LESSON 7 RATIONAL ZEROS (ROOTS) OF POLYNOMIALS

Recall that a rational number is a quotient of integers. That is, a rational number is of the form $\frac{a}{b}$, where $a$ and $b$ are integers. A rational number $\frac{a}{b}$ is said to be in reduced form if the greatest common divisor (GCD) of $a$ and $b$ is one.

Examples $-\frac{3}{6}, \frac{4}{5}, \frac{17}{9}, \text{ and } \frac{8}{12}$ are rational numbers. The numbers $\frac{4}{5}$ and $\frac{17}{9}$ are in reduced form. The numbers $-\frac{3}{6}$ and $\frac{8}{12}$ are not in reduced form. Of course, we can write $-\frac{3}{6}$ and $\frac{17}{9}$ as $-\frac{1}{2}$ and $-\frac{17}{9}$ respectively. In reduced form, $-\frac{3}{6}$ is $-\frac{1}{2}$ and $\frac{8}{12}$ is $\frac{2}{3}$.

Examples $\frac{\sqrt{7}}{4}$ and $\frac{\pi}{6}$ are not rational numbers since $\sqrt{7}$ and $\pi$ are not integers.

Theorem Let $p$ be a polynomial with integer coefficients. If $\frac{c}{d}$ is a rational zero (root) in reduced form of

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where the $a_i$'s are integers for $i = 1, 2, 3, \ldots, n$ and $a_n \neq 0$ and $a_0 \neq 0$, then $c$ is a factor of $a_0$ and $d$ is a factor of $a_n$.

Theorem (Bounds for Real Zeros (Roots) of Polynomials) Let $p$ be a polynomial with real coefficients and positive leading coefficient.
1. If \( p(x) \) is synthetically divided by \( x - a \), where \( a > 0 \), and all the numbers in the third row of the division process are either positive or zero, then \( a \) is an upper bound for the real solutions of the equation \( p(x) = 0 \).

2. If \( p(x) \) is synthetically divided by \( x - a \), where \( a < 0 \), and all the numbers in the third row of the division process are alternately positive and negative (and a 0 can be considered to be either positive or negative as needed), then \( a \) is a lower bound for the real solutions of the equation \( p(x) = 0 \).

**Examples** Find the zeros (roots) of the following polynomials. Also, give a factorization for the polynomial.

1. \( f(x) = x^3 - 5x^2 - 2x + 24 \)

   To find the zeros (roots) of \( f \), we want to solve the equation \( f(x) = 0 \) \( \Rightarrow \)

   \[ x^3 - 5x^2 - 2x + 24 = 0 \]. The expression \( x^3 - 5x^2 - 2x + 24 \) can not be factored by grouping.

   We need to find one rational zero (root) for the polynomial \( f \). This will produce a linear factor for the polynomial and the other factor will be quadratic.

   Factors of \( 24 \): \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24 \)

   Factors of \( 1 \): \( 1 \)

   Possible rational zeros (roots): \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24 \)

   \[
   \begin{array}{c|ccc|c}
   \text{Coeff of } x^3 & -5 & -2 & 24 & 1 \\
   \hline
   1 & & & & 1 \\
   \hline
   1 & -4 & -6 & 18
   \end{array}
   \]

   Trying 1:

   \[
   \begin{array}{c|ccc|c}
   & 1 & -4 & -6 & 18 \\
   \hline
   1 & & & & 1 \\
   \hline
   \end{array}
   \]

   Thus, \( f(1) = 18 \neq 0 \) \( \Rightarrow x - 1 \) is not a factor of \( f \) and 1 is not a zero (root) of \( f \).
Trying $-1$:

\[
\begin{array}{c|ccc}
\text{Coeff of } x^3 - 5x^2 - 2x + 24 & 1 & -5 & -2 & 24 \\
\hline
1 & -1 & 6 & -4 \\
1 & -6 & 4 & 20 \\
\end{array}
\]

Thus, $f(-1) = 20 \neq 0 \Rightarrow x + 1$ is not a factor of $f$ and $-1$ is not a zero (root) of $f$.

Trying 2:

\[
\begin{array}{c|ccc}
\text{Coeff of } x^3 - 5x^2 - 2x + 24 & 1 & -5 & -2 & 24 \\
\hline
2 & 2 & -6 & -16 \\
1 & 3 & -8 & 8 \\
\end{array}
\]

Thus, $f(2) = 8 \neq 0 \Rightarrow x - 2$ is not a factor of $f$ and $2$ is not a zero (root) of $f$.

Trying $-2$:

\[
\begin{array}{c|ccc}
\text{Coeff of } x^3 - 5x^2 - 2x + 24 & 1 & -5 & -2 & 24 \\
\hline
-2 & -2 & 14 & -24 \\
1 & -7 & 12 & 0 \\
\end{array}
\]

Thus, $f(-2) = 0 \Rightarrow x + 2$ is a factor of $f$ and $-2$ is a zero (root) of $f$.

NOTE: By the Bound Theorem above, $-2$ is a lower bound for the negative zeros (roots) of $f$ since we alternate from positive 1 to negative 7 to positive 12 to negative 0 in the third row of the synthetic division.

The third row in the synthetic division gives us the coefficients of the other factor starting with $x^2$. Thus, the other factor is $x^2 - 7x + 12$.

Thus, we have that $x^3 - 5x^2 - 2x + 24 = (x + 2)(x^2 - 7x + 12)$.

Now, we can try to find a factorization for the expression $x^2 - 7x + 12$: $x^2 - 7x + 12 = (x - 3)(x - 4)$.
Thus, we have that \( x^3 - 5x^2 - 2x + 24 = (x + 2)(x^2 - 7x + 12) = (x + 2)(x - 3)(x - 4) \)

Thus, \( x^3 - 5x^2 - 2x + 24 = 0 \Rightarrow (x + 2)(x - 3)(x - 4) = 0 \Rightarrow \)
\[ x = -2, \ x = 3, \ x = 4 \]

**Answer:** Zeros (Roots): \( -2, \ 3, \ 4 \)

Factorization: \( x^3 - 5x^2 - 2x + 24 = (x + 2)(x - 3)(x - 4) \)

2. \( g(x) = 3x^3 - 23x^2 + 57x - 45 \)

To find the zeros (roots) of \( g \), we want to solve the equation \( g(x) = 0 \Rightarrow \)
\[ 3x^3 - 23x^2 + 57x - 45 = 0. \] The expression \( 3x^3 - 23x^2 + 57x - 45 \) can not be factored by grouping.

We need to find one rational zero (root) for the polynomial \( g \). This will produce a linear factor for the polynomial and the other factor will be quadratic.

Factors of \(-45\): \( \pm 1, \ pm 3, \ pm 5, \ pm 9, \ pm 15, \ pm 45 \)

Factors of 3: 1, 3

The rational numbers obtained using the factors of \( -45 \) for the numerator and the 1 as the factor of 3 for the denominator:

\[ \pm 1, \ pm 3, \ pm 5, \ pm 9, \ pm 15, \ pm 45 \]

The rational numbers obtained using the factors of \( -45 \) for the numerator and the 3 as the factor of 3 for the denominator:
\[\pm \frac{1}{3}, \pm 1, \pm \frac{5}{3}, \pm 3, \pm 5, \pm 15\]

NOTE: \[\pm \frac{3}{3} = \pm 1, \pm \frac{9}{3} = \pm 3, \pm \frac{15}{3} = \pm 5, \text{ and } \pm \frac{45}{3} = \pm 15\]

Possible rational zeros (roots): \(\pm \frac{1}{3}, \pm 1, \pm \frac{5}{3}, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45\)

\[
\begin{array}{c|ccc}
\text{Coeff of } 3x^3 - 23x^2 + 57x - 45 & 3 & -23 & 57 & -45 \\
\hline
1
\end{array}
\]

Trying 1:
\[
\begin{array}{c|ccc}
\text{Coeff of } 3x^3 - 23x^2 + 57x - 45 & 3 & -20 & 37 \\
\hline
3 & -20 & 37 & -8
\end{array}
\]

Thus, \(g(1) = -8 \neq 0 \Rightarrow x - 1\) is not a factor of \(g\) and 1 is not a zero (root) of \(g\).

\[
\begin{array}{c|ccc}
\text{Coeff of } 3x^3 - 23x^2 + 57x - 45 & 3 & -23 & 57 & -45 \\
\hline
-1
\end{array}
\]

Trying -1:
\[
\begin{array}{c|ccc}
\text{Coeff of } 3x^3 - 23x^2 + 57x - 45 & 3 & -26 & 83 \\
\hline
3 & -26 & 83 & -128
\end{array}
\]

Thus, \(g(-1) = -128 \neq 0 \Rightarrow x + 1\) is not a factor of \(g\) and -1 is not a zero (root) of \(g\).

NOTE: By the Bound Theorem above, \(-1\) is a lower bound for the negative zeros (roots) of \(g\) since we alternate from positive 3 to negative 26 to positive 83 to negative 128 in the third row of the synthetic division. Thus, \(-\frac{5}{3}, -3, -5, -9, -15, \text{ and } -45\) cannot be rational zeros (roots) of \(g\).

\[
\begin{array}{c|ccc}
\text{Coeff of } 3x^3 - 23x^2 + 57x - 45 & 3 & -23 & 57 & -45 \\
\hline
2
\end{array}
\]

Trying 2:
\[
\begin{array}{c|ccc}
\text{Coeff of } 3x^3 - 23x^2 + 57x - 45 & 3 & -17 & 23 \\
\hline
6 & -34 & 46
\end{array}
\]
Thus, \( g(2) = 1 \neq 0 \Rightarrow x - 2 \) is not a factor of \( g \) and 2 is not a zero (root) of \( g \).

\[
\begin{array}{c|ccc}
\text{Coeff of } 3x^3 - 23x^2 + 57x - 45 & 3 & -23 & 57 - 45 \\
9 & -42 & 45 \\
3 & -14 & 15 & 0 \\
\end{array}
\]

Thus, \( g(3) = 0 \Rightarrow x - 3 \) is a factor of \( g \) and 3 is a zero (root) of \( g \).

The third row in the synthetic division gives us the coefficients of the other factor starting with \( x^2 \). Thus, the other factor is \( 3x^2 - 14x + 15 \).

Thus, we have that \( 3x^3 - 23x^2 + 57x - 45 = (x - 3)(3x^2 - 14x + 15) \).

Now, we can try to find a factorization for the expression \( 3x^2 - 14x + 15 \):
\[
3x^2 - 14x + 15 = (x - 3)(3x - 5)
\]

Thus, we have that \( 3x^3 - 23x^2 + 57x - 45 = (x - 3)(3x^2 - 14x + 15) = (x - 3)(x - 3)(3x - 5) = (x - 3)^2(3x - 5) \)

Thus, \( 3x^3 - 23x^2 + 57x - 45 = 0 \Rightarrow (x - 3)^2(3x - 5) = 0 \Rightarrow \)
\[
x = 3, \quad x = \frac{5}{3}
\]

Answer: Zeros (Roots): \( \frac{5}{3}, \ 3 \) (multiplicity 2)

Factorization: \( 3x^3 - 23x^2 + 57x - 45 = (x - 3)^2(3x - 5) \)
3. \( h(t) = 4t^3 - 4t^2 - 9t + 30 \)

To find the zeros (roots) of \( h \), we want to solve the equation \( h(t) = 0 \implies 4t^3 - 4t^2 - 9t + 30 = 0 \). The expression \( 4t^3 - 4t^2 - 9t + 30 \)

can not be factored by grouping.

We need to find one rational zero (root) for the polynomial \( h \). This will produce a linear factor for the polynomial and the other factor will be quadratic.

Factors of 30: \( \pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30 \)

Factors of 4: 1, 2, 4

The rational numbers obtained using the factors of \(-30\) for the numerator and the 1 as the factor of 4 for the denominator:

\[ \pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30 \]

The rational numbers obtained using the factors of \(-30\) for the numerator and the 2 as the factor of 4 for the denominator:

\[ \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm 3, \pm 5, \pm \frac{15}{2}, \pm 15 \]

Eliminating the ones that are already listed above, we have

\[ \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2} \]

The rational numbers obtained using the factors of \(-30\) for the numerator and the 4 as the factor of 4 for the denominator:

\[ \pm \frac{1}{4}, \pm \frac{1}{2}, \pm \frac{3}{4}, \pm \frac{5}{4}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{4}, \pm \frac{15}{2} \]

Eliminating the ones that are already listed above, we have
\[\pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{5}{4}, \pm \frac{15}{4}\]

Possible rational zeros (roots): \[\pm \frac{1}{4}, \pm \frac{1}{2}, \pm \frac{3}{4}, \pm 1, \pm \frac{5}{4}, \pm \frac{3}{2}, \pm 2, \pm \frac{5}{2}, \pm 3, \pm \frac{15}{4}, \pm 5, \pm 6, \pm \frac{15}{2}, \pm 10, \pm 15, \pm 30\]

\[
\begin{array}{c|cccc}
\text{Coeff of } 4t^3 - 4t^2 - 9t + 30 & 4 & -4 & -9 & 30 \\
\hline
1
\end{array}
\]

Trying 1:
\[
\begin{array}{cccc}
4 & 0 & -9 & 21 \\
\end{array}
\]

Thus, \(h(1) = 21 \neq 0 \Rightarrow t - 1\) is not a factor of \(h\) and \(1\) is not a zero (root) of \(h\).

\[
\begin{array}{c|cccc}
\text{Coeff of } 4t^3 - 4t^2 - 9t + 30 & 4 & -4 & -9 & 30 \\
\hline
2
\end{array}
\]

Trying 2:
\[
\begin{array}{cccc}
8 & 8 & -2 &  \ \\
\end{array}
\]

\[
\begin{array}{cccc}
4 & 4 & -1 & 28 \\
\end{array}
\]

Thus, \(h(2) = 28 \neq 0 \Rightarrow t - 2\) is not a factor of \(h\) and \(2\) is not a zero (root) of \(h\).

\[
\begin{array}{c|cccc}
\text{Coeff of } 4t^3 - 4t^2 - 9t + 30 & 4 & -4 & -9 & 30 \\
\hline
3
\end{array}
\]

Trying 3:
\[
\begin{array}{cccc}
12 & 24 & 45 &  \ \\
\end{array}
\]

\[
\begin{array}{cccc}
4 & 8 & 15 & 75 \\
\end{array}
\]

Thus, \(h(3) = 75 \neq 0 \Rightarrow t - 3\) is not a factor of \(h\) and \(3\) is not a zero (root) of \(h\).
NOTE: By the Bound Theorem above, 3 is an upper bound for the positive zeros (roots) of $h$ since all the numbers are positive in the third row of the synthetic division. Thus, $\frac{15}{4}$, 5, 6, $\frac{15}{2}$, 10, 15, and 30 can not be rational zeros (roots) of $h$.

\[
\begin{array}{c|ccccc}
\text{Coeff of } 4t^3 - 4t^2 - 9t + 30 & \multicolumn{4}{c}{-1} \\
4 & -4 & -9 & 30 & \\
\hline
\end{array}
\]

Trying $-1$:
\[
\begin{array}{c|ccccc}
\text{Coeff of } 4t^3 - 4t^2 - 9t + 30 & -1 & \\
4 & -4 & 8 & 1 & \\
\hline
-8 & 24 & -30 & \\
4 & -12 & 15 & 0 & \\
\end{array}
\]

Thus, $h(-1) = 31 \neq 0 \Rightarrow t + 1$ is not a factor of $h$ and $-1$ is not a zero (root) of $h$.

\[
\begin{array}{c|ccccc}
\text{Coeff of } 4t^3 - 4t^2 - 9t + 30 & -2 & \\
4 & -4 & -9 & 30 & \\
\hline
-8 & 24 & -30 & \\
4 & -12 & 15 & 0 & \\
\end{array}
\]

Thus, $h(-2) = 0 \Rightarrow t + 2$ is factor of $h$ and $-2$ is a zero (root) of $h$.

The third row in the synthetic division gives us the coefficients of the other factor starting with $t^2$. Thus, the other factor is $4t^2 - 12t + 15$.

Thus, we have that $4t^3 - 4t^2 - 9t + 30 = (t + 2)(4t^2 - 12t + 15)$.

Now, we can try to find a factorization for the expression $4t^2 - 12t + 15$. However, it does not factor.

Thus, we have that $4t^3 - 4t^2 - 9t + 30 = (t + 2)(4t^2 - 12t + 15)$

Thus, $4t^3 - 4t^2 - 9t + 30 = 0 \Rightarrow (t + 2)(4t^2 - 12t + 15) = 0 \Rightarrow t = -2$, $4t^2 - 12t + 15 = 0$
We will need to use the Quadratic Formula to solve \( 4t^2 - 12t + 15 = 0 \).

Thus, 
\[
t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{12 \pm \sqrt{12 \cdot 12 - 4(4)15}}{8} =
\]
\[
12 \pm \sqrt{4 \cdot 4 \cdot [3 \cdot 3 - 1(1)15]} = 12 \pm \sqrt{9 - 15} = 12 \pm 4 \sqrt{-6} =
\]
\[
12 \pm 4i \sqrt{6} = 3 \pm i \sqrt{6}
\]

Answer: Zeros (Roots): \(-2, \frac{3 + i \sqrt{6}}{2}, \frac{3 - i \sqrt{6}}{2}\)

Factorization: \(4t^3 - 4t^2 - 9t + 30 = (t + 2)(4t^2 - 12t + 15)\)

4. \(p(x) = x^4 - 6x^3 - 3x^2 + 16x + 12\)

To find the zeros (roots) of \(p\), we want to solve the equation \(p(x) = 0 \Rightarrow \)
\(x^4 - 6x^3 - 3x^2 + 16x + 12 = 0\).

We need to find two rational zeros (roots) for the polynomial \(p\). This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of 12: \(\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\)

Factors of 1: 1

Possible rational zeros (roots): \(\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\)
Thus, \( p(1) = 20 \neq 0 \Rightarrow x - 1 \) is not a factor of \( p \) and 1 is not a zero (root) of \( p \).

Thus, \( p(-1) = 0 \Rightarrow x + 1 \) is a factor of \( p \) and \(-1\) is a zero (root) of \( p \).

The third row in the synthetic division gives us the coefficients of the other factor starting with \( x^3 \). Thus, the other factor is \( x^3 - 7x^2 + 4x + 12 \).

Thus, we have that \( x^4 - 6x^3 - 3x^2 + 16x + 12 = (x + 1)(x^3 - 7x^2 + 4x + 12) \).

Note that the remaining zeros of the polynomial \( p \) must also be zeros (roots) of the quotient polynomial \( q(x) = x^3 - 7x^2 + 4x + 12 \). We will use this polynomial to find the remaining zeros (roots) of \( p \), including another zero (root) of \(-1\).

The remainder is 0. Thus, \( x + 1 \) is a factor of the quotient polynomial \( q(x) = x^3 - 7x^2 + 4x + 12 \) and \(-1\) is a zero (root) of multiplicity of the polynomial \( p \).
Thus, we have that \( x^3 - 7x^2 + 4x + 12 = (x + 1)(x^2 - 8x + 12) \).

Thus, we have that \( x^4 - 6x^3 - 3x^2 + 16x + 12 = (x + 1)(x^2 - 7x^2 + 4x + 12) = (x + 1)(x + 1)(x^2 - 8x + 12) = (x + 1)^2(x^2 - 8x + 12) \).

Now, we can try to find a factorization for the expression \( x^2 - 8x + 12 \):
\[
x^2 - 8x + 12 = (x - 2)(x - 6)
\]

Thus, we have that \( x^4 - 6x^3 - 3x^2 + 16x + 12 = (x + 1)^2(x^2 - 8x + 12) = (x + 1)^2(x - 2)(x - 6) \)

Thus, \( x^4 - 6x^3 - 3x^2 + 16x + 12 = 0 \implies \)
\[
(x + 1)^2(x - 2)(x - 6) = 0 \implies x = -1, \ x = 2, \ x = 6
\]

**Answer**: Zeros (Roots): \(-1\) (multiplicity 2), 2, 6

Factorization: \( x^4 - 6x^3 - 3x^2 + 16x + 12 = (x + 1)^2(x - 2)(x - 6) \)

5. \( f(z) = 6z^4 - 11z^3 - 53z^2 + 108z - 36 \)

To find the zeros (roots) of \( f \), we want to solve the equation \( f(z) = 0 \implies \)
\[
6z^4 - 11z^3 - 53z^2 + 108z - 36 = 0.
\]

We need to find two rational zeros (roots) for the polynomial \( f \). This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of \(-36\): \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36 \)

Factors of 6: 1, 2, 3, 6
The rational numbers obtained using the factors of \(-36\) for the numerator and the 1 as the factor of 6 for the denominator:

\[\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36\]

The rational numbers obtained using the factors of \(-36\) for the numerator and the 2 as the factor of 6 for the denominator:

\[\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \pm 3, \pm \frac{9}{2}, \pm 6, \pm 9, \pm 18\]

Eliminating the ones that are already listed above, we have

\[\pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}\]

The rational numbers obtained using the factors of \(-36\) for the numerator and the 3 as the factor of 6 for the denominator:

\[\pm \frac{1}{3}, \pm \frac{2}{3}, \pm 1, \pm \frac{4}{3}, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\]

Eliminating the ones that are already listed above, we have

\[\pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}\]

The rational numbers obtained using the factors of \(-36\) for the numerator and the 6 as the factor of 6 for the denominator:

\[\pm \frac{1}{6}, \pm \frac{1}{3}, \pm \frac{1}{2}, \pm \frac{2}{3}, \pm 1, \pm \frac{3}{2}, \pm 2, \pm 3, \pm 6\]

Eliminating the ones that are already listed above, we have \[\pm \frac{1}{6}\]
Possible rational zeros (roots): \[ \pm \frac{1}{6}, \pm \frac{1}{3}, \pm \frac{1}{2}, \pm \frac{2}{3}, \pm 1, \pm \frac{4}{3}, \pm \frac{3}{2}, \pm 2, \pm 3, \pm 4, \pm \frac{9}{2}, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36 \]

\[
\begin{array}{rrrrr}
\text{Coeff of } 6z^4 - 11z^3 - 53z^2 + 108z - 36 & | & 1 \\
6 & -11 & -53 & 108 & -36 \\
\end{array}
\]

Trying 1:
\[
\begin{array}{rrrrr}
6 & -5 & -58 & 50 & 14 \\
6 & -5 & -58 & 50 & 14 \\
\end{array}
\]

Thus, \( f(1) = 14 \neq 0 \Rightarrow z - 1 \) is not a factor of \( f \) and 1 is not a zero (root) of \( f \).

\[
\begin{array}{rrrrr}
\text{Coeff of } 6z^4 - 11z^3 - 53z^2 + 108z - 36 & | & -1 \\
6 & -11 & -53 & 108 & -36 \\
\end{array}
\]

Trying \(-1\):
\[
\begin{array}{rrrrr}
-6 & 17 & 36 & -144 \\
6 & -17 & -36 & 144 & -180 \\
\end{array}
\]

Thus, \( f(-1) = -180 \Rightarrow z + 1 \) is not a factor of \( f \) and \(-1 \) is not a zero (root) of \( f \).

\[
\begin{array}{rrrrr}
\text{Coeff of } 6z^4 - 11z^3 - 53z^2 + 108z - 36 & | & 2 \\
6 & -11 & -53 & 108 & -36 \\
\end{array}
\]

Trying 2:
\[
\begin{array}{rrrrr}
12 & 2 & -102 & 12 \\
6 & 1 & -51 & 6 & -24 \\
\end{array}
\]

Thus, \( f(2) = -24 \neq 0 \Rightarrow z - 2 \) is not a factor of \( f \) and 2 is not a zero (root) of \( f \).

\[
\begin{array}{rrrrr}
\text{Coeff of } 6z^4 - 11z^3 - 53z^2 + 108z - 36 & | & -2 \\
6 & -11 & -53 & 108 & -36 \\
\end{array}
\]

Trying \(-2\):
\[
\begin{array}{rrrrr}
-12 & 46 & 14 & -244 \\
6 & -23 & -7 & 122 & -280 \\
\end{array}
\]
Thus, \( f(-2) = -280 \Rightarrow z + 2 \) is not a factor of \( p \) and \(-2\) is not a zero (root) of \( f \).

\[
\begin{array}{cccc|c}
\text{Coeff of } 6z^4 - 11z^3 - 53z^2 + 108z - 36 & 3 \\
6 & -11 & -53 & 108 & -36 \\
\hline
18 & 21 & -96 & 36 \\
6 & 7 & -32 & 12 & 0 \\
\end{array}
\]

Thus, \( f(3) = 0 \Rightarrow z - 3 \) is a factor of \( f \) and \( 3 \) is a zero (root) of \( f \).

The third row in the synthetic division gives us the coefficients of the other factor starting with \( z^3 \). Thus, the other factor is \( 6z^3 + 7z^2 - 32z + 12 \).

Thus, we have that \( 6z^4 - 11z^3 - 53z^2 + 108z - 36 = (z - 3)(6z^3 + 7z^2 - 32z + 12) \).

Note that the remaining zeros of the polynomial \( f \) must also be zeros (roots) of the quotient polynomial \( q(z) = 6z^3 + 7z^2 - 32z + 12 \). We will use this polynomial to find the remaining zeros (roots) of \( f \), including another zero (root) of \( 3 \).

\[
\begin{array}{cccc|c}
\text{Coeff of } 6z^3 + 7z^2 - 32z + 12 & 3 \\
6 & 7 & -32 & 12 \\
\hline
18 & 75 & 129 \\
6 & 25 & 43 & 141 \\
\end{array}
\]

The remainder is 141 and not 0. Thus, \( z - 3 \) is not a factor of the quotient polynomial \( q(z) = 6z^3 + 7z^2 - 32z + 12 \) and \( 3 \) is not a zero (root) of \( q \). Thus, the multiplicity of the zero (root) of \( 3 \) is one.

**NOTE:** By the Bound Theorem above, \( 3 \) is an upper bound for the positive zeros (roots) of the quotient polynomial \( q(z) = 6z^3 + 7z^2 - 32z + 12 \) since all the numbers are positive in the third row of the synthetic division.
Thus, \( \frac{9}{2}, 6, 9, 12, 18, \) and \( 36 \) can not be rational zeros (roots) of the quotient polynomial \( q(z) = 6z^3 + 7z^2 - 32z + 12 \) nor of the polynomial \( f(z) = 6z^4 - 11z^3 - 53z^2 + 108z - 36 = (z - 3)(6z^3 + 7z^2 - 32z + 12) \).

\[
\begin{array}{cccc|c}
\text{Coeff of } 6z^3 - 32z + 12 & \text{ } & \text{ } & \text{ } & - 3 \\
6 & 7 & -32 & 12 & \\
-18 & 33 & -3 & \\
6 & -11 & 1 & 9 & \\
\end{array}
\]

Trying \(- 3\):

\[
\begin{array}{cccc|c}
\text{Coeff of } 6z^3 - 32z + 12 & \text{ } & \text{ } & \text{ } & - 4 \\
6 & 7 & -32 & 12 & \\
-24 & 68 & -144 & \\
6 & -17 & 36 & -132 & \\
\end{array}
\]

The remainder is \(-132\) and not 0. Thus, \( z + 3 \) is not a factor of the quotient polynomial \( q(z) = 6z^3 + 7z^2 - 32z + 12 \) and \(- 3\) is not a zero (root) of \( q \).

NOTE: By the Bound Theorem above, \(- 4\) is a lower bound for the negative zeros (roots) of the quotient polynomial \( q \) since we alternate from positive 6 to negative 17 to positive 36 to negative 132 in the third row of the synthetic division. Thus, \(- \frac{9}{2}, - 6, - 9, - 12, - 18, \) and \(- 36\) can not be rational zeros (roots) of \( q \).

Thus, the only possible rational zeros (roots) which are left to be checked are \( \pm \frac{1}{6}, \pm \frac{1}{3}, \pm \frac{1}{2}, \pm \frac{2}{3}, \pm \frac{4}{3}, \) and \( \pm \frac{3}{2} \).
The remainder is 0. Thus, \( z - \frac{3}{2} \) is a factor of the quotient polynomial \( q(z) = 6z^3 + 7z^2 - 32z + 12 \) and \( \frac{3}{2} \) is a zero (root) of \( q \).

Thus, we have that
\[
6z^3 + 7z^2 - 32z + 12 = \left(z - \frac{3}{2}\right) \left(6z^2 + 16z - 8\right) = \left(z - \frac{3}{2}\right) \left(2z^2 + 8z - 4\right) = (2z - 3)(3z^2 + 8z - 4).
\]

Thus, we have that
\[
6z^4 - 11z^3 - 53z^2 + 108z - 36 = (z - 3)(6z^3 + 7z^2 - 32z + 12) = (z - 3)(2z - 3)(3z^2 + 8z - 4).
\]

Now, we can try to find a factorization for the expression \( 3z^2 + 8z - 4 \). However, it does not factor.

Thus, we have that
\[
6z^4 - 11z^3 - 53z^2 + 108z - 36 = (z - 3)(2z - 3)(3z^2 + 8z - 4).
\]

Thus, \( 6z^4 - 11z^3 - 53z^2 + 108z - 36 = 0 \) \( \Rightarrow \)
\[
(z - 3)(2z - 3)(3z^2 + 8z - 4) = 0 \Rightarrow z = 3, \ z = \frac{3}{2},
\]
\[
3z^2 + 8z - 4 = 0
\]

We will need to use the Quadratic Formula to solve \( 3z^2 + 8z - 4 = 0 \).
Thus, 
\[
z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{64 - 4(3)(-4)}}{6} = \frac{-8 \pm \sqrt{16(4 + 3)}}{6} \]
\[
= \frac{-8 \pm 4\sqrt{7}}{6} = \frac{-4 \pm 2\sqrt{7}}{3}
\]

**Answer:** Zeros (Roots): \( \frac{-4 - 2\sqrt{7}}{3}, \frac{-4 + 2\sqrt{7}}{3}, \frac{3}{2}, 3 \)

Factorization: 
\[
6z^4 - 11z^3 - 53z^2 + 108z - 36 = (z - 3)(2z - 3)(3z^2 + 8z - 4)
\]

6. \( g(x) = 9x^4 + 18x^3 - 43x^2 - 32x + 48 \)

To find the zeros (roots) of \( g \), we want to solve the equation \( g(x) = 0 \) \( \Rightarrow \)
\[
9x^4 + 18x^3 - 43x^2 - 32x + 48 = 0.
\]

We need to find two rational zeros (roots) for the polynomial \( f \). This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of 48: \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48 \)

Factors of 9: 1, 3, 9

The rational numbers obtained using the factors of 48 for the numerator and the 1 as the factor of 9 for the denominator:
\[
\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48
\]

The rational numbers obtained using the factors of 48 for the numerator and the 3 as the factor of 9 for the denominator:
\[
\pm \frac{1}{3}, \pm \frac{2}{3}, \pm 1, \pm \frac{4}{3}, \pm 2, \pm \frac{8}{3}, \pm 4, \pm \frac{16}{3}, \pm 8, \pm 16
\]
Eliminating the ones that are already listed above, we have

\[ \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3}, \pm \frac{16}{3} \]

The rational numbers obtained using the factors of 48 for the numerator and the 9 as the factor of 9 for the denominator:

\[ \pm \frac{1}{9}, \pm \frac{2}{9}, \pm \frac{1}{3}, \pm \frac{4}{9}, \pm \frac{2}{3}, \pm \frac{8}{9}, \pm \frac{4}{3}, \pm \frac{16}{9}, \pm \frac{8}{3}, \pm \frac{16}{3} \]

Eliminating the ones that are already listed above, we have

\[ \pm \frac{1}{9}, \pm \frac{2}{9}, \pm \frac{4}{9}, \pm \frac{8}{9}, \pm \frac{16}{9} \]

Possible rational zeros (roots):
\[ \pm \frac{1}{9}, \pm \frac{2}{9}, \pm \frac{1}{3}, \pm \frac{4}{9}, \pm \frac{2}{3}, \pm \frac{8}{9}, \pm 1, \]
\[ \pm \frac{4}{3}, \pm \frac{16}{9}, \pm 2, \pm \frac{8}{3}, \pm 3, \pm 4, \pm \frac{16}{3}, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48 \]

\[
\begin{array}{cccccc}
\text{Coeff of } 9x^4 + 18x^3 - 43x^2 - 32x + 48 & | & 1 \\
9 & 18 & -43 & -32 & 48 \\
\end{array}
\]

Trying 1:

\[
\begin{array}{cccc}
& 9 & 27 & -16 & -48 \\
9 & 27 & -16 & -48 & 0 \\
\end{array}
\]

Thus, \( g(1) = 0 \) \( \Rightarrow \) \( x - 1 \) is a factor of \( g \) and 1 is a zero (root) of \( g \).

The third row in the synthetic division gives us the coefficients of the other factor starting with \( x^3 \). Thus, the other factor is \( 9x^3 + 27x^2 - 16x - 48 \).

Thus, we have that
\[
9x^4 + 18x^3 - 43x^2 - 32x + 48 = (x - 1)(9x^3 + 27x^2 - 16x - 48).
\]
Note that the remaining zeros of the polynomial $g$ must also be zeros (roots) of the quotient polynomial $q(x) = 9x^3 + 27x^2 - 16x - 48$. We will use this polynomial to find the remaining zeros (roots) of $g$, including another zero (root) of 1.

NOTE: The expression $9x^3 + 27x^2 - 16x - 48$ can be factored by grouping:

$$9x^3 + 27x^2 - 16x - 48 = 9x^2(x + 3) - 16(x + 3) = (x + 3)(9x^2 - 16) = (x + 3)(3x + 4)(3x - 4)$$

Thus, we have that $9x^4 + 18x^3 - 43x^2 - 32x + 48 = (x - 1)(9x^3 + 27x^2 - 16x - 48) = (x - 1)(x + 3)(3x + 4)(3x - 4)$.

Thus, $9x^4 + 18x^3 - 43x^2 - 32x + 48 = 0 \Rightarrow (x - 1)(x + 3)(3x + 4)(3x - 4) = 0 \Rightarrow x = 1, \ x = -3, \ x = -\frac{4}{3}$,

$$x = \frac{4}{3}$$

**Answer:** Zeros (Roots): $-3, -\frac{4}{3}, 1, \frac{4}{3}$

**Factorization:** $9x^4 + 18x^3 - 43x^2 - 32x + 48 = (x - 1)(x + 3)(3x + 4)(3x - 4)$

7. $h(x) = x^4 + 8x^3 + 11x^2 - 40x - 80$

To find the zeros (roots) of $h$, we want to solve the equation $h(x) = 0 \Rightarrow x^4 + 8x^3 + 11x^2 - 40x - 80 = 0$. 
We need to find two rational zeros (roots) for the polynomial \( h \). This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of \(-80\): \( \pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 16, \pm 20, \pm 40, \pm 80\)

Factors of 1: 1

Possible rational zeros (roots): \( \pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 16, \pm 20, \pm 40, \pm 80\)

\[
\begin{array}{cccc}
\text{Coeff of } x^4 + 8x^3 + 11x^2 - 40x - 80 & | & 1 \\
1 & 8 & 11 & -40 & -80 \\
\end{array}
\]

Trying 1:

\[
\begin{array}{cccc}
1 & 9 & 20 & -20 \\
1 & 9 & 20 & -20 & -100 \\
\end{array}
\]

Thus, \( h(1) = -100 \neq 0 \Rightarrow x - 1 \) is not a factor of \( h \) and 1 is not a zero (root) of \( h \).

\[
\begin{array}{cccc}
\text{Coeff of } x^4 + 8x^3 + 11x^2 - 40x - 80 & | & -1 \\
1 & 8 & 11 & -40 & -80 \\
\end{array}
\]

Trying \(-1\):

\[
\begin{array}{cccc}
-1 & 7 & 4 & -44 \\
1 & 7 & 4 & -44 & -36 \\
\end{array}
\]

Thus, \( h(-1) = -36 \neq 0 \Rightarrow x + 1 \) is not a factor of \( h \) and \(-1\) is not a zero (root) of \( h \).

\[
\begin{array}{cccc}
\text{Coeff of } x^4 + 8x^3 + 11x^2 - 40x - 80 & | & 2 \\
1 & 8 & 11 & -40 & -80 \\
\end{array}
\]

Trying 2:

\[
\begin{array}{cccc}
2 & 20 & 62 & 44 \\
1 & 10 & 31 & 22 & -36 \\
\end{array}
\]

Thus, \( h(2) = -36 \neq 0 \Rightarrow x - 2 \) is not a factor of \( h \) and 2 is not a zero (root) of \( h \).
Trying $-2$:

\[
\begin{array}{cccccc}
1 & 8 & 11 & -40 & -80 \\
-2 & -12 & 2 & 76 \\
1 & 6 & -1 & -38 & -4
\end{array}
\]

Thus, $h(-2) = -4 \neq 0 \Rightarrow x + 2$ is not a factor of $h$ and $-2$ is not a zero (root) of $h$.

Trying $3$:

\[
\begin{array}{cccccc}
1 & 8 & 11 & -40 & -80 \\
3 & 33 & 132 & 276 \\
1 & 11 & 44 & 92 & 196
\end{array}
\]

Thus, $h(3) = 196 \neq 0 \Rightarrow x - 3$ is not a factor of $h$ and $3$ is not a zero (root) of $h$.

NOTE: By the Bound Theorem above, $3$ is an upper bound for the positive zeros (roots) of $h$ since all the numbers are positive in the third row of the synthetic division. Thus, $4$, $5$, $8$, $10$ $16$, $20$, $40$, and $80$ can not be rational zeros (roots) of $h$.

Trying $-3$:

\[
\begin{array}{cccccc}
1 & 8 & 11 & -40 & -80 \\
-3 & -15 & 12 & 84 \\
1 & 5 & -4 & -28 & 4
\end{array}
\]

Thus, $h(-3) = 4 \neq 0 \Rightarrow x + 3$ is not a factor of $h$ and $-3$ is not a zero (root) of $h$.
Thus,  \( h(-4) = 0 \Rightarrow x + 4 \) is a factor of  \( h \) and  \(-4\) is a zero (root) of  \( h \).

The third row in the synthetic division gives us the coefficients of the other factor starting with  \( x^3 \). Thus, the other factor is  \( x^3 + 4x^2 - 5x - 20 \).

Thus, we have that  \( x^4 + 8x^3 + 11x^2 - 40x - 80 = (x + 4)(x^3 + 4x^2 - 5x - 20) \).

Note that the remaining zeros of the polynomial  \( g \) must also be zeros (roots) of the quotient polynomial  \( q(x) = x^3 + 4x^2 - 5x - 20 \). We will use this polynomial to find the remaining zeros (roots) of  \( g \), including another zero (root) of  \(-4\).

\[
\begin{array}{cccc|c}
\text{Coeff of } x^3 + 4x^2 - 5x - 20 & -4 \\
1 & 4 & -5 & -20 & 1 \\
-4 & 0 & 20 & 0 \\
1 & 0 & -5 & 0 \\
\end{array}
\]

The remainder is 0. Thus,  \( x + 4 \) is a factor of the quotient polynomial  \( q(x) = x^3 + 4x^2 - 5x - 20 \) and  \(-4\) is a zero (root) of multiplicity of the polynomial  \( h \).

Thus, we have that  \( x^3 + 4x^2 - 5x - 20 = (x + 4)(x^2 - 5) \).

Thus, we have that  \( x^4 + 8x^3 + 11x^2 - 40x - 80 = (x + 4)(x^3 + 4x^2 - 5x - 20) \Rightarrow (x + 4)(x^2 - 5) = (x + 4)(x + 4)(x^2 - 5) = (x + 4)^2(x^2 - 5) \).

Thus,  \( x^4 + 8x^3 + 11x^2 - 40x - 80 = 0 \Rightarrow (x + 4)^2(x^2 - 5) = 0 \Rightarrow x = -4, \quad x^2 - 5 = 0 \)

\[ x^2 - 5 = 0 \Rightarrow x^2 = 5 \Rightarrow x = \pm \sqrt{5} \]
Answer: Zeros (Roots): \(-4\) (multiplicity 2), \(-\sqrt{5}, \sqrt{5}\)

Factorization: \(x^4 + 8x^3 + 11x^2 - 40x - 80 = (x + 4)^2(x^2 - 5)\)

8. \(p(t) = t^4 - 12t^3 + 54t^2 - 108t + 81\)

To find the zeros (roots) of \(p\), we want to solve the equation \(p(t) = 0 \Rightarrow t^4 - 12t^3 + 54t^2 - 108t + 81 = 0\).

We need to find two rational zeros (roots) for the polynomial \(p\). This will produce two linear factors for the polynomial and the other factor will be quadratic.

Factors of 81: \(±1, ±3, ±9, ±27, ±81\)

Factors of 1: 1

Possible rational zeros (roots): \(±1, ±3, ±9, ±27, ±81\)

\[
\begin{array}{cccc|c}
\text{Coeff of } t^4 - 12t^3 + 54t^2 - 108t + 81 & 1 & -12 & 54 & -108 & 81 \\
\end{array}
\]

Trying 1:

\[
\begin{array}{cccc|c}
1 & -12 & 54 & -108 & 81 \\
& 1 & -11 & 43 & -65 \\
1 & -11 & 43 & -65 & 16 \\
\end{array}
\]

Thus, \(p(1) = 16 \neq 0 \Rightarrow t - 1\) is not a factor of \(p\) and 1 is not a zero (root) of \(p\).

\[
\begin{array}{cccc|c}
\text{Coeff of } t^4 - 12t^3 + 54t^2 - 108t + 81 & 1 & -12 & 54 & -108 & 81 \\
\end{array}
\]

Trying \(-1:\)

\[
\begin{array}{cccc|c}
1 & -12 & 54 & -108 & 81 \\
-1 & 13 & -67 & 175 \\
1 & -13 & 67 & -175 & 256 \\
\end{array}
\]

Thus, \(p(-1) = 256 \neq 0 \Rightarrow t + 1\) is not a factor of \(p\) and \(-1\) is not a zero (root) of \(p\).
NOTE: By the Bound Theorem above, \(-1\) is a lower bound for the negative zeros (roots) of \(p\) since we alternate from positive 1 to negative 13 to positive 67 to negative 175 to positive 256 in the third row of the synthetic division. Thus, \(-3\), \(-9\), \(-27\), and \(-81\) can not be rational zeros (roots) of \(p\).

\[
\begin{array}{c|ccccc}
\text{Coeff of } t^4 - 12t^3 + 54t^2 - 108t + 81 & 1 & -12 & 54 & -108 & 81 \\
\hline
\text{Trying 3:} & 3 & -27 & 81 & -81 \\
& 1 & -9 & 27 & -27 & 0
\end{array}
\]

Thus, \(p(3) = 0 \Rightarrow t - 3\) is a factor of \(p\) and 3 is a zero (root) of \(p\).

The third row in the synthetic division gives us the coefficients of the other factor starting with \(t^3\). Thus, the other factor is \(t^3 - 9t^2 + 27t - 27\).

Thus, we have that \(t^4 - 12t^3 + 54t^2 - 108t + 81 = (t - 3)(t^3 - 9t^2 + 27t - 27)\).

Note that the remaining zeros of the polynomial \(p\) must also be zeros (roots) of the quotient polynomial \(q(t) = t^3 - 9t^2 + 27t - 27\). We will use this polynomial to find the remaining zeros (roots) of \(p\), including another zero (root) of 3.

\[
\begin{array}{c|cc}
\text{Coeff of } t^3 - 9t^2 + 27t - 27 & 1 & -9 & 27 & -27 \\
\hline
\text{Trying 3 again:} & 3 & -18 & 27 \\
& 1 & -6 & 9 & 0
\end{array}
\]

The remainder is 0. Thus, \(t - 3\) is a factor of the quotient polynomial \(q(t) = t^3 - 9t^2 + 27t - 27\) and 3 is a zero (root) of multiplicity of the polynomial \(p\).

Thus, we have that \(t^3 - 9t^2 + 27t - 27 = (t - 3)(t^2 - 6t + 9)\).

Thus, we have that \(t^4 - 12t^3 + 54t^2 - 108t + 81 = \)
\[(t - 3)(t^3 - 9t^2 + 27t - 27) = (t - 3)(t - 3)(t^2 - 6t + 9) = (t - 3)^2(t^2 - 6t + 9).\]

Since \(t^2 - 6t + 9 = (t - 3)^2\), then we have that
\[t^4 - 12t^3 + 54t^2 - 108t + 81 = (t - 3)^2(t^2 - 6t + 9) = (t - 3)^2(t - 3)^2 = (t - 3)^4\]

Thus, \(t^4 - 12t^3 + 54t^2 - 108t + 81 = 0 \Rightarrow (t - 3)^4 = 0 \Rightarrow t = 3\)

**Answer:** Zeros (Roots): 3 (multiplicity 4)

Factorization: \(t^4 - 12t^3 + 54t^2 - 108t + 81 = (t - 3)^4\)