LESSON 6 LONG DIVISION AND SYNTHETIC DIVISION

Example Find $(2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20) \div (x^2 - 4x + 3)$.

$$\begin{array}{r} 2x^3 + 5x^2 - 3 \\ x^2 - 4x + 3 \overline{\smash{)}\, 2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20} \\ \underline{2x^5 - 8x^4 + 6x^3} \qquad\qquad\qquad\qquad \\ 5x^4 - 20x^3 + 12x^2 \qquad\qquad\quad \\ \underline{5x^4 - 20x^3 + 15x^2} \qquad\qquad\quad \\ -3x^2 + 16x - 20 \\ \underline{-3x^2 + 12x - 9} \\ 4x - 11 \end{array}$$

NOTE: $2x^3(x^2 - 4x + 3) = 2x^5 - 8x^4 + 6x^3$

$5x^2(x^2 - 4x + 3) = 5x^4 - 20x^3 + 15x^2$

$-3(x^2 - 4x + 3) = -3x^2 + 12x - 9$

The expression $x^2 - 4x + 3$ is called the divisor in the division. The function $b(x) = x^2 - 4x + 3$ is called the divisor function.

The expression $2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20$ is called the dividend in the division. The function $a(x) = 2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20$ is called the dividend function.

The expression $2x^3 + 5x^2 - 3$ is called the quotient in the division. The function $q(x) = 2x^3 + 5x^2 - 3$ is called the quotient function.

The expression $4x - 11$ is called the remainder in the division. The function $r(x) = 4x - 11$ is called the remainder function.

We have that

$$\frac{2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20}{x^2 - 4x + 3} = 2x^3 + 5x^2 - 3 + \frac{4x - 11}{x^2 - 4x + 3}$$

Multiplying both sides of this equation by $x^2 - 4x + 3$, we have that
$2x^5 - 3x^4 - 14x^3 + 12x^2 + 16x - 20 =$

$$(x^2 - 4x + 3)(2x^3 + 5x^2 - 3) + (4x - 11)$$

Let a and b be polynomials. Then $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$. The degree of the remainder polynomial r is less than the degree of divisor polynomial b, written $\deg r < \deg b$.

Multiplying both sides of the equation $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ by $r(x)$, we have that $a(x) = b(x)q(x) + r(x)$.

<u>Example</u> Find $(12x^4 - 17x^3 - 21x^2 + 7) \div (3x - 2)$.

$$\begin{array}{r|l} & 4x^3 - 3x^2 - 9x - 6 \\ \hline 3x - 2 & 12x^4 - 17x^3 - 21x^2 + 0x + 7 \\ & \underline{12x^4 - 8x^3} \\ & -9x^3 - 21x^2 \\ & \underline{-9x^3 + 6x^2} \\ & -27x^2 + 0x \\ & \underline{-27x^2 + 18x} \\ & -18x + 7 \\ & \underline{-18x + 12} \\ & -5 \end{array}$$

NOTE: $4x^3(3x - 2) = 12x^4 - 8x^3$

$-3x^2(3x - 2) = -9x^3 + 6x^2$

$-9x(3x - 2) = -27x^2 + 18x$

$-6(3x - 2) = -18x + 12$

The quotient function is $q(x) = 4x^3 - 3x^2 - 9x - 6$ and the remainder function is $r(x) = -5$. We have that

$$12x^4 - 17x^3 - 21x^2 + 7 = (3x - 2)(4x^3 - 3x^2 - 9x - 6) + (-5).$$

Example Find $(12x^4 - 17x^3 - 21x^2 + 7) \div \left(x - \frac{2}{3}\right)$.

$$\begin{array}{r|l} & \quad 12x^3 - 9x^2 - 27x - 18 \\ \hline x - \frac{2}{3} & 12x^4 - 17x^3 - 21x^2 + 0x + 7 \\ & \underline{12x^4 - 8x^3} \\ & \qquad -9x^3 - 21x^2 \\ & \qquad \underline{-9x^3 + 6x^2} \\ & \qquad\qquad -27x^2 + 0x \\ & \qquad\qquad \underline{-27x^2 + 18x} \\ & \qquad\qquad\qquad -18x + 7 \\ & \qquad\qquad\qquad \underline{-18x + 12} \\ & \qquad\qquad\qquad\qquad -5 \end{array}$$

The quotient function is $q(x) = 12x^3 - 9x^2 - 27x - 18$ and the remainder function is $r(x) = -5$. We have that

$$12x^4 - 17x^3 - 21x^2 + 7 = \left(x - \frac{2}{3}\right)(12x^3 - 9x^2 - 27x - 18) + (-5).$$

NOTE: In the example above, we had that
$12x^4 - 17x^3 - 21x^2 + 7 = (3x - 2)(4x^3 - 3x^2 - 9x - 6) + (-5)$. Thus,

$$12x^4 - 17x^3 - 21x^2 + 7 = (3x - 2)(4x^3 - 3x^2 - 9x - 6) - 5 =$$

$$3\left(x - \frac{2}{3} \right)(4x^3 - 3x^2 - 9x - 6) - 5 =$$

$$\left(x - \frac{2}{3} \right)(12x^3 - 9x^2 - 27x - 18) - 5$$

Example Find $(x^5 + 4x^4 + 72x^2 - 8x - 20) \div (x + 6)$.

$$\begin{array}{r|l} & x^4 - 2x^3 + 12x^2 - 8 \\ \hline x + 6 & x^5 + 4x^4 + 0x^3 + 72x^2 - 8x - 20 \\ & \underline{x^5 + 6x^4} \\ & -2x^4 + 0x^3 \\ & \underline{-2x^4 - 12x^3} \\ & 12x^3 + 72x^2 \\ & \underline{12x^3 + 72x^2} \\ & -8x - 20 \\ & \underline{-8x - 48} \\ & 28 \end{array}$$

The quotient function is $q(x) = x^4 - 2x^3 + 12x^2 - 8$ and the remainder function is $r(x) = 28$. We have that

$$x^5 + 4x^4 + 72x^2 - 8x - 20 = (x + 6)(x^4 - 2x^3 + 12x^2 - 8) + 28.$$

Consider the following.

$$\begin{array}{cccccc|l} \multicolumn{6}{c}{\overbrace{\hphantom{1 \quad 4 \quad 0 \quad 72 \quad -8 \quad -20}}^{\textit{Coefficients of } x^5 + 4x^4 + 72x^2 - 8x - 20}} & \\ 1 & 4 & 0 & 72 & -8 & -20 & -6 \\ & -6 & 12 & -72 & 0 & 48 & \\ \hline 1 & -2 & 12 & 0 & -8 & 28 & \end{array}$$

What do the numbers in the third row represent?

$$\begin{array}{cccccc|l} 1 & 4 & 0 & 72 & -8 & -20 & -6 \\ & -6 & 12 & -72 & 0 & 48 & \\ \hline \multicolumn{5}{c}{\underbrace{1 \quad -2 \quad 12 \quad 0 \quad -8}_{\textit{Coefficients of the quotient function starting with } x^4}} & \underbrace{28}_{\text{Remainder}} & \end{array}$$

Thus, the quotient function is $q(x) = x^4 - 2x^3 + 12x^2 - 8$ and the remainder function is $r(x) = 28$. These are the same answers that we obtained above using long division.

This process is called synthetic division. Synthetic division can only be used to divide a polynomial by another polynomial of degree one with a leading coefficient of one. Thus, you can NOT use synthetic division to find $(12x^4 - 17x^3 - 21x^2 + 7) \div (3x - 2)$. However, we can do the following division.

<u>**Example**</u> Use synthetic division to find $(12x^4 - 17x^3 - 21x^2 + 7) \div \left(x - \dfrac{2}{3} \right)$.

$$\begin{array}{ccccc|l} \multicolumn{5}{c}{\overbrace{\hphantom{12 \quad -17 \quad -21 \quad 0 \quad 7}}^{\textit{Coefficients of } 12x^4 - 17x^3 - 21x^2 + 7}} & \\ 12 & -17 & -21 & 0 & 7 & \frac{2}{3} \\ & 8 & -6 & -18 & -12 & \\ \hline \multicolumn{4}{c}{\underbrace{12 \quad -9 \quad -27 \quad -18}_{\textit{Coeff of quotient function starting with } x^3}} & -5 & \end{array}$$

Thus, the quotient function is $q(x) = 12x^3 - 9x^2 - 27x - 18$ and the remainder function is $r(x) = -5$. These are the same answers that we obtained above using long division.

Example If $f(x) = x^5 + 4x^4 + 72x^2 - 8x - 20$, then find $f(-6)$.

$f(-6) = (-6)^5 + 4(-6)^4 + 72(-6)^2 - 8(-6) - 20 =$

$-7776 + 4(1296) + 72(36) - 8(-6) - 20 =$

$-7776 + 5184 + 2592 + 48 - 20 = 28$

This calculation would have been faster (and easier) using the fact that

$x^5 + 4x^4 + 72x^2 - 8x - 20 = (x + 6)(x^4 - 2x^3 + 12x^2 - 8) + 28$

that we obtained in the example above. Thus, $f(x) = (x + 6)q(x) + 28$, where $q(x) = x^4 - 2x^3 + 12x^2 - 8$.

Thus, $f(-6) = (-6 + 6)q(-6) + 28 = 0 \cdot q(-6) + 28 = 0 + 28 = 28$.

This result can be explained by the following theorem.

Theorem (The Remainder Theorem) Let p be a polynomial. If $p(x)$ is divided by $x - a$, then the remainder is $p(a)$.

Proof If $p(x)$ is divided by $x - a$, then $p(x) = (x - a)q(x) + r(x)$. Thus, $p(a) = (a - a)q(a) + r(a) = 0 \cdot q(a) + r(a) = 0 + r(a) = r(a)$.

Example If $g(x) = 12x^4 - 17x^3 - 21x^2 + 7$, then find $g\left(\frac{2}{3}\right)$.

Using synthetic division to find $(12x^4 - 17x^3 - 21x^2 + 7) \div \left(x - \frac{2}{3}\right)$, we have that

$$\begin{array}{rrrrr|l} \multicolumn{5}{c}{\overbrace{\hphantom{12 \quad -17 \quad -21 \quad 0 \quad 7}}^{\textit{Coefficients of } 12x^4 - 17x^3 - 21x^2 + 7}} & \\ 12 & -17 & -21 & 0 & 7 & \frac{2}{3} \\ & 8 & -6 & -18 & -12 & \\ \hline 12 & -9 & -27 & -18 & -5 & \end{array}$$

Thus, the remainder is -5. Thus, $g\left(\frac{2}{3}\right) = -5$.

Example If $h(x) = x^5 - 8x^4 + 12x^3 + 23x^2 - 16x - 37$, then find $h(3)$.

Using synthetic division to find $(x^5 - 8x^4 + 12x^3 + 23x^2 - 16x - 37) \div (x - 3)$, we have that

$$\begin{array}{rrrrrr|l} \multicolumn{6}{c}{\overbrace{\hphantom{1 \quad -8 \quad 12 \quad 23 \quad -16 \quad -37}}^{\textit{Coefficients of } x^5 - 8x^4 + 12x^3 + 23x^2 - 16x - 37}} & \\ 1 & -8 & 12 & 23 & -16 & -37 & \ 3 \\ & 3 & -15 & -9 & 42 & 78 & \\ \hline 1 & -5 & -3 & 14 & 26 & 41 & \end{array}$$

Thus, the remainder is 41. Thus, $h(3) = 41$.

Theorem (The Factor Theorem) Let p be a polynomial. The expression $x - a$ is a factor of $p(x)$ if and only if $p(a) = 0$.

Proof $(\Rightarrow)$ Suppose that $x - a$ is a factor of $p(x)$. Then the remainder upon division by $x - a$ must be zero. By the Remainder Theorem, $p(a) = 0$.

$(\Leftarrow)$ Suppose that $p(a) = 0$. By the Remainder Theorem, we have that $p(x) = (x - a)q(x) + p(a)$. Thus, $p(x) = (x - a)q(x)$. Thus, $x - a$ is a factor of $p(x)$.

Example Show that $x + 4$ is a factor of $p(x) = 5x^3 + 12x^2 - 20x + 48$.

We will use the Factor Theorem and show that $p(-4) = 0$. We will use the Remainder Theorem and synthetic division to find $p(-4)$.

$$\begin{array}{rrrr|l} \multicolumn{4}{c}{\overbrace{\hphantom{5 \quad 12 \quad -20 \quad 48}}^{\textit{Coeff of } 5x^3 + 12x^2 - 20x + 48}} & \\ 5 & 12 & -20 & 48 & \ -4 \\ & -20 & 32 & -48 & \\ \hline 5 & -8 & 12 & 0 & \end{array}$$

Thus, $p(-4) = 0$. Thus, by the Factor Theorem, $x + 4$ is a factor of $p(x) = 5x^3 + 12x^2 - 20x + 48$.

NOTE: The third row in the synthetic division gives us the coefficients of the other factor starting with x^2. Thus, the other factor is $5x^2 - 8x + 12$.

Thus, we have that $5x^3 + 12x^2 - 20x + 48 = (x + 4)(5x^2 - 8x + 12)$.

Example Show that $t - 6$ is not a factor of $q(t) = 2t^4 - 7t^2 + 15$.

We will use the Factor Theorem and show that $q(6) \neq 0$. We will use the Remainder Theorem and synthetic division to find $q(6)$.

$$\begin{array}{ccccc|c} \multicolumn{5}{c}{\overbrace{\hspace{6cm}}^{\textit{Coefficients of } 2t^4 - 7t^2 + 15}} & \\ 2 & 0 & -7 & 0 & 15 & 6 \\ & 12 & 72 & 390 & 2340 & \\ \hline 2 & 12 & 65 & 390 & 2355 & \end{array}$$

Thus, $q(6) = 2355 \neq 0$. Thus, by the Factor Theorem, $t - 6$ is not a factor of $q(t) = 2t^4 - 7t^2 + 15$.

Example Find the value(s) of c so that $x + 3$ is a factor of $f(x) = 2x^4 - x^3 - 9x^2 + 22x + c$.

By the Factor Theorem, $x + 3$ is a factor of the polynomial f if and only if $f(-3) = 0$.

$$f(-3) = 162 + 27 - 81 - 66 + c = c + 42$$

Thus, $f(-3) = 0 \Rightarrow c + 42 = 0 \Rightarrow c = -42$.

Using the Remainder Theorem and synthetic division to find $f(-3)$, we have

$$\begin{array}{ccccc|l} \multicolumn{5}{c}{\overbrace{}^{Coeff\ of\ 2x^4 - x^3 - 9x^2 + 22x + c}} & \\ 2 & -1 & -9 & 22 & c & -3 \\ & -6 & 21 & -36 & 42 & \\ \hline 2 & -7 & 12 & -14 & c + 42 & \end{array}$$

By the Remainder Theorem, $f(-3) = c + 42$. Thus, $f(-3) = 0 \Rightarrow$

$c + 42 = 0 \Rightarrow c = -42$.

Answer: -42

<u>Example</u> Find the value(s) of c so that $t - 2$ is a factor of $g(t) = t^5 + 5t^3 - 6t^2 + ct - 64$.

By the Factor Theorem, $t - 2$ is a factor of the polynomial g if and only if $g(2) = 0$.

$$g(2) = 32 + 40 - 24 + 2c - 64 = 2c - 16$$

Thus, $g(2) = 0 \Rightarrow 2c - 16 = 0 \Rightarrow c = 8$.

Using the Remainder Theorem and synthetic division to find $g(2)$, we have

$$\begin{array}{cccccc|l} \multicolumn{6}{c}{\overbrace{}^{Coeff\ of\ t^5 + 5t^3 - 6t^2 + ct - 64}} & \\ 1 & 0 & 5 & -6 & c & -64 & 2 \\ & 2 & 4 & 18 & 24 & 2c + 48 & \\ \hline 1 & 2 & 9 & 12 & c + 24 & 2c - 16 & \end{array}$$

By the Remainder Theorem, $g(2) = 2c - 16$. Thus, $g(2) = 0 \Rightarrow$

$2c - 16 = 0 \Rightarrow c = 8$.

Answer: 8