}$

The $y$-intercept of the graph of the function is the point $(0, 1)$.

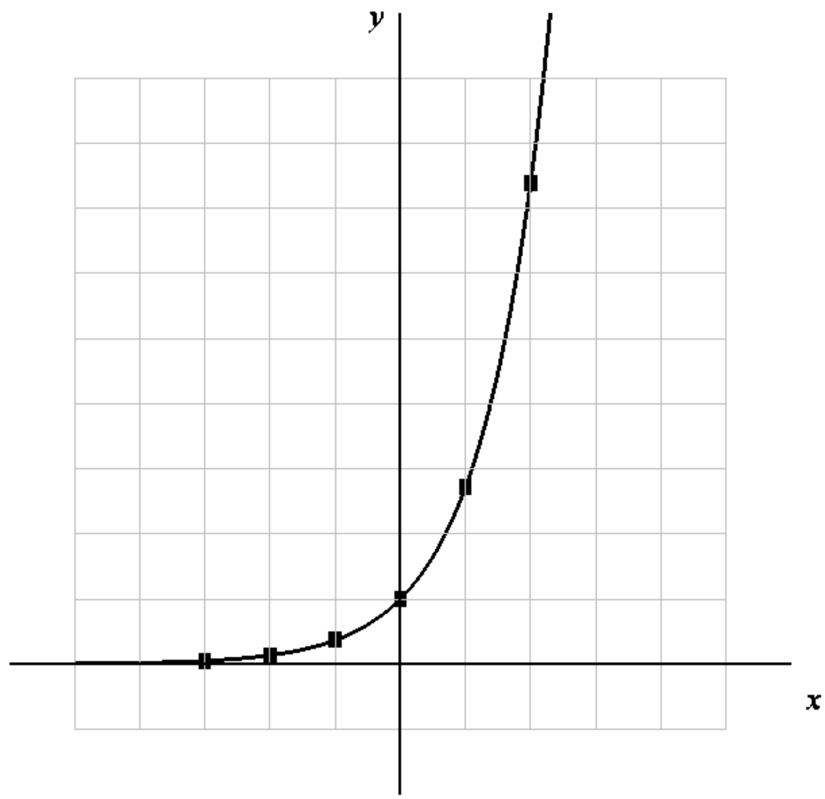
Note that as $x \to \infty$, $y = \left(\frac{3}{5}\right)^{-x} \to \infty$ and as $x \to -\infty$, $y = \left(\frac{3}{5}\right)^{-x} \to 0$.

Since as $x \to -\infty$, $y = \left(\frac{3}{5}\right)^{-x} \to 0$, then the horizontal line of $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

5. $f(x) = e^{x}$

In order to graph the function $f$ given by $f(x) = e^{x}$, we set $f(x) = y$ and graph the equation $y = e^{x}$.

| $x$ | $y$ |
|---|---|
| $-3$ | $e^{-3} \approx 0.04979$ |
| $-2$ | $e^{-2} \approx 0.13534$ |
| $-1$ | $e^{-1} \approx 0.36788$ |
| $0$ | $1$ |
| $1$ | $e \approx 2.71828$ |
| $2$ | $e^{2} \approx 7.38906$ |
| $3$ | $e^{3} \approx 20.08554$ |




The Drawing of this Graph

The $y$-intercept of the graph of the function is the point $(0, 1)$.

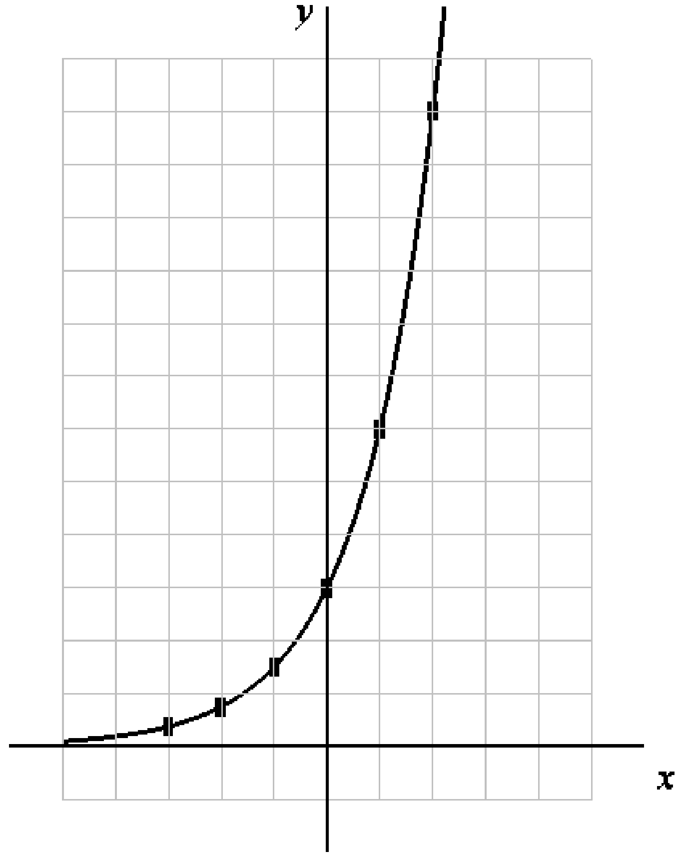
Note that as $x \to \infty$, $y = e^{x} \to \infty$ and as $x \to -\infty$, $y = e^{x} \to 0$.

Since as $x \to -\infty$, $y = e^{x} \to 0$, then the horizontal line of $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

6. $g(x) = 3(2^{x})$

In order to graph the function $g$ given by $g(x) = 3(2^{x})$, we set $g(x) = y$ and graph the equation $y = 3(2^{x})$.

| $x$ | $y$ |
|---|---|
| $-3$ | $\frac{3}{8}$ |
| $-2$ | $\frac{3}{4}$ |
| $-1$ | $\frac{3}{2}$ |
| $0$ | $3$ |
| $1$ | $6$ |
| $2$ | $12$ |
| $3$ | $24$ |




The Drawing of this Graph

The $y$-intercept of the graph of the function is the point $(0, 3)$.

Note that as $x \to \infty$, $y = 3(2^x) \to \infty$ and as $x \to -\infty$, $y = 3(2^x) \to 0$.

Since as $x \to -\infty$, $y = 3(2^x) \to 0$, then the horizontal line of $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

7. $h(x) = -4\left(\frac{2}{3}\right)^x$

In order to graph the function $h$ given by $h(x) = -4\left(\frac{2}{3}\right)^x$, we set $h(x) = y$ and graph the equation $y = -4\left(\frac{2}{3}\right)^x$.

NOTE: $h(-3) = -4\left(\frac{2}{3}\right)^{-3} = -4\left(\frac{3}{2}\right)^{3} = -4\left(\frac{27}{8}\right) = -\frac{27}{2}$

$$h(-2) = -4\left(\frac{2}{3}\right)^{-2} = -4\left(\frac{3}{2}\right)^{2} = -4\left(\frac{9}{4}\right) = -9$$

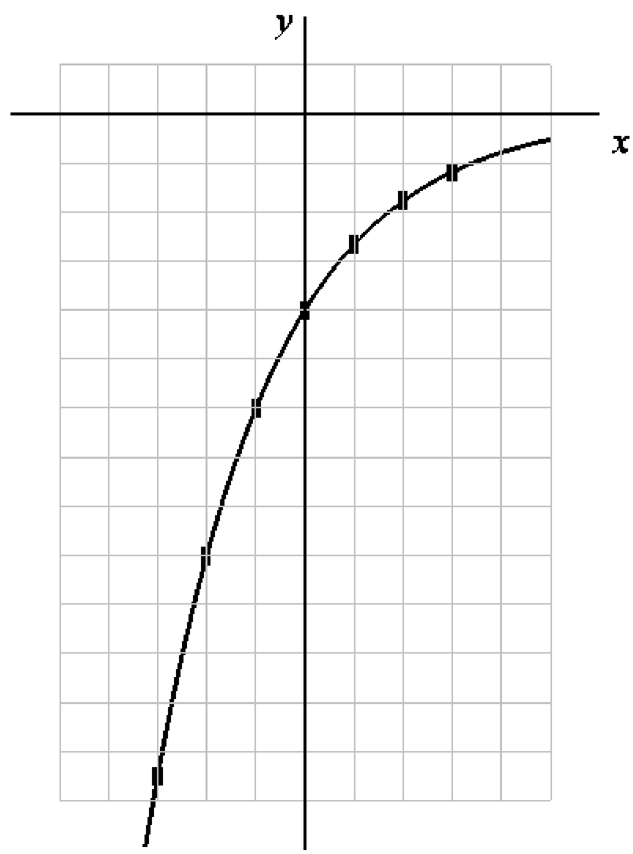
$$h(-1) = -4\left(\frac{2}{3}\right)^{-1} = -4\left(\frac{3}{2}\right) = -6$$

$$h(1) = -4\left(\frac{2}{3}\right) = -\frac{8}{3}$$

$$h(2) = -4\left(\frac{2}{3}\right)^{2} = -4\left(\frac{4}{9}\right) = -\frac{16}{9}$$

$$h(3) = -4\left(\frac{2}{3}\right)^{3} = -4\left(\frac{8}{27}\right) = -\frac{32}{27}$$

| $x$ | $y$ |
|---|---|
| $-3$ | $-\frac{27}{2}$ |
| $-2$ | $-9$ |
| $-1$ | $-6$ |
| $0$ | $-4$ |
| $1$ | $-\frac{8}{3}$ |
| $2$ | $-\frac{16}{9}$ |
| $3$ | $-\frac{32}{27}$ |




The Drawing of this Graph

The $y$-intercept of the graph of the function is the point $(0, -4)$.

Note that as $x \to \infty$, $y = -4\left(\frac{2}{3}\right)^x \to 0$ and as $x \to -\infty$, $y = -4\left(\frac{2}{3}\right)^x \to -\infty$.

Since $x \to \infty$, $y = -4\left(\frac{2}{3}\right)^x \to 0$, then the horizontal line of $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

8. $f(t) = -\frac{3}{7}\left(\frac{5}{4}\right)^t$

In order to graph the function $f$ given by $f(t) = -\frac{3}{7}\left(\frac{5}{4}\right)^t$, we set $f(t) = y$ and graph the equation $y = -\frac{3}{7}\left(\frac{5}{4}\right)^t$.

NOTE: $f(-3) = -\frac{3}{7}\left(\frac{5}{4}\right)^{-3} = -\frac{3}{7}\left(\frac{4}{5}\right)^{3} = -\frac{3}{7}\left(\frac{64}{125}\right) = -\frac{192}{875}$

$$f(-2) = -\frac{3}{7}\left(\frac{5}{4}\right)^{-2} = -\frac{3}{7}\left(\frac{4}{5}\right)^{2} = -\frac{3}{7}\left(\frac{16}{25}\right) = -\frac{48}{175}$$

$$f(-1) = -\frac{3}{7}\left(\frac{5}{4}\right)^{-1} = -\frac{3}{7}\left(\frac{4}{5}\right) = -\frac{12}{35}$$

$$f(1) = -\frac{3}{7}\left(\frac{5}{4}\right) = -\frac{15}{28}$$

$$f(2) = -\frac{3}{7}\left(\frac{5}{4}\right)^{2} = -\frac{3}{7}\left(\frac{25}{16}\right) = -\frac{75}{112}$$

$$f(3) = -\frac{3}{7}\left(\frac{5}{4}\right)^{3} = -\frac{3}{7}\left(\frac{125}{64}\right) = -\frac{375}{448}$$

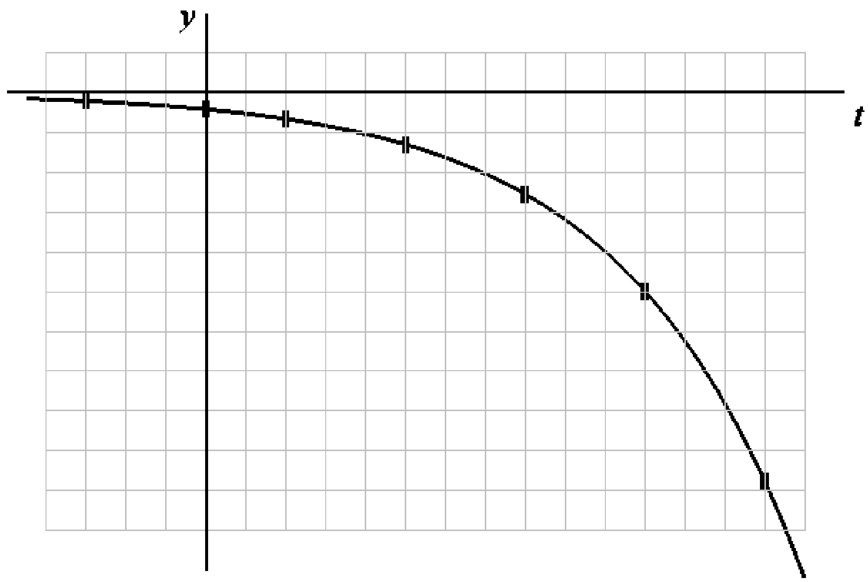
$$f(5) = -\frac{3}{7}\left(\frac{5}{4}\right)^{5} = -\frac{3}{7}\left(\frac{3125}{1024}\right) = -\frac{9375}{7168}$$

$$f(8) = -\frac{3}{7}\left(\frac{5}{4}\right)^{8} = -\frac{3}{7}\left(\frac{390625}{65536}\right) = -\frac{1171875}{458752} \approx -2.55$$

$$f(11) = -\frac{3}{7}\left(\frac{5}{4}\right)^{11} = -\frac{3}{7}\left(\frac{48828125}{4194304}\right) = -\frac{146484375}{29360128} \approx -4.99$$

$$f(14) = -\frac{3}{7}\left(\frac{5}{4}\right)^{14} = -\frac{18310546875}{1879048192} \approx -9.74$$

| $t$ | $y$ |
|---|---|
| $-3$ | $-\frac{192}{875}$ |
| $0$ | $-\frac{3}{7}$ |
| $2$ | $-\frac{75}{112}$ |
| $5$ | $-\frac{9375}{7168}$ |
| $8$ | $-\frac{1171875}{458752}$ |
| $11$ | $-\frac{146484375}{29360128}$ |
| $14$ | $-\frac{18310546875}{1879048192}$ |




The Drawing of this Graph

The $y$-intercept of the graph of the function is the point $\left(0, -\frac{3}{7}\right)$.

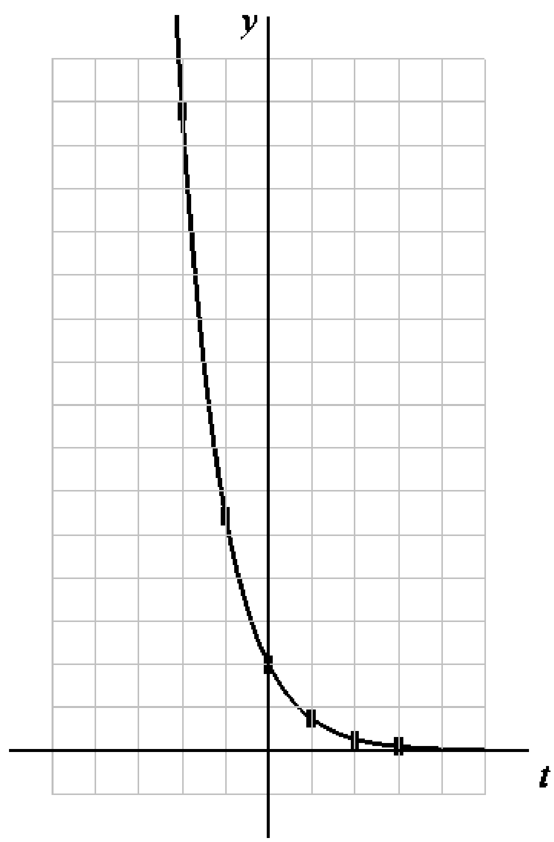
Note that as $t \to \infty$, $y = -\frac{3}{7}\left(\frac{5}{4}\right)^t \to \infty$ and as $t \to -\infty$, $y = -\frac{3}{7}\left(\frac{5}{4}\right)^t \to 0$.

Since as $t \to -\infty$, $y = -\frac{3}{7}\left(\frac{5}{4}\right)^t \to 0$, then the horizontal line of $y = 0$, which is the $t$-axis, is a horizontal asymptote of the graph of the function.

9. $g(t) = 2e^{-t}$

In order to graph the function $g$ given by $g(t) = 2e^{-t}$, we set $g(t) = y$ and graph the equation $y = 2e^{-t}$.

| $t$ | $y$ |
|---|---|
| $-3$ | $2e^{3} \approx 40.1711$ |
| $-2$ | $2e^{2} \approx 14.7781$ |
| $-1$ | $2e \approx 5.4366$ |
| $0$ | $2$ |
| $1$ | $2e^{-1} \approx 0.7258$ |
| $2$ | $2e^{-2} \approx 0.2707$ |
| $3$ | $2e^{-3} \approx 0.0996$ |




The Drawing of this Graph

The $y$-intercept of the graph of the function is the point $(0, 2)$.

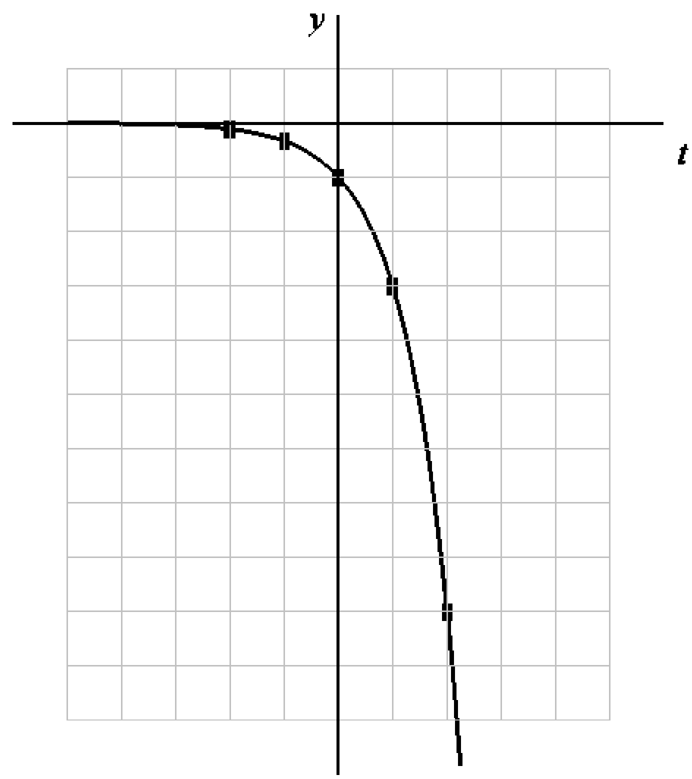
Note that as $t \to \infty$, $y = 2e^{-t} \to 0$ and as $t \to -\infty$, $y = 2e^{-t} \to \infty$.

Since as $t \to \infty$, $y = 2e^{-t} \to 0$, then the horizontal line of $y = 0$, which is the $t$-axis, is a horizontal asymptote of the graph of the function.

10. $h(t) = -3^t$

In order to graph the function $h$ given by $h(t) = -3^t$, we set $h(t) = y$ and graph the equation $y = -3^t$.

| $t$ | $y$ |
|---|---|
| $-2$ | $-\frac{1}{9}$ |
| $-1$ | $-\frac{1}{3}$ |
| $0$ | $-1$ |
| $1$ | $-3$ |
| $2$ | $-9$ |



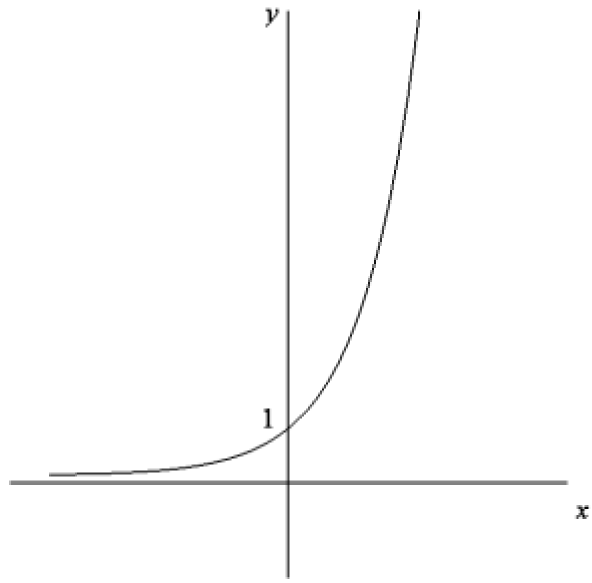

The Drawing of this Graph

The $y$-intercept of the graph of the function is the point $(0, -1)$.

Note that as $t \to \infty$, $y = -3^t \to -\infty$ and as $t \to -\infty$, $y = -3^t \to 0$.

Since as $t \to -\infty$, $y = -3^t \to 0$, then the horizontal line of $y = 0$, which is the $t$-axis, is a horizontal asymptote of the graph of the function.

The sketch of the graph of $y = b^{x}$, where $b > 1$ **OR** the sketch of the graph of $y = b^{-x}$, where $0 < b < 1$:



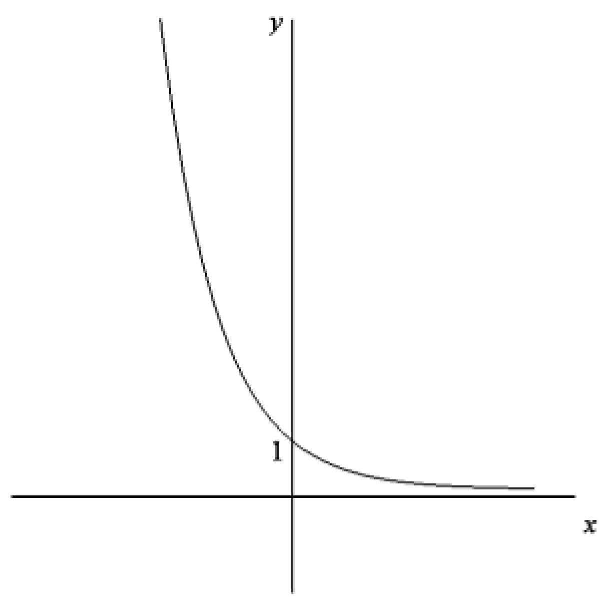

NOTE: $b^{-x} = (b^{-1})^{x} = \left(\frac{1}{b}\right)^{x}$

If $0 < b < 1$, then $\frac{1}{b} > 1$.

The $y$-intercept of the graph of the function is the point $(0, 1)$.

The horizontal line $y = 0$, which is the *x*-axis, is a horizontal asymptote of the graph of the function.

The sketch of the graph of $y = b^{x}$, where $0 < b < 1$ **OR** the sketch of the graph of $y = b^{-x}$, where $b > 1$:




NOTE: $b^{-x} = (b^{-1})^{x} = \left(\frac{1}{b}\right)^{x}$

If $b > 1$, then $0 < \frac{1}{b} < 1$.

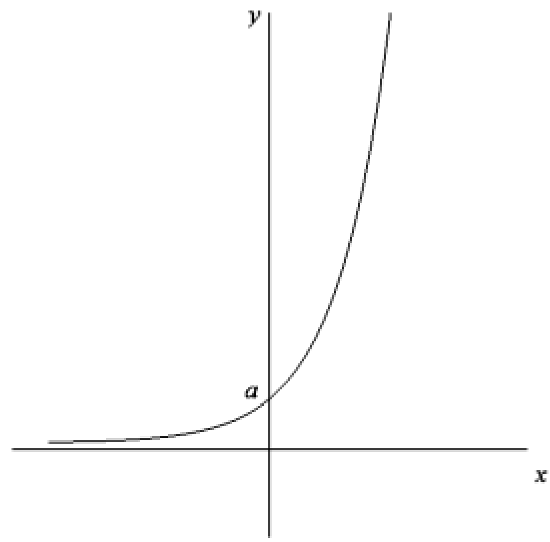
The $y$-intercept of the graph of the function is the point $(0, 1)$.

The horizontal line $y = 0$, which is the *x*-axis, is a horizontal asymptote of the graph of the function.

Let $a > 0$. The sketch of the graph of $y = a(b^x)$, where $b > 1$ **OR** the sketch of the graph of $y = a(b^{-x})$, where $0 < b < 1$:

NOTE: $b^{-x} = (b^{-1})^x = \left(\dfrac{1}{b}\right)^x$

If $0 < b < 1$, then $\dfrac{1}{b} > 1$.



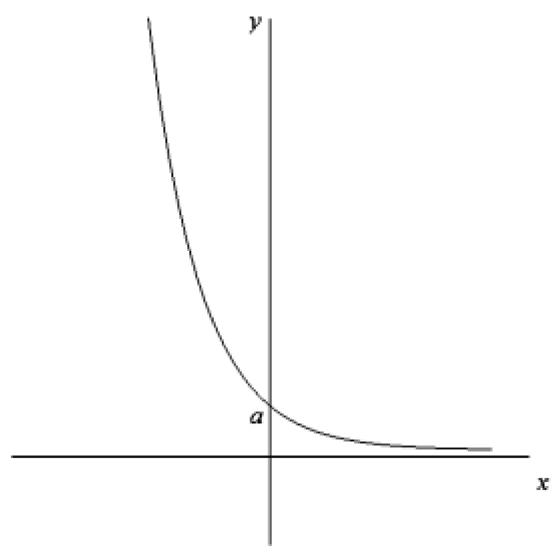

The $y$-intercept of the graph of the function is the point $(0, a)$.

The horizontal line $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

Let $a > 0$. The sketch of the graph of $y = a(b^x)$, where $0 < b < 1$ **OR** the sketch of the graph of $y = a(b^{-x})$, where $b > 1$:

NOTE: $b^{-x} = (b^{-1})^x = \left(\dfrac{1}{b}\right)^x$

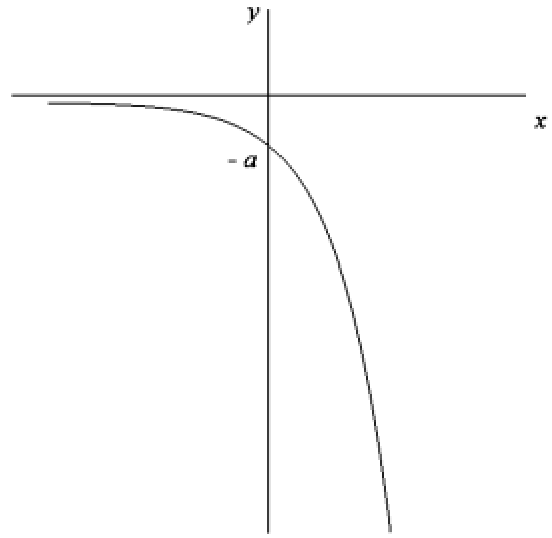
If $b > 1$, then $0 < \dfrac{1}{b} < 1$.




The $y$-intercept of the graph of the function is the point $(0, a)$.

The horizontal line $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

Let $a > 0$. The sketch of the graph of $y = -a(b^x)$, where $b > 1$ **OR** the sketch of the graph of $y = -a(b^{-x})$, where $0 < b < 1$:



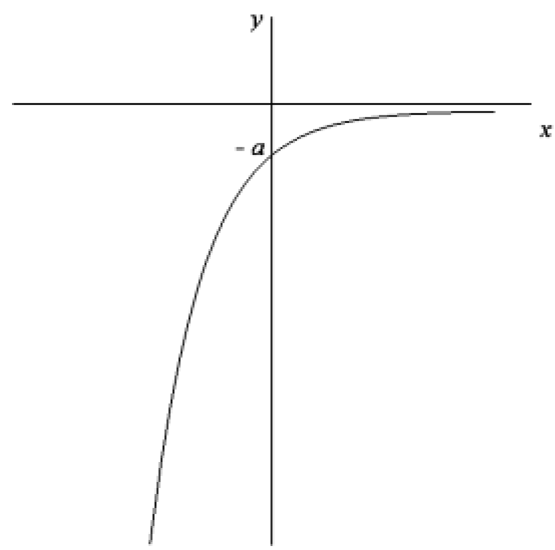

NOTE: $b^{-x} = (b^{-1})^x = \left(\frac{1}{b}\right)^x$

If $0 < b < 1$, then $\frac{1}{b} > 1$.

The $y$-intercept of the graph of the function is the point $(0, -a)$.

The horizontal line $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

Let $a > 0$. The sketch of the graph of $y = -a(b^x)$, where $0 < b < 1$ **OR** the sketch of the graph of $y = -a(b^{-x})$, where $b > 1$:




NOTE: $b^{-x} = (b^{-1})^x = \left(\frac{1}{b}\right)^x$

If $b > 1$, then $0 < \frac{1}{b} < 1$.

The $y$-intercept of the graph of the function is the point $(0, -a)$.

The horizontal line $y = 0$, which is the $x$-axis, is a horizontal asymptote of the graph of the function.

**Examples** Sketch the graph of the following functions. State the domain of the function and use the sketch to state the range of the function.

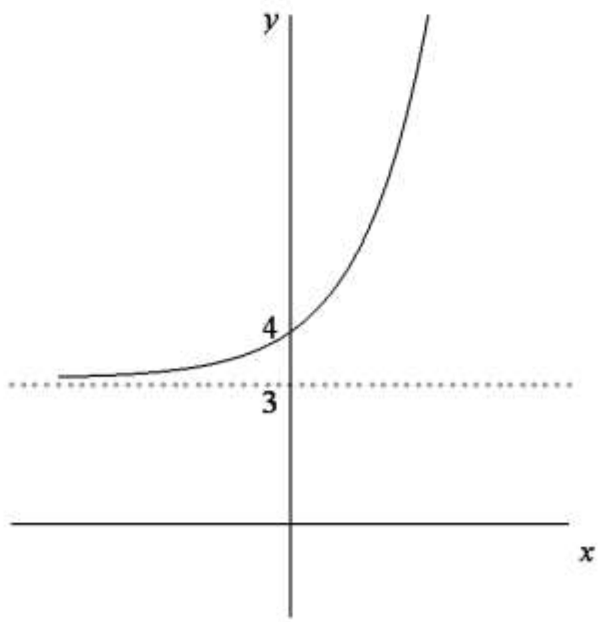
1. $f(x) = 5^x + 3$

The domain of $f$ is the set of all real numbers.

To graph the function $f$, we set $f(x) = y$ and graph the equation $y = 5^x + 3$.

$$y = 5^x + 3 \Rightarrow y - 3 = 5^x$$

The graph of $y - 3 = 5^x$ is the graph of $y = 5^x$ shifted 3 units upward.




The Drawing of this Sketch

The range of $f$ is $(3, \infty)$. Note that the $y$-intercept is the point $(0, 4)$.

NOTE: The vertical shift of 3 units upward is determined from the expression $y - 3$ in the equation $y - 3 = 5^x$.

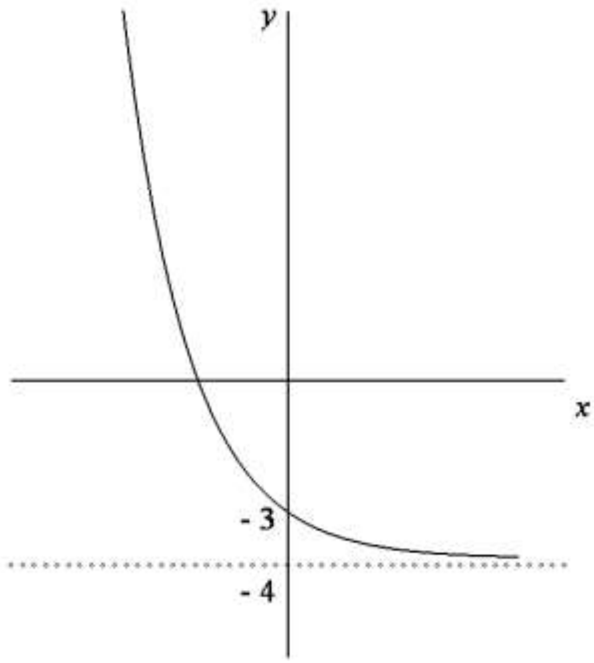
2. $g(x) = e^{-x} - 4$

The domain of $g$ is the set of all real numbers.

To graph the function $g$, we set $g(x) = y$ and graph the equation $y = e^{-x} - 4$.

$$y = e^{-x} - 4 \Rightarrow y + 4 = e^{-x}$$

The graph of $y + 4 = e^{-x}$ is the graph of $y = e^{-x}$ shifted 4 units downward.




The Drawing of this Sketch

The range of $g$ is $(-4, \infty)$. Note that the *y*-intercept is the point $(0, -3)$.

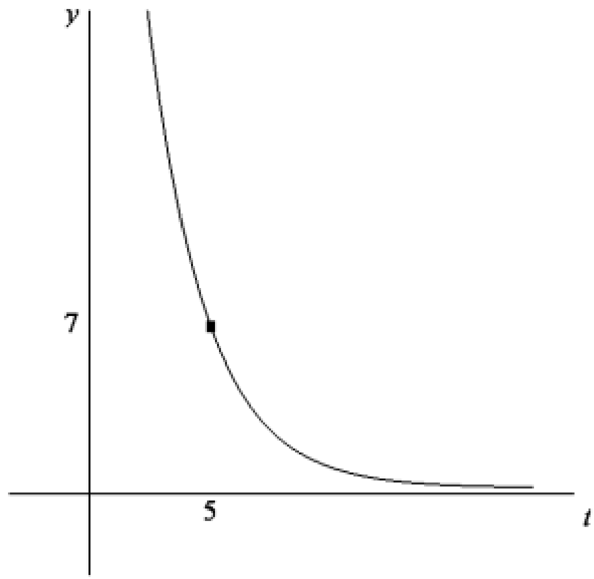
NOTE: The vertical shift of 4 units downward is determined from the expression $y + 4$ in the equation $y + 4 = e^{-x}$.

3. $h(t) = 7\left(\frac{3}{4}\right)^{t - 5}$

The domain of $h$ is the set of all real numbers.

To graph the function $h$, we set $h(t) = y$ and graph the equation $y = 7\left(\frac{3}{4}\right)^{t - 5}$.

The graph of $y = 7\left(\frac{3}{4}\right)^{t-5}$ is the graph of $y = 7\left(\frac{3}{4}\right)^{t}$ shifted 5 units to the right.




The Drawing of this Sketch

The range of $h$ is $(0, \infty)$.

The $y$-coordinate of the $y$-intercept is obtained by setting $t = 0$ in the equation $y = 7\left(\frac{3}{4}\right)^{t-5}$. Thus, we have that $y = 7\left(\frac{3}{4}\right)^{-5} = 7\left(\frac{4}{3}\right)^{5} =$ $7\left(\frac{1024}{243}\right) = \frac{7168}{243}$. Thus, the $y$-intercept is the point $\left(0, \frac{7168}{243}\right)$.

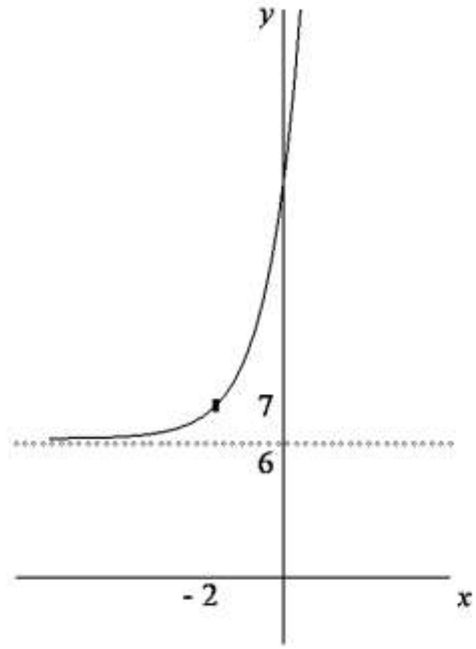
NOTE: The horizontal shift of 5 units to the right is determined from the expression $t - 5$ in the equation $y = 7\left(\frac{3}{4}\right)^{t-5}$.

4. $y = 8^{x+2} + 6$

The domain of the function is the set of all real numbers.

$$y = 8^{x+2} + 6 \Rightarrow y - 6 = 8^{x+2}$$

The graph of $y - 6 = 8^{x + 2}$ is the graph of $y = 8^x$ shifted 2 units to the left and 6 units upward.




The Drawing of this Sketch

The range of the function is $(6, \infty)$.

The $y$-coordinate of the $y$-intercept is obtained by setting $x = 0$ in the equation $y = 8^{x + 2} + 6$. Thus, we have that $y = 8^2 + 6 = 64 + 6 = 70$. Thus, the $y$-intercept is the point $(0, 70)$.

NOTE: The horizontal shift of 2 units to the left is determined from the expression $x + 2$ in the equation $y - 6 = 8^{x + 2}$ and the vertical shift of 6 units upward is determined from the expression $y - 6$ in the equation.

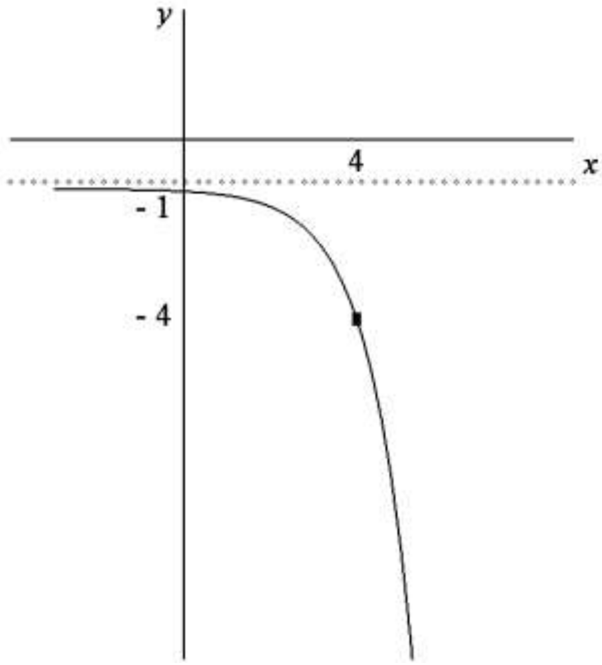
5. $f(x) = -3e^{x - 4} - 1$

The domain of $f$ is the set of all real numbers.

To graph the function $f$, we set $f(x) = y$ and graph the equation $y = -3e^{x - 4} - 1$.

$$y = -3e^{x - 4} - 1 \Rightarrow y + 1 = -3e^{x - 4}$$

The graph of $y + 1 = -3e^{x - 4}$ is the graph of $y = -3e^{x}$ shifted 4 units to the right and 1 unit downward.




The Drawing of this Sketch

The range of the function is $(-\infty, -1)$.

The *y*-coordinate of the *y*-intercept is obtained by setting $x = 0$ in the equation $y = -3e^{x - 4} - 1$. Thus, we have that $y = -3e^{-4} - 1$. Thus, the *y*-intercept is the point $(0, -3e^{-4} - 1)$.

NOTE: The horizontal shift of 4 units to the right is determined from the expression $x - 4$ in the equation $y + 1 = -3e^{x - 4}$ and the vertical shift of 1 unit downward is determined from the expression $y + 1$ in the equation.

6. $g(x) = 2(0.35)^{x + 6} - 9$

7. $h(t) = -4e^{7 - t} + 15$

8. $f(x) = \pi^{3x + 8}$

9. $g(x) = 5(3^{x/2}) - 12$

10. $h(x) = -\left(\frac{11}{7}\right)^{2x - 6} + 25$