LESSON 7 RATIONAL ZEROS (ROOTS) OF POLYNOMIALS

Recall that a rational number is a quotient of integers. That is, a rational number is of the form $\frac{a}{b}$, where a and b are integers. A rational number $\frac{a}{b}$ is said to be in reduced form if the greatest common divisor (GCD) of a and b is one.

Examples $\frac{-3}{6}$, $\frac{4}{5}$, $\frac{17}{-9}$, and $\frac{8}{12}$ are rational numbers. The numbers $\frac{4}{5}$ and $\frac{17}{-9}$ are in reduced form. The numbers $\frac{-3}{6}$ and $\frac{8}{12}$ are not in reduced form. Of course, we can write $\frac{-3}{6}$ and $\frac{17}{-9}$ as $-\frac{3}{6}$ and $-\frac{17}{9}$ respectively. In reduced form, $-\frac{3}{6}$ is $-\frac{1}{2}$ and $\frac{8}{12}$ is $\frac{2}{3}$.

Examples $\frac{\sqrt{7}}{4}$ and $\frac{\pi}{6}$ are not rational numbers since $\sqrt{7}$ and π are not integers.

Theorem Let p be a polynomial with integer coefficients. If $\frac{c}{d}$ is a rational zero (root) in reduced form of

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where the a_i's are integers for $i = 1, 2, 3, \ldots, n$ and $a_n \neq 0$ and $a_0 \neq 0$, then c is a factor of a_0 and d is a factor of a_n.

Theorem (Bounds for Real Zeros (Roots) of Polynomials) Let p be a polynomial with real coefficients and positive leading coefficient.

1. If $p(x)$ is synthetically divided by $x - a$, where $a > 0$, and all the numbers in the third row of the division process are either positive or zero, then a is an upper bound for the real solutions of the equation $p(x) = 0$.

2. If $p(x)$ is synthetically divided by $x - a$, where $a < 0$, and all the numbers in the third row of the division process are alternately positive and negative (and a 0 can be considered to be either positive or negative as needed), then a is a lower bound for the real solutions of the equation $p(x) = 0$.

Examples Find the zeros (roots) of the following polynomials. Also, give a factorization for the polynomial.

1. $f(x) = x^3 - 5x^2 - 2x + 24$

To find the zeros (roots) of *f*, we want to solve the equation $f(x) = 0 \Rightarrow$

$x^3 - 5x^2 - 2x + 24 = 0$. The expression $x^3 - 5x^2 - 2x + 24$ can

not be factored by grouping.

We need to find one rational zero (root) for the polynomial *f*. This will produce a linear factor for the polynomial and the other factor will be quadratic.

Factors of 24: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 8,\ \pm 12,\ \pm 24$

Factors of 1: 1

Possible rational zeros (roots): $\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 8,\ \pm 12,\ \pm 24$

Trying 1:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{Coeff\ of\ x^3 - 5x^2 - 2x + 24}} & \\ 1 & -5 & -2 & 24 & 1 \\ & 1 & -4 & -6 & \\ \hline 1 & -4 & -6 & 18 & \end{array}$$

Thus, $f(1) = 18 \neq 0 \Rightarrow x - 1$ is not a factor of f and 1 is not a zero (root) of *f*.

Trying -1:

$$\begin{array}{rrrr|r} \multicolumn{4}{c}{\overbrace{\hspace{4cm}}^{Coeff\ of\ x^3 - 5x^2 - 2x + 24}} & \\ 1 & -5 & -2 & 24 & -1 \\ & -1 & 6 & -4 & \\ \hline 1 & -6 & 4 & 20 & \end{array}$$

Thus, $f(-1) = 20 \neq 0 \Rightarrow x + 1$ is not a factor of f and -1 is not a zero (root) of f.

Trying 2:

$$\begin{array}{rrrr|r} \multicolumn{4}{c}{\overbrace{\hspace{4cm}}^{Coeff\ of\ x^3 - 5x^2 - 2x + 24}} & \\ 1 & -5 & -2 & 24 & 2 \\ & 2 & -6 & -16 & \\ \hline 1 & -3 & -8 & 8 & \end{array}$$

Thus, $f(2) = 8 \neq 0 \Rightarrow x - 2$ is not a factor of f and 2 is not a zero (root) of f.

Trying -2:

$$\begin{array}{rrrr|r} \multicolumn{4}{c}{\overbrace{\hspace{4cm}}^{Coeff\ of\ x^3 - 5x^2 - 2x + 24}} & \\ 1 & -5 & -2 & 24 & -2 \\ & -2 & 14 & -24 & \\ \hline 1 & -7 & 12 & 0 & \end{array}$$

Thus, $f(-2) = 0 \Rightarrow x + 2$ is a factor of f and -2 is a zero (root) of f.

NOTE: By the Bound Theorem above, -2 is a lower bound for the negative zeros (roots) of f since we alternate from **positive** 1 to **negative** 7 to **positive** 12 to **negative** 0 in the third row of the synthetic division.

The third row in the synthetic division gives us the coefficients of the other factor starting with x^2. Thus, the other factor is $x^2 - 7x + 12$.

Thus, we have that $x^3 - 5x^2 - 2x + 24 = (x + 2)(x^2 - 7x + 12)$.

Now, we can try to find a factorization for the expression $x^2 - 7x + 12$:
$x^2 - 7x + 12 = (x - 3)(x - 4)$

Thus, we have that $x^3 - 5x^2 - 2x + 24 = (x + 2)(x^2 - 7x + 12) =$ $(x + 2)(x - 3)(x - 4)$

Thus, $x^3 - 5x^2 - 2x + 24 = 0 \Rightarrow (x + 2)(x - 3)(x - 4) = 0 \Rightarrow$

$x = -2, \; x = 3, \; x = 4$

Answer: Zeros (Roots): -2, 3, 4

Factorization: $x^3 - 5x^2 - 2x + 24 = (x + 2)(x - 3)(x - 4)$

2. $g(x) = 3x^3 - 23x^2 + 57x - 45$

To find the zeros (roots) of g, we want to solve the equation $g(x) = 0 \Rightarrow$

$3x^3 - 23x^2 + 57x - 45 = 0$. The expression $3x^3 - 23x^2 + 57x - 45$

can not be factored by grouping.

We need to find one rational zero (root) for the polynomial g. This will produce a linear factor for the polynomial and the other factor will be quadratic.

Factors of -45: $\pm 1, \; \pm 3, \; \pm 5, \; \pm 9, \; \pm 15, \; \pm 45$

Factors of 3: 1, 3

The rational numbers obtained using the factors of -45 for the numerator and the 1 as the factor of 3 for the denominator:

$$\pm 1, \; \pm 3, \; \pm 5, \; \pm 9, \; \pm 15, \; \pm 45$$

The rational numbers obtained using the factors of -45 for the numerator and the 3 as the factor of 3 for the denominator:

$$\pm\frac{1}{3},\ \pm 1,\ \pm\frac{5}{3},\ \pm 3,\ \pm 5,\ \pm 15$$

NOTE: $\pm\frac{3}{3} = \pm 1,\ \pm\frac{9}{3} = \pm 3,\ \pm\frac{15}{3} = \pm 5$, and $\pm\frac{45}{3} = \pm 15$

Possible rational zeros (roots): $\pm\frac{1}{3}, \pm 1, \pm\frac{5}{3}, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45$

Trying 1:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\text{Coeff of } 3x^3 - 23x^2 + 57x - 45}} & \\ 3 & -23 & 57 & -45 & \underline{1} \\ & 3 & -20 & 37 & \\ \hline 3 & -20 & 37 & -8 & \end{array}$$

Thus, $g(1) = -8 \neq 0 \Rightarrow x - 1$ is not a factor of g and 1 is not a zero (root) of g.

Trying -1:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\text{Coeff of } 3x^3 - 23x^2 + 57x - 45}} & \\ 3 & -23 & 57 & -45 & \underline{-1} \\ & -3 & 26 & -83 & \\ \hline 3 & -26 & 83 & -128 & \end{array}$$

Thus, $g(-1) = -128 \neq 0 \Rightarrow x + 1$ is not a factor of g and -1 is not a zero (root) of g.

NOTE: By the Bound Theorem above, -1 is a lower bound for the negative zeros (roots) of g since we alternate from **positive** 3 to **negative** 26 to **positive** 83 to **negative** 128 in the third row of the synthetic division. Thus, $-\frac{5}{3}, -3, -5, -9, -15$, and -45 can not be rational zeros (roots) of g.

Trying 2:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\text{Coeff of } 3x^3 - 23x^2 + 57x - 45}} & \\ 3 & -23 & 57 & -45 & \underline{2} \\ & 6 & -34 & 46 & \\ \hline 3 & -17 & 23 & 1 & \end{array}$$

Thus, $g(2) = 1 \neq 0 \Rightarrow x - 2$ is not a factor of g and 2 is not a zero (root) of g.

Trying 3:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{3 \quad -23 \quad 57 \quad -45}}^{Coeff\ of\ 3x^3 - 23x^2 + 57x - 45}} & \\ 3 & -23 & 57 & -45 & \underline{3} \\ & 9 & -42 & 45 & \\ \hline 3 & -14 & 15 & 0 & \end{array}$$

Thus, $g(3) = 0 \Rightarrow x - 3$ is a factor of g and 3 is a zero (root) of g.

The third row in the synthetic division gives us the coefficients of the other factor starting with x^2. Thus, the other factor is $3x^2 - 14x + 15$.

Thus, we have that $3x^3 - 23x^2 + 57x - 45 = (x - 3)(3x^2 - 14x + 15)$.

Now, we can try to find a factorization for the expression $3x^2 - 14x + 15$:
$3x^2 - 14x + 15 = (x - 3)(3x - 5)$

Thus, we have that $3x^3 - 23x^2 + 57x - 45 = (x - 3)(3x^2 - 14x + 15)$
$= (x - 3)(x - 3)(3x - 5) = (x - 3)^2(3x - 5)$

Thus, $3x^3 - 23x^2 + 57x - 45 = 0 \Rightarrow (x - 3)^2(3x - 5) = 0 \Rightarrow$

$x = 3, \; x = \dfrac{5}{3}$

Answer: Zeros (Roots): $\dfrac{5}{3}$, 3 (multiplicity 2)

Factorization: $3x^3 - 23x^2 + 57x - 45 = (x - 3)^2(3x - 5)$

3. $h(t) = 4t^3 - 4t^2 - 9t + 30$

To find the zeros (roots) of h, we want to solve the equation $h(t) = 0 \Rightarrow$

$4t^3 - 4t^2 - 9t + 30 = 0$. The expression $4t^3 - 4t^2 - 9t + 30$

can not be factored by grouping.

We need to find one rational zero (root) for the polynomial h. This will produce a linear factor for the polynomial and the other factor will be quadratic.

Factors of 30: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 5,\ \pm 6,\ \pm 10,\ \pm 15,\ \pm 30$

Factors of 4: 1, 2, 4

The rational numbers obtained using the factors of -30 for the numerator and the 1 as the factor of 4 for the denominator:

$$\pm 1,\ \pm 2,\ \pm 3,\ \pm 5,\ \pm 6,\ \pm 10,\ \pm 15,\ \pm 30$$

The rational numbers obtained using the factors of -30 for the numerator and the 2 as the factor of 4 for the denominator:

$$\pm \frac{1}{2},\ \pm 1,\ \pm \frac{3}{2},\ \pm \frac{5}{2},\ \pm 3,\ \pm 5,\ \pm \frac{15}{2},\ \pm 15$$

Eliminating the ones that are already listed above, we have

$$\pm \frac{1}{2},\ \pm \frac{3}{2},\ \pm \frac{5}{2},\ \pm \frac{15}{2}$$

The rational numbers obtained using the factors of -30 for the numerator and the 4 as the factor of 4 for the denominator:

$$\pm \frac{1}{4},\ \pm \frac{1}{2},\ \pm \frac{3}{4},\ \pm \frac{5}{4},\ \pm \frac{3}{2},\ \pm \frac{5}{2},\ \pm \frac{15}{4},\ \pm \frac{15}{2}$$

Eliminating the ones that are already listed above, we have

$$\pm \frac{1}{4}, \ \pm \frac{3}{4}, \ \pm \frac{5}{4}, \ \pm \frac{15}{4}$$

Possible rational zeros (roots): $\pm \frac{1}{4}, \ \pm \frac{1}{2}, \ \pm \frac{3}{4}, \ \pm 1, \ \pm \frac{5}{4}, \ \pm \frac{3}{2}, \ \pm 2,$ $\pm \frac{5}{2}, \ \pm 3, \ \pm \frac{15}{4}, \ \pm 5, \ \pm 6, \ \pm \frac{15}{2}, \ \pm 10, \ \pm 15, \ \pm 30$

Trying 1:

$$\begin{array}{cccc|c} \multicolumn{4}{c}{\overbrace{\textit{Coeff of } 4t^3 - 4t^2 - 9t + 30}} & \\ 4 & -4 & -9 & 30 & 1 \\ & 4 & 0 & -9 & \\ \hline 4 & 0 & -9 & 21 & \end{array}$$

Thus, $h(1) = 21 \neq 0 \Rightarrow t - 1$ is not a factor of h and 1 is not a zero (root) of h.

Trying 2:

$$\begin{array}{cccc|c} \multicolumn{4}{c}{\overbrace{\textit{Coeff of } 4t^3 - 4t^2 - 9t + 30}} & \\ 4 & -4 & -9 & 30 & 2 \\ & 8 & 8 & -2 & \\ \hline 4 & 4 & -1 & 28 & \end{array}$$

Thus, $h(2) = 28 \neq 0 \Rightarrow t - 2$ is not a factor of h and 2 is not a zero (root) of h.

Trying 3:

$$\begin{array}{cccc|c} \multicolumn{4}{c}{\overbrace{\textit{Coeff of } 4t^3 - 4t^2 - 9t + 30}} & \\ 4 & -4 & -9 & 30 & 3 \\ & 12 & 24 & 45 & \\ \hline 4 & 8 & 15 & 75 & \end{array}$$

Thus, $h(3) = 75 \neq 0 \Rightarrow t - 3$ is not a factor of h and 3 is not a zero (root) of h.

NOTE: By the Bound Theorem above, 3 is an upper bound for the positive zeros (roots) of h since all the numbers are positive in the third row of the synthetic division. Thus, $\frac{15}{4}$, 5, 6, $\frac{15}{2}$, 10, 15, and 30 can not be rational zeros (roots) of h.

Trying -1:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{Coeff\ of\ 4t^3 - 4t^2 - 9t + 30}} & \\ 4 & -4 & -9 & 30 & \underline{-1} \\ & -4 & 8 & 1 & \\ \hline 4 & -8 & -1 & 31 & \end{array}$$

Thus, $h(-1) = 31 \neq 0 \Rightarrow t + 1$ is not a factor of h and -1 is not a zero (root) of h.

Trying -2:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{Coeff\ of\ 4t^3 - 4t^2 - 9t + 30}} & \\ 4 & -4 & -9 & 30 & \underline{-2} \\ & -8 & 24 & -30 & \\ \hline 4 & -12 & 15 & 0 & \end{array}$$

Thus, $h(-2) = 0 \Rightarrow t + 2$ is factor of h and -2 is a zero (root) of h.

The third row in the synthetic division gives us the coefficients of the other factor starting with t^2. Thus, the other factor is $4t^2 - 12t + 15$.

Thus, we have that $4t^3 - 4t^2 - 9t + 30 = (t + 2)(4t^2 - 12t + 15)$.

Now, we can try to find a factorization for the expression $4t^2 - 12t + 15$. However, it does not factor.

Thus, we have that $4t^3 - 4t^2 - 9t + 30 = (t + 2)(4t^2 - 12t + 15)$

Thus, $4t^3 - 4t^2 - 9t + 30 = 0 \Rightarrow (t + 2)(4t^2 - 12t + 15) = 0 \Rightarrow$

$t = -2,\ 4t^2 - 12t + 15 = 0$

We will need to use the Quadratic Formula to solve $4t^2 - 12t + 15 = 0$.

Thus, $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{12 \pm \sqrt{12 \cdot 12 - 4(4)15}}{8} =$

$\dfrac{12 \pm \sqrt{4 \cdot 4[3 \cdot 3 - 1(1)15]}}{8} = \dfrac{12 \pm 4\sqrt{9 - 15}}{8} = \dfrac{12 \pm 4\sqrt{-6}}{8} =$

$\dfrac{12 \pm 4i\sqrt{6}}{8} = \dfrac{3 \pm i\sqrt{6}}{2}$

Answer: Zeros (Roots): $-2, \dfrac{3 + i\sqrt{6}}{2}, \dfrac{3 - i\sqrt{6}}{2}$

Factorization: $4t^3 - 4t^2 - 9t + 30 = (t + 2)(4t^2 - 12t + 15)$

4. $p(x) = x^4 - 6x^3 - 3x^2 + 16x + 12$

To find the zeros (roots) of p, we want to solve the equation $p(x) = 0 \Rightarrow$

$x^4 - 6x^3 - 3x^2 + 16x + 12 = 0$.

We need to find two rational zeros (roots) for the polynomial p. This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of 12: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Factors of 1: 1

Possible rational zeros (roots): $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Trying 1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad -6 \quad -3 \quad 16 \quad 12}}^{Coeff\ of\ x^4 - 6x^3 - 3x^2 + 16x + 12}} & \\ 1 & -6 & -3 & 16 & 12 & \ 1 \\ & 1 & -5 & -8 & 8 & \\ \hline 1 & -5 & -8 & 8 & 20 & \end{array}$$

Thus, $p(1) = 20 \neq 0 \Rightarrow x - 1$ is not a factor of p and 1 is not a zero (root) of p.

Trying -1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad -6 \quad -3 \quad 16 \quad 12}}^{Coeff\ of\ x^4 - 6x^3 - 3x^2 + 16x + 12}} & \\ 1 & -6 & -3 & 16 & 12 & \ -1 \\ & -1 & 7 & -4 & -12 & \\ \hline 1 & -7 & 4 & 12 & 0 & \end{array}$$

Thus, $p(-1) = 0 \Rightarrow x + 1$ is a factor of p and -1 is a zero (root) of p.

The third row in the synthetic division gives us the coefficients of the other factor starting with x^3. Thus, the other factor is $x^3 - 7x^2 + 4x + 12$.

Thus, we have that $x^4 - 6x^3 - 3x^2 + 16x + 12 =$ $(x + 1)(x^3 - 7x^2 + 4x + 12)$.

Note that the remaining zeros of the polynomial p must also be zeros (roots) of the quotient polynomial $q(x) = x^3 - 7x^2 + 4x + 12$. We will use this polynomial to find the remaining zeros (roots) of p, including another zero (root) of -1.

Trying -1 again:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{1 \quad -7 \quad 4 \quad 12}}^{Coeff\ of\ x^3 - 7x^2 + 4x + 12}} & \\ 1 & -7 & 4 & 12 & \ -1 \\ & -1 & 8 & -12 & \\ \hline 1 & -8 & 12 & 0 & \end{array}$$

The remainder is 0. Thus, $x + 1$ is a factor of the quotient polynomial $q(x) = x^3 - 7x^2 + 4x + 12$ and -1 is a zero (root) of multiplicity of the polynomial p.

Thus, we have that $x^3 - 7x^2 + 4x + 12 = (x + 1)(x^2 - 8x + 12)$.

Thus, we have that $x^4 - 6x^3 - 3x^2 + 16x + 12 =$
$(x + 1)(x^3 - 7x^2 + 4x + 12) = (x + 1)(x + 1)(x^2 - 8x + 12) =$
$(x + 1)^2 (x^2 - 8x + 12)$.

Now, we can try to find a factorization for the expression $x^2 - 8x + 12$:
$x^2 - 8x + 12 = (x - 2)(x - 6)$

Thus, we have that $x^4 - 6x^3 - 3x^2 + 16x + 12 =$
$(x + 1)^2 (x^2 - 8x + 12) = (x + 1)^2 (x - 2)(x - 6)$

Thus, $x^4 - 6x^3 - 3x^2 + 16x + 12 = 0 \Rightarrow$

$(x + 1)^2 (x - 2)(x - 6) = 0 \Rightarrow x = -1, \; x = 2, \; x = 6$

Answer: Zeros (Roots): -1 (multiplicity 2), 2, 6

Factorization: $x^4 - 6x^3 - 3x^2 + 16x + 12 = (x + 1)^2 (x - 2)(x - 6)$

5. $f(z) = 6z^4 - 11z^3 - 53z^2 + 108z - 36$

To find the zeros (roots) of *f*, we want to solve the equation $f(z) = 0 \Rightarrow$

$6z^4 - 11z^3 - 53z^2 + 108z - 36 = 0$.

We need to find two rational zeros (roots) for the polynomial *f*. This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of -36: $\pm 1, \; \pm 2, \; \pm 3, \; \pm 4, \; \pm 6, \; \pm 9, \; \pm 12, \; \pm 18, \pm 36$

Factors of 6: 1, 2, 3, 6

The rational numbers obtained using the factors of -36 for the numerator and the 1 as the factor of 6 for the denominator:

$$\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 9,\ \pm 12,\ \pm 18,\ \pm 36$$

The rational numbers obtained using the factors of -36 for the numerator and the 2 as the factor of 6 for the denominator:

$$\pm \frac{1}{2},\ \pm 1,\ \pm \frac{3}{2},\ \pm 2,\ \pm 3,\ \pm \frac{9}{2},\ \pm 6,\ \pm 9,\ \pm 18$$

Eliminating the ones that are already listed above, we have

$$\pm \frac{1}{2},\ \pm \frac{3}{2},\ \pm \frac{9}{2}$$

The rational numbers obtained using the factors of -36 for the numerator and the 3 as the factor of 6 for the denominator:

$$\pm \frac{1}{3},\ \pm \frac{2}{3},\ \pm 1,\ \pm \frac{4}{3},\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 12$$

Eliminating the ones that are already listed above, we have

$$\pm \frac{1}{3},\ \pm \frac{2}{3},\ \pm \frac{4}{3}$$

The rational numbers obtained using the factors of -36 for the numerator and the 6 as the factor of 6 for the denominator:

$$\pm \frac{1}{6},\ \pm \frac{1}{3},\ \pm \frac{1}{2},\ \pm \frac{2}{3},\ \pm 1,\ \pm \frac{3}{2},\ \pm 2,\ \pm 3,\ \pm 6$$

Eliminating the ones that are already listed above, we have $\pm \frac{1}{6}$

Possible rational zeros (roots): $\pm\frac{1}{6}$, $\pm\frac{1}{3}$, $\pm\frac{1}{2}$, $\pm\frac{2}{3}$, ± 1, $\pm\frac{4}{3}$, $\pm\frac{3}{2}$, ± 2, ± 3, ± 4, $\pm\frac{9}{2}$, ± 6, ± 9, ± 12, ± 18, ± 36

Trying 1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hspace{8cm}}^{Coeff\ of\ 6z^4 - 11z^3 - 53z^2 + 108z - 36}} \\ 6 & -11 & -53 & 108 & -36 & \underline{1} \\ & 6 & -5 & -58 & 50 & \\ \hline 6 & -5 & -58 & 50 & 14 & \end{array}$$

Thus, $f(1) = 14 \neq 0 \Rightarrow z - 1$ is not a factor of f and 1 is not a zero (root) of f.

Trying -1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hspace{8cm}}^{Coeff\ of\ 6z^4 - 11z^3 - 53z^2 + 108z - 36}} \\ 6 & -11 & -53 & 108 & -36 & \underline{-1} \\ & -6 & 17 & 36 & -144 & \\ \hline 6 & -17 & -36 & 144 & -180 & \end{array}$$

Thus, $f(-1) = -180 \Rightarrow z + 1$ is not a factor of p and -1 is not a zero (root) of f.

Trying 2:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hspace{8cm}}^{Coeff\ of\ 6z^4 - 11z^3 - 53z^2 + 108z - 36}} \\ 6 & -11 & -53 & 108 & -36 & \underline{2} \\ & 12 & 2 & -102 & 12 & \\ \hline 6 & 1 & -51 & 6 & -24 & \end{array}$$

Thus, $f(2) = -24 \neq 0 \Rightarrow z - 2$ is not a factor of f and 2 is not a zero (root) of f.

Trying -2:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hspace{8cm}}^{Coeff\ of\ 6z^4 - 11z^3 - 53z^2 + 108z - 36}} \\ 6 & -11 & -53 & 108 & -36 & \underline{-2} \\ & -12 & 46 & 14 & -244 & \\ \hline 6 & -23 & -7 & 122 & -280 & \end{array}$$

Thus, $f(-2) = -280 \Rightarrow z + 2$ is not a factor of p and -2 is not a zero (root) of f.

Trying 3:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\textit{Coeff of } 6z^4 - 11z^3 - 53z^2 + 108z - 36}} & \\ 6 & -11 & -53 & 108 & -36 & \underline{3} \\ & 18 & 21 & -96 & 36 & \\ \hline 6 & 7 & -32 & 12 & 0 & \end{array}$$

Thus, $f(3) = 0 \Rightarrow z - 3$ is a factor of f and 3 is a zero (root) of f.

The third row in the synthetic division gives us the coefficients of the other factor starting with z^3. Thus, the other factor is $6z^3 + 7z^2 - 32z + 12$.

Thus, we have that $6z^4 - 11z^3 - 53z^2 + 108z - 36 =$
$(z - 3)(6z^3 + 7z^2 - 32z + 12)$.

Note that the remaining zeros of the polynomial f must also be zeros (roots) of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$. We will use this polynomial to find the remaining zeros (roots) of f, including another zero (root) of 3.

Trying 3 again:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\textit{Coeff of } 6z^3 + 7z^2 - 32z + 12}} & \\ 6 & 7 & -32 & 12 & \underline{3} \\ & 18 & 75 & 129 & \\ \hline 6 & 25 & 43 & 141 & \end{array}$$

The remainder is 141 and not 0. Thus, $z - 3$ is not a factor of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ and 3 is not a zero (root) of q. Thus, the multiplicity of the zero (root) of 3 is one.

NOTE: By the Bound Theorem above, 3 is an upper bound for the positive zeros (roots) of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ since all the numbers are positive in the third row of the synthetic division.

Thus, 4, $\frac{9}{2}$, 6, 9, 12, 18, and 36 can not be rational zeros (roots) of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ nor of the polynomial $f(z) = 6z^4 - 11z^3 - 53z^2 + 108z - 36 =$
$(z - 3)(6z^3 + 7z^2 - 32z + 12)$.

Trying -3:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hspace{6em}}^{Coeff\ of\ 6z^3 + 7z^2 - 32z + 12}} & \\ 6 & 7 & -32 & 12 & \underline{-3} \\ & -18 & 33 & -3 & \\ \hline 6 & -11 & 1 & 9 & \end{array}$$

The remainder is 9 and not 0. Thus, $z + 3$ is not a factor of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ and -3 is not a zero (root) of q.

Trying -4:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hspace{6em}}^{Coeff\ of\ 6z^3 + 7z^2 - 32z + 12}} & \\ 6 & 7 & -32 & 12 & \underline{-4} \\ & -24 & 68 & -144 & \\ \hline 6 & -17 & 36 & -132 & \end{array}$$

The remainder is -132 and not 0. Thus, $z + 4$ is not a factor of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ and -4 is not a zero (root) of q.

NOTE: By the Bound Theorem above, -4 is a lower bound for the negative zeros (roots) of the quotient polynomial q since we alternate from **positive** 6 to **negative** 17 to **positive** 36 to **negative** 132 in the third row of the synthetic division. Thus, $-\frac{9}{2}$, -6, -9, -12, -18, and -36 can not be rational zeros (roots) of q.

Thus, the only possible rational zeros (roots) which are left to be checked are $\pm\frac{1}{6}$, $\pm\frac{1}{3}$, $\pm\frac{1}{2}$, $\pm\frac{2}{3}$, $\pm\frac{4}{3}$, and $\pm\frac{3}{2}$.

Trying $\frac{3}{2}$:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hspace{6cm}}^{Coeff\ of\ 6z^3 + 7z^2 - 32z + 12}} & \\ 6 & 7 & -32 & 12 & \frac{3}{2} \\ & 9 & 24 & -12 & \\ \hline 6 & 16 & -8 & 0 & \end{array}$$

The remainder is 0. Thus, $z - \frac{3}{2}$ is a factor of the quotient polynomial $q(z) = 6z^3 + 7z^2 - 32z + 12$ and $\frac{3}{2}$ is a zero (root) of q.

Thus, we have that $6z^3 + 7z^2 - 32z + 12 = \left(z - \frac{3}{2}\right)(6z^2 + 16z - 8) = \left(z - \frac{3}{2}\right)2(3z^2 + 8z - 4) = (2z - 3)(3z^2 + 8z - 4)$.

Thus, we have that $6z^4 - 11z^3 - 53z^2 + 108z - 36 = (z - 3)(6z^3 + 7z^2 - 32z + 12) = (z - 3)(2z - 3)(3z^2 + 8z - 4)$.

Now, we can try to find a factorization for the expression $3z^2 + 8z - 4$. However, it does not factor.

Thus, we have that $6z^4 - 11z^3 - 53z^2 + 108z - 36 = (z - 3)(2z - 3)(3z^2 + 8z - 4)$.

Thus, $6z^4 - 11z^3 - 53z^2 + 108z - 36 = 0 \Rightarrow$

$(z - 3)(2z - 3)(3z^2 + 8z - 4) = 0 \Rightarrow z = 3, \; z = \frac{3}{2},$

$3z^2 + 8z - 4 = 0$

We will need to use the Quadratic Formula to solve $3z^2 + 8z - 4 = 0$.

Thus, $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-8 \pm \sqrt{64 - 4(3)(-4)}}{6} =$

$$\frac{-8 \pm \sqrt{16(4 + 3)}}{6} = \frac{-8 \pm 4\sqrt{7}}{6} = \frac{-4 \pm 2\sqrt{7}}{3}$$

Answer: Zeros (Roots): $\dfrac{-4 - 2\sqrt{7}}{3}$, $\dfrac{-4 + 2\sqrt{7}}{3}$, $\dfrac{3}{2}$, 3

Factorization: $6z^4 - 11z^3 - 53z^2 + 108z - 36 =$
$(z - 3)(2z - 3)(3z^2 + 8z - 4)$

6. $g(x) = 9x^4 + 18x^3 - 43x^2 - 32x + 48$

To find the zeros (roots) of g, we want to solve the equation $g(x) = 0 \Rightarrow$

$9x^4 + 18x^3 - 43x^2 - 32x + 48 = 0$.

We need to find two rational zeros (roots) for the polynomial f. This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of 48: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48$

Factors of 9: 1, 3, 9

The rational numbers obtained using the factors of 48 for the numerator and the 1 as the factor of 9 for the denominator:

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48$$

The rational numbers obtained using the factors of 48 for the numerator and the 3 as the factor of 9 for the denominator:

$$\pm \frac{1}{3}, \pm \frac{2}{3}, \pm 1, \pm \frac{4}{3}, \pm 2, \pm \frac{8}{3}, \pm 4, \pm \frac{16}{3}, \pm 8, \pm 16$$

Eliminating the ones that are already listed above, we have

$$\pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{8}{3},\ \pm\frac{16}{3}$$

The rational numbers obtained using the factors of 48 for the numerator and the 9 as the factor of 9 for the denominator:

$$\pm\frac{1}{9},\ \pm\frac{2}{9},\ \pm\frac{1}{3},\ \pm\frac{4}{9},\ \pm\frac{2}{3},\ \pm\frac{8}{9},\ \pm\frac{4}{3},\ \pm\frac{16}{9},\ \pm\frac{8}{3},\ \pm\frac{16}{3}$$

Eliminating the ones that are already listed above, we have

$$\pm\frac{1}{9},\ \pm\frac{2}{9},\ \pm\frac{4}{9},\ \pm\frac{8}{9},\ \pm\frac{16}{9}$$

Possible rational zeros (roots): $\pm\frac{1}{9},\ \pm\frac{2}{9},\ \pm\frac{1}{3},\ \pm\frac{4}{9},\ \pm\frac{2}{3},\ \pm\frac{8}{9},\ \pm 1,\ \pm\frac{4}{3},\ \pm\frac{16}{9},\ \pm 2,\ \pm\frac{8}{3},\ \pm 3,\ \pm 4,\ \pm\frac{16}{3},\ \pm 6,\ \pm 8,\ \pm 12,\ \pm 16,\ \pm 24,\ \pm 48$

Trying 1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\qquad\textit{Coeff of } 9x^4 + 18x^3 - 43x^2 - 32x + 48\qquad}} & \\ 9 & 18 & -43 & -32 & 48 & 1 \\ & 9 & 27 & -16 & -48 & \\ \hline 9 & 27 & -16 & -48 & 0 & \end{array}$$

Thus, $g(1) = 0 \Rightarrow x - 1$ is a factor of g and 1 is a zero (root) of g.

The third row in the synthetic division gives us the coefficients of the other factor starting with x^3. Thus, the other factor is $9x^3 + 27x^2 - 16x - 48$.

Thus, we have that $9x^4 + 18x^3 - 43x^2 - 32x + 48 =$
$(x - 1)(9x^3 + 27x^2 - 16x - 48)$.

Note that the remaining zeros of the polynomial g must also be zeros (roots) of the quotient polynomial $q(x) = 9x^3 + 27x^2 - 16x - 48$. We will use this polynomial to find the remaining zeros (roots) of g, including another zero (root) of 1.

NOTE: The expression $9x^3 + 27x^2 - 16x - 48$ can be factored by grouping:

$$9x^3 + 27x^2 - 16x - 48 = 9x^2(x + 3) - 16(x + 3) =$$

$$(x + 3)(9x^2 - 16) = (x + 3)(3x + 4)(3x - 4)$$

Thus, we have that $9x^4 + 18x^3 - 43x^2 - 32x + 48 =$
$(x - 1)(9x^3 + 27x^2 - 16x - 48) = (x - 1)(x + 3)(3x + 4)(3x - 4)$.

Thus, $9x^4 + 18x^3 - 43x^2 - 32x + 48 = 0 \Rightarrow$

$$(x - 1)(x + 3)(3x + 4)(3x - 4) = 0 \Rightarrow x = 1, \; x = -3, \; x = -\frac{4}{3},$$

$$x = \frac{4}{3}$$

Answer: Zeros (Roots): $-3, \; -\frac{4}{3}, \; 1, \; \frac{4}{3}$

Factorization: $9x^4 + 18x^3 - 43x^2 - 32x + 48 =$
$(x - 1)(x + 3)(3x + 4)(3x - 4)$

7. $h(x) = x^4 + 8x^3 + 11x^2 - 40x - 80$

To find the zeros (roots) of h, we want to solve the equation $h(x) = 0 \Rightarrow$
$x^4 + 8x^3 + 11x^2 - 40x - 80 = 0$.

We need to find two rational zeros (roots) for the polynomial h. This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of -80: ± 1, ± 2, ± 4, ± 5, ± 8, ± 10 ± 16, ± 20, ± 40, ± 80

Factors of 1: 1

Possible rational zeros (roots): ± 1, ± 2, ± 4, ± 5, ± 8, ± 10 ± 16, ± 20, ± 40, ± 80

Trying 1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad 8 \quad 11 \quad -40 \quad -80}}^{Coeff\ of\ x^4 + 8x^3 + 11x^2 - 40x - 80}} & \\ 1 & 8 & 11 & -40 & -80 & \ 1 \\ & 1 & 9 & 20 & -20 & \\ \hline 1 & 9 & 20 & -20 & -100 & \end{array}$$

Thus, $h(1) = -100 \neq 0 \Rightarrow x - 1$ is not a factor of h and 1 is not a zero (root) of h.

Trying -1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad 8 \quad 11 \quad -40 \quad -80}}^{Coeff\ of\ x^4 + 8x^3 + 11x^2 - 40x - 80}} & \\ 1 & 8 & 11 & -40 & -80 & \ -1 \\ & -1 & -7 & -4 & 44 & \\ \hline 1 & 7 & 4 & -44 & -36 & \end{array}$$

Thus, $h(-1) = -36 \neq 0 \Rightarrow x + 1$ is not a factor of h and -1 is not a zero (root) of h.

Trying 2:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad 8 \quad 11 \quad -40 \quad -80}}^{Coeff\ of\ x^4 + 8x^3 + 11x^2 - 40x - 80}} & \\ 1 & 8 & 11 & -40 & -80 & \ 2 \\ & 2 & 20 & 62 & 44 & \\ \hline 1 & 10 & 31 & 22 & -36 & \end{array}$$

Thus, $h(2) = -36 \neq 0 \Rightarrow x - 2$ is not a factor of h and 2 is not a zero (root) of h.

Trying -2:

$$\begin{array}{ccccc|l} \multicolumn{5}{c}{\overbrace{\textit{Coeff of } x^4 + 8x^3 + 11x^2 - 40x - 80}} & \\ 1 & 8 & 11 & -40 & -80 & -2 \\ & -2 & -12 & 2 & 76 & \\ \hline 1 & 6 & -1 & -38 & -4 & \end{array}$$

Thus, $h(-2) = -4 \neq 0 \Rightarrow x + 2$ is not a factor of h and -2 is not a zero (root) of h.

Trying 3:

$$\begin{array}{ccccc|l} \multicolumn{5}{c}{\overbrace{\textit{Coeff of } x^4 + 8x^3 + 11x^2 - 40x - 80}} & \\ 1 & 8 & 11 & -40 & -80 & 3 \\ & 3 & 33 & 132 & 276 & \\ \hline 1 & 11 & 44 & 92 & 196 & \end{array}$$

Thus, $h(3) = 196 \neq 0 \Rightarrow x - 3$ is not a factor of h and 3 is not a zero (root) of h.

NOTE: By the Bound Theorem above, 3 is an upper bound for the positive zeros (roots) of h since all the numbers are positive in the third row of the synthetic division. Thus, 4, 5, 8, 10 16, 20, 40, and 80 can not be rational zeros (roots) of h.

Trying -3:

$$\begin{array}{ccccc|l} \multicolumn{5}{c}{\overbrace{\textit{Coeff of } x^4 + 8x^3 + 11x^2 - 40x - 80}} & \\ 1 & 8 & 11 & -40 & -80 & -3 \\ & -3 & -15 & 12 & 84 & \\ \hline 1 & 5 & -4 & -28 & 4 & \end{array}$$

Thus, $h(-3) = 4 \neq 0 \Rightarrow x + 3$ is not a factor of h and -3 is not a zero (root) of h.

Trying -4:

$$\begin{array}{ccccc|l} \multicolumn{5}{c}{\overbrace{\textit{Coeff of } x^4 + 8x^3 + 11x^2 - 40x - 80}} & \\ 1 & 8 & 11 & -40 & -80 & -4 \\ & -4 & -16 & 20 & 80 & \\ \hline 1 & 4 & -5 & -20 & 0 & \end{array}$$

Thus, $h(-4) = 0 \Rightarrow x + 4$ is a factor of h and -4 is a zero (root) of h.

The third row in the synthetic division gives us the coefficients of the other factor starting with x^3. Thus, the other factor is $x^3 + 4x^2 - 5x - 20$.

Thus, we have that $x^4 + 8x^3 + 11x^2 - 40x - 80 =$ $(x + 4)(x^3 + 4x^2 - 5x - 20)$.

Note that the remaining zeros of the polynomial g must also be zeros (roots) of the quotient polynomial $q(x) = x^3 + 4x^2 - 5x - 20$. We will use this polynomial to find the remaining zeros (roots) of g, including another zero (root) of -4.

Trying -4 again:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{1 \quad 4 \quad -5 \quad -20}}^{Coeff\ of\ x^3 + 4x^2 - 5x - 20}} & \\ 1 & 4 & -5 & -20 & -4 \\ & -4 & 0 & 20 & \\ \hline 1 & 0 & -5 & 0 & \end{array}$$

The remainder is 0. Thus, $x + 4$ is a factor of the quotient polynomial $q(x) = x^3 + 4x^2 - 5x - 20$ and -4 is a zero (root) of multiplicity of the polynomial h.

Thus, we have that $x^3 + 4x^2 - 5x - 20 = (x + 4)(x^2 - 5)$.

Thus, we have that $x^4 + 8x^3 + 11x^2 - 40x - 80 =$ $(x + 4)(x^3 + 4x^2 - 5x - 20) = (x + 4)(x + 4)(x^2 - 5) = (x + 4)^2(x^2 - 5)$.

Thus, $x^4 + 8x^3 + 11x^2 - 40x - 80 = 0 \Rightarrow$

$(x + 4)^2(x^2 - 5) = 0 \Rightarrow x = -4,\ x^2 - 5 = 0$

$x^2 - 5 = 0 \Rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5}$

Answer: Zeros (Roots): -4 (multiplicity 2), $-\sqrt{5}$, $\sqrt{5}$

Factorization: $x^4 + 8x^3 + 11x^2 - 40x - 80 = (x+4)^2(x^2 - 5)$

8. $p(t) = t^4 - 12t^3 + 54t^2 - 108t + 81$

To find the zeros (roots) of p, we want to solve the equation $p(t) = 0 \Rightarrow$ $t^4 - 12t^3 + 54t^2 - 108t + 81 = 0$.

We need to find two rational zeros (roots) for the polynomial p. This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of 81: ± 1, ± 3, ± 9, ± 27, ± 81

Factors of 1: 1

Possible rational zeros (roots): ± 1, ± 3, ± 9, ± 27, ± 81

Trying 1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\textit{Coeff of } t^4 - 12t^3 + 54t^2 - 108t + 81}} & \\ 1 & -12 & 54 & -108 & 81 & \underline{1} \\ & 1 & -11 & 43 & -65 & \\ \hline 1 & -11 & 43 & -65 & 16 & \end{array}$$

Thus, $p(1) = 16 \neq 0 \Rightarrow t - 1$ is not a factor of p and 1 is not a zero (root) of p.

Trying -1:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\textit{Coeff of } t^4 - 12t^3 + 54t^2 - 108t + 81}} & \\ 1 & -12 & 54 & -108 & 81 & \underline{-1} \\ & -1 & 13 & -67 & 175 & \\ \hline 1 & -13 & 67 & -175 & 256 & \end{array}$$

Thus, $p(-1) = 256 \neq 0 \Rightarrow t + 1$ is not a factor of p and -1 is not a zero (root) of p.

NOTE: By the Bound Theorem above, -1 is a lower bound for the negative zeros (roots) of p since we alternate from **positive** 1 to **negative** 13 to **positive** 67 to **negative** 175 to **positive** 256 in the third row of the synthetic division. Thus, -3, -9, -27, and -81 can not be rational zeros (roots) of p.

Trying 3:

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hspace{6cm}}^{Coeff\ of\ t^4 - 12t^3 + 54t^2 - 108t + 81}} & \\ 1 & -12 & 54 & -108 & 81 & \underline{3} \\ & 3 & -27 & 81 & -81 & \\ \hline 1 & -9 & 27 & -27 & 0 & \end{array}$$

Thus, $p(3) = 0 \Rightarrow t - 3$ is a factor of p and 3 is a zero (root) of p.

The third row in the synthetic division gives us the coefficients of the other factor starting with t^3. Thus, the other factor is $t^3 - 9t^2 + 27t - 27$.

Thus, we have that $t^4 - 12t^3 + 54t^2 - 108t + 81 =$
$(t - 3)(t^3 - 9t^2 + 27t - 27)$.

Note that the remaining zeros of the polynomial p must also be zeros (roots) of the quotient polynomial $q(t) = t^3 - 9t^2 + 27t - 27$. We will use this polynomial to find the remaining zeros (roots) of p, including another zero (root) of 3.

Trying 3 again:

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hspace{4.5cm}}^{Coeff\ of\ t^3 - 9t^2 + 27t - 27}} & \\ 1 & -9 & 27 & -27 & \underline{3} \\ & 3 & -18 & 27 & \\ \hline 1 & -6 & 9 & 0 & \end{array}$$

The remainder is 0. Thus, $t - 3$ is a factor of the quotient polynomial $q(t) = t^3 - 9t^2 + 27t - 27$ and 3 is a zero (root) of multiplicity of the polynomial p.

Thus, we have that $t^3 - 9t^2 + 27t - 27 = (t - 3)(t^2 - 6t + 9)$.

Thus, we have that $t^4 - 12t^3 + 54t^2 - 108t + 81 =$

$(t - 3)(t^3 - 9t^2 + 27t - 27) = (t - 3)(t - 3)(t^2 - 6t + 9) =$
$(t - 3)^2(t^2 - 6t + 9)$.

Since $t^2 - 6t + 9 = (t - 3)^2$, then we have that
$t^4 - 12t^3 + 54t^2 - 108t + 81 = (t - 3)^2(t^2 - 6t + 9) =$
$(t - 3)^2(t - 3)^2 = (t - 3)^4$

Thus, $t^4 - 12t^3 + 54t^2 - 108t + 81 = 0 \Rightarrow (t - 3)^4 = 0 \Rightarrow t = 3$

Answer: Zeros (Roots): 3 (multiplicity 4)

Factorization: $t^4 - 12t^3 + 54t^2 - 108t + 81 = (t - 3)^4$