LESSON 5 POLYNOMIALS

Definition A polynomial p is a function of the form where

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where n is a positive integer. The a_i's can be real or complex numbers for $i = 0, 1, 2, 3, \ldots, n$ and $a_n \neq 0$.

NOTE: In this class, we will restrict our discussion to polynomials with real (number) coefficients.

Notation: The coefficient a_n is called the leading coefficient and n is called the degree of the polynomial.

COMMENT: Any polynomial is a "continuous" function. The concept of continuity will be defined in calculus. The continuity of a polynomial implies that its graph can be drawn without lifting your pencil. Thus, there are no points missing in the graph, and there are no jumps or breaks in the graph. In general, the graph of any continuous function can be drawn without lifting your pencil.

COMMENT: The graph of any polynomial is "smooth." The concept of smooth will also be defined in calculus. The smoothness of the graph means that there are no sharp turns in the graph.

Definition A real or complex number z is called a zero or a root of the polynomial p if and only if $p(z) = 0$.

Theorem If $z = a + bi$ is a zero (root) of a polynomial with real coefficients, then the conjugate $\bar{z} = a - bi$ is also a zero (root) of the polynomial.

Proof Recall the following properties for conjugates.

1. $\overline{z + w} = \overline{z} + \overline{w}$

2. By mathematical induction,

$$\overline{z_1 + z_2 + z_3 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \overline{z_3} + \cdots + \overline{z_n}$$

3. $\overline{zw} = \overline{z}\,\overline{w}$

4. By mathematical induction, $\overline{z_1 z_2 z_3 \cdots z_n} = \overline{z_1}\,\overline{z_2}\,\overline{z_3} \cdots \overline{z_n}$

5. $\overline{z^n} = \overline{z}^{\,n}$

6. If a is a real number, then $\overline{a} = a$.

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$.

Since z is a zero (root) of the polynomial, then $p(z) = 0$. We want to show that $p(\overline{z}) = 0$. $p(z) = 0 \Rightarrow$

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0 = 0 \Rightarrow$$

$$\overline{a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0} = \overline{0} \Rightarrow$$

$$\overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \overline{a_{n-2} z^{n-2}} + \cdots + \overline{a_2 z^2} + \overline{a_1 z} + \overline{a_0} = 0 \Rightarrow$$

$$\overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \overline{a_{n-2}}\,\overline{z^{n-2}} + \cdots + \overline{a_2}\,\overline{z^2} + \overline{a_1}\,\overline{z} + \overline{a_0} = 0 \Rightarrow$$

$$a_n \overline{z}^{\,n} + a_{n-1} \overline{z}^{\,n-1} + a_{n-2} \overline{z}^{\,n-2} + \cdots + a_2 \overline{z}^{\,2} + a_1 \overline{z} + a_0 = 0 \Rightarrow$$

$p(\overline{z}) = 0$. Thus, $\overline{z}$ is a zero (root) of the polynomial.

Note, since the coefficients of the polynomial are real numbers, then $\overline{a_i} = a_i$ for $i = 0, 1, 2, 3, \ldots, n$.

NOTE: This proof is easier to read (and type) if we make use of summation notation

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 =$$

$$\sum_{i=0}^{n} a_i x^i. \text{ Then } p(z) = 0 \Rightarrow \sum_{i=0}^{n} a_i z^i = 0 \Rightarrow \overline{\sum_{i=0}^{n} a_i z^i} = \overline{0} \Rightarrow$$

$$\sum_{i=0}^{n} \overline{a_i z^i} = 0 \Rightarrow \sum_{i=0}^{n} \overline{a_i}\, \overline{z^i} = 0 \Rightarrow \sum_{i=0}^{n} a_i \overline{z}^{\,i} = 0 \Rightarrow p(\overline{z}) = 0.$$

Theorem Let a be a real number. Let p be any polynomial. Then the following statements are equivalent.

1. $x = a$ is a zero (root) of the polynomial p

2. $p(a) = 0$

3. $x - a$ is a factor of $p(x)$

4. $(a, 0)$ is an x-intercept of the graph of $y = p(x)$

Definition Let p be any polynomial. If $(x - z)^m$, where m is positive integer, is a factor of $p(x)$, then $x = z$ is called a zero (root) of multiplicity m.

Theorem Let p be any polynomial. Let a be a real number. If $(x - a)^m$ is a factor of $p(x)$, then

1. if m is odd, then the graph of $y = p(x)$ crosses the x-axis at the x-intercept $(a, 0)$,

2. if m is even, then the graph of $y = p(x)$ touches the x-axis at the x-intercept $(a, 0)$ but does not cross the x-axis.

Theorem (The Fundamental Theorem of Algebra) An *n*th degree polynomial has at most n zeros (roots). If you count the multiplicity of the zero (root), then an *n*th degree polynomial has exactly n zeros (roots).

Theorem An *n*th degree polynomial has at most $n - 1$ local extremum points (turning points).

Information about local extremum points can be obtained using calculus.

Theorem If a and b are real zeros (roots) of a polynomial and $a \neq b$, then the polynomial has at least one local extremum point whose *x*-coordinate is between a and b.

Now, we will look at the behavior of a polynomial for numerically large numbers.

Recall that the mathematical symbol $\Rightarrow$ means implies. For example, $x^2 = 3 \Rightarrow x = \pm\sqrt{3}$. That is, $x^2 = 3$ implies x is positive square root of 3 or x is negative square root of 3.

The mathematical symbol $\rightarrow$ means approaches. For example, $x \rightarrow 5$ means that x approaches 5.

Real Number Line: $-\infty \longleftrightarrow \infty$

Then $x \rightarrow \infty$ means x approaches positive infinity. That is, x grows positively without bound. Also, $x \rightarrow -\infty$ means x approaches negative infinity. That is, x grows negatively without bound.

Now, consider

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdot \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

for numerically large values of x positive or negative. In order to do this, we will need to rewrite $p(x)$. Factoring out x^n, we obtain that

$$p(x) = x^n\left(\frac{a_n x^n}{x^n} + \frac{a_{n-1}x^{n-1}}{x^n} + \frac{a_{n-2}x^{n-2}}{x^n} + \cdot\cdot\cdot\cdot + \frac{a_2 x^2}{x^n} + \frac{a_1 x}{x^n} + \frac{a_0}{x^n}\right) =$$

$$x^n\left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdot\cdot\cdot\cdot + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)$$

As $x \to \infty$ or $x \to -\infty$, we have that $\dfrac{a_{n-1}}{x} \to 0$, $\dfrac{a_{n-2}}{x^2} \to 0, \ldots.,$

$\dfrac{a_2}{x^{n-2}} \to 0$, $\dfrac{a_1}{x^{n-1}} \to 0$, and $\dfrac{a_0}{x^n} \to 0$.

Thus, we have that $p(x) \approx a_n x^n$ for numerically large values of x positive or negative.

Thus, as $x \to \infty$, $p(x) \to \begin{cases} \infty, & a_n > 0 \\ -\infty, & a_n < 0 \end{cases}$

and as $x \to -\infty$, $p(x) \to \begin{cases} -\infty, & n \text{ is odd and } a_n > 0 \\ -\infty, & n \text{ is even and } a_n < 0 \\ \infty, & n \text{ is odd and } a_n < 0 \\ \infty, & n \text{ is even and } a_n > 0 \end{cases}$

<u>**Theorem**</u> (Leading Coefficient) If the leading coefficient of a polynomial p is positive, then $p(x) \to \infty$ as $x \to \infty$. If the leading coefficient of a polynomial p is negative, then $p(x) \to -\infty$ as $x \to \infty$. **That is, the sign of the leading coefficient tells you the sign of the infinity that $p(x)$ approaches as $x \to \infty$.**

<u>Theorem</u> Let p be a polynomial whose degree is **even**.

1. If $p(x) \to \infty$ as $x \to \infty$, then $p(x) \to \infty$ as $x \to -\infty$.
2. If $p(x) \to -\infty$ as $x \to \infty$, then $p(x) \to -\infty$ as $x \to -\infty$.

That is, for an even degree polynomial *p*, $p(x)$ approaches the same signed infinity as $x \to \infty$ and $x \to -\infty$.

<u>Theorem</u> Let p be a polynomial whose degree is **odd**.

1. If $p(x) \to \infty$ as $x \to \infty$, then $p(x) \to -\infty$ as $x \to -\infty$.
2. If $p(x) \to -\infty$ as $x \to \infty$, then $p(x) \to \infty$ as $x \to -\infty$.

That is, for an odd degree polynomial *p*, $p(x)$ approaches different signed infinities as $x \to \infty$ and $x \to -\infty$.

<u>Definition</u> A function f is said to be even if and only if $f(-x) = f(x)$ for all x in the domain of *f*. A function f is said to be odd if and only if $f(-x) = -f(x)$ for all x in the domain of *f*.

COMMENT: The graph of an even function is symmetry about the *y*-axis, and the graph of an odd function is symmetry through the origin.

<u>Examples</u> Find the zeros (roots) and their multiplicities. Discuss the implication of the multiplicity on the graph of the polynomial. Determine the sign of the infinity that the polynomial values approaches as x or t approaches positive infinity and negative infinity. Use this information to determine the number of local extremum points (turning points) that the graph of the polynomial has. Determine whether the polynomial is even, odd, or neither in order to make use of symmetry if possible.

1. $f(x) = 2x^3 - 72x$

Zeros (Roots) of f: $f(x) = 0 \Rightarrow 2x^3 - 72x = 0 \Rightarrow$

$2x(x^2 - 36) = 0 \Rightarrow 2x(x+6)(x-6) = 0 \Rightarrow x = 0,\ x = -6,$

$x = 6$

Since the factor x produces the zero (root) of 0, its multiplicity is one. Since the factor $x + 6$ produces the zero (root) of -6, its multiplicity is one. Since the factor $x - 6$ produces the zero (root) of 6, its multiplicity is one.

Zero (Root)	Multiplicity	Implication for the Graph of the Polynomial
-6	1	Crosses the *x*-axis at $(-6, 0)$
0	1	Crosses the *x*-axis at $(0, 0)$
6	1	Crosses the *x*-axis at $(6, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

$x \to \infty$:
Use the Leading Coefficient Theorem. Since the leading coefficient of $f(x) = 2x^3 - 72x$ is $2 > 0$, then $f(x) \to \infty$ as $x \to \infty$.

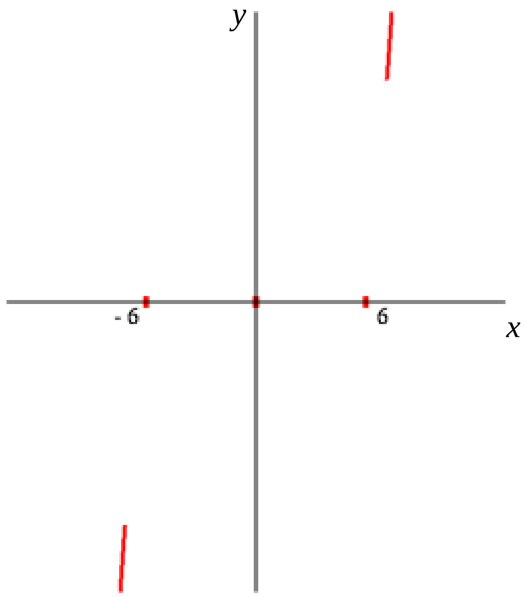
$x \to -\infty$:
The degree of $f(x) = 2x^3 - 72x$ is 3, which is odd. Since $f(x) \to \infty$ as $x \to \infty$, then $f(x) \to -\infty$ as $x \to -\infty$.

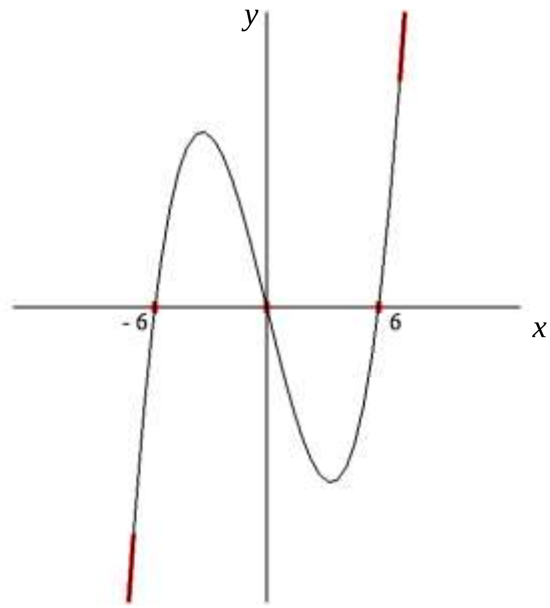
The polynomial is odd: $f(x) = 2x^3 - 72x \Rightarrow$

$$f(-x) = 2(-x)^3 - 72(-x) = -2x^3 + 72x = -(2x^3 - 72x) = -f(x)$$

Thus, the graph of f is symmetry through the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the point $(-6, 0)$. Then the graph must cross the *x*-axis at the origin. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -6 and 0. Then the graph must cross the *x*-axis at the point $(6, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between 0 and 6. We would need calculus in order to obtain information about the *x*-coordinate of these two local extremum points.

Thus, the graph of the polynomial has two local extremum points (turning points).

2. $g(x) = 6x^3 - 7x^2$

Zeros (Roots) of g: $g(x) = 0 \Rightarrow 6x^3 - 7x^2 = 0 \Rightarrow$

$$x^2(6x - 7) = 0 \Rightarrow x = 0, \ x = \frac{7}{6}$$

Since the factor x^2 produces the zero (root) of 0, its multiplicity is two. Since the factor $6x - 7$ produces the zero (root) of $\frac{7}{6}$, its multiplicity is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	2	Touches the *x*-axis at $(0, 0)$
$\frac{7}{6}$	1	Crosses the *x*-axis at $\left(\frac{7}{6}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

$x \to \infty$:

Use the Leading Coefficient Theorem. Since the leading coefficient of $g(x) = 6x^3 - 7x^2$ is $6 > 0$, then $g(x) \to \infty$ as $x \to \infty$.

$x \to -\infty$:

The degree of $g(x) = 6x^3 - 7x^2$ is 3, which is odd. Since $g(x) \to \infty$ as $x \to \infty$, then $g(x) \to -\infty$ as $x \to -\infty$.

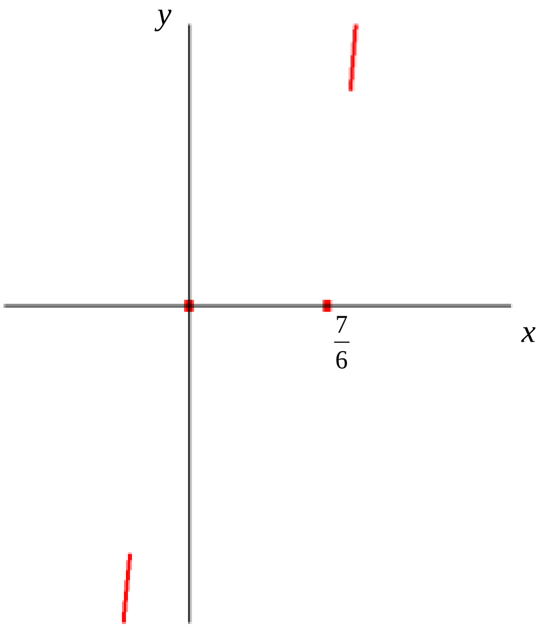
The polynomial is neither even nor odd: $g(x) = 6x^3 - 7x^2 \Rightarrow$

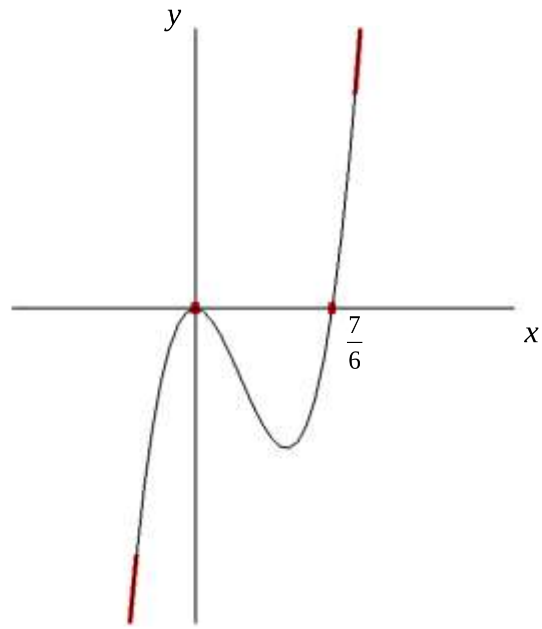
$$g(-x) = 6(-x)^3 - 7(-x)^2 = -6x^3 - 7x^2$$

Thus, $g(-x) \neq g(x)$ and $g(-x) \neq -g(x)$.

Thus, the graph of the polynomial is not symmetry with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

Since the graph touches at the origin, then there is a local extremum point (turning point) at the origin. Since the graph of the continuous polynomial must cross the x-axis at the point $\left(\frac{7}{6}, 0\right)$, then there is another local extremum point (turning point) whose x-coordinate is between 0 and $\frac{7}{6}$. We would need calculus in order to obtain information about this x-coordinate.

Thus, the graph of the polynomial has two local extremum points (turning points).

3. $h(x) = 48 + 32x - 3x^2 - 2x^3$

To solve for the zeros (roots) of this polynomial, you we need to recall a technique of factoring called "factor by grouping."

Zeros (Roots) of h: $h(x) = 0 \Rightarrow 48 + 32x - 3x^2 - 2x^3 = 0 \Rightarrow$

$16(3 + 2x) - x^2(3 + 2x) = 0 \Rightarrow (3 + 2x)(16 - x^2) = 0 \Rightarrow$

$$(3 + 2x)(4 + x)(4 - x) = 0 \Rightarrow x = -\frac{3}{2},\ x = -4,\ x = 4$$

Since the factor $3 + 2x$ produces the zero (root) of $-\frac{3}{2}$, its multiplicity is one. Since the factor $4 + x$ produces the zero (root) of -4, its multiplicity is one. Since the factor $4 - x$ produces the zero (root) of 4, its multiplicity is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
-4	1	Crosses the *x*-axis at $(-4, 0)$
$-\frac{3}{2}$	1	Crosses the *x*-axis at $\left(-\frac{3}{2}, 0\right)$
4	1	Crosses the *x*-axis at $(4, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

$x \to \infty$:
Use the Leading Coefficient Theorem. Since the leading coefficient of $h(x) = 48 + 32x - 3x^2 - 2x^3$ is $-2 < 0$, then $h(x) \to -\infty$ as $x \to \infty$.

$x \to -\infty$:
The degree of $h(x) = 48 + 32x - 3x^2 - 2x^3$ is 3, which is odd. Since $h(x) \to -\infty$ as $x \to \infty$, then $h(x) \to \infty$ as $x \to -\infty$.

The polynomial is neither even nor odd:
$h(x) = 48 + 32x - 3x^2 - 2x^3 \Rightarrow$

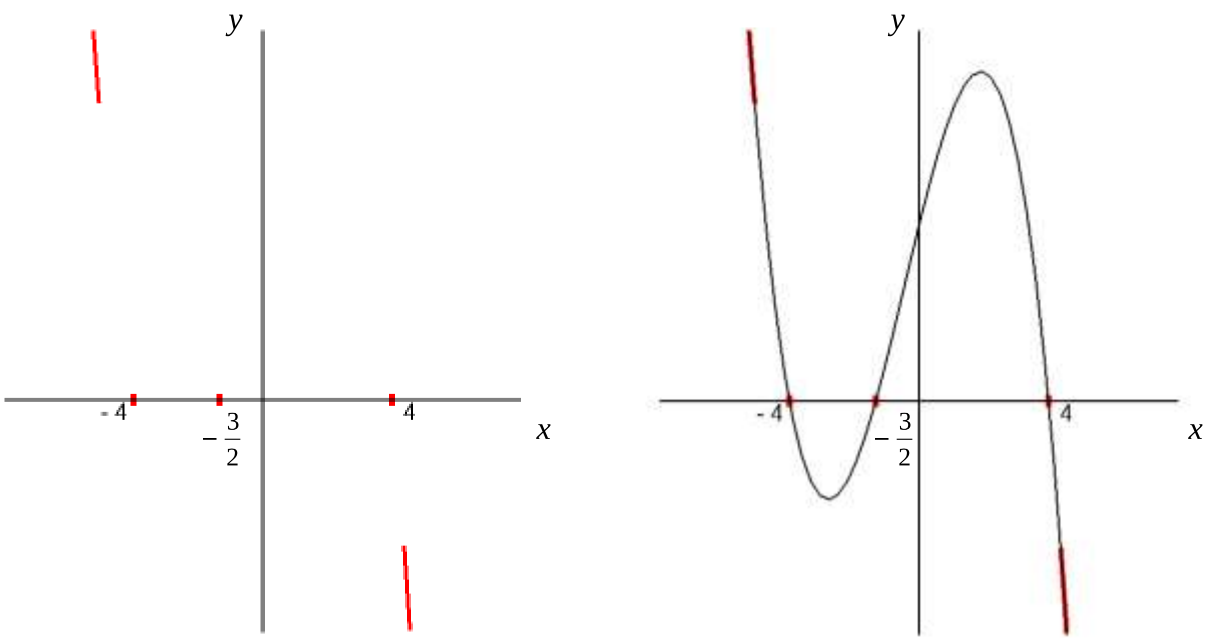
$h(-x) = 48 + 32(-x) - 3(-x)^2 - 2(-x)^3 =$

$48 - 32x - 3x^2 + 2x^3$

Thus, $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$.

Thus, the graph of the polynomial is not symmetry with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.



The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the point $(-4, 0)$. Then the graph must cross the *x*-axis at the point $\left(-\frac{3}{2}, 0\right)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -4 and $-\frac{3}{2}$. Then the graph must cross the *x*-axis at the point $(4, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between $-\frac{3}{2}$ and 4.

We would need calculus in order to obtain information about the *x*-coordinate of these two local extremum points.

Thus, the graph of the polynomial has two local extremum points (turning points).

4. $p(x) = 5x^3 + 20x$

Zeros (Roots) of p: $p(x) = 0 \Rightarrow 5x^3 + 20x = 0 \Rightarrow$

$5x(x^2 + 4) = 0 \Rightarrow x = 0, \; x = \pm 2i$

NOTE: $x^2 + 4 = 0 \Rightarrow x^2 = -4 \Rightarrow x = \pm \sqrt{-4} = \pm 2i$

Since the factor x produces the zero (root) of 0, its multiplicity is one. Since the factor $x^2 + 4$ produces the zeros (roots) of $\pm 2i$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	1	Crosses the *x*-axis at $(0, 0)$
$-2i$	1	No implication
$2i$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

$x \to \infty$:
Use the Leading Coefficient Theorem. Since the leading coefficient of $p(x) = 5x^3 + 20x$ is $5 > 0$, then $p(x) \to \infty$ as $x \to \infty$.

$x \to -\infty$:

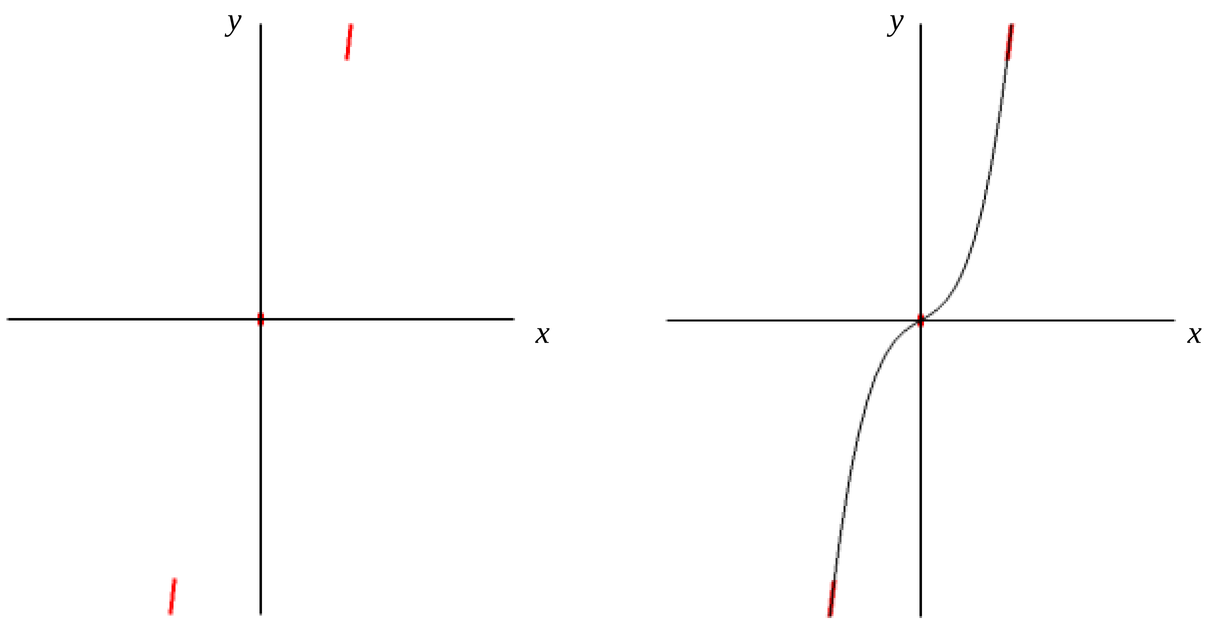
The degree of $p(x) = 5x^3 + 20x$ is 3, which is odd. Since $p(x) \to \infty$ as $x \to \infty$, then $p(x) \to -\infty$ as $x \to -\infty$.

The polynomial is odd: $p(x) = 5x^3 + 20x \Rightarrow$

$$p(-x) = 5(-x)^3 + 20(-x) = -5x^3 - 20x = -(5x^3 - 20x) = -p(x)$$

Thus, the graph of p is symmetry through the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.



The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the origin. If the graph had a local extremum point (turning point) for some $x < 0$, then it would have to have a second one in order to pass through the origin. Because the graph is symmetry through the origin, the graph would have to have two local extremum points (turning points) for $x > 0$. Thus, the polynomial would have four local extremum points (turning points).

Thus, the graph of the polynomial has no local extremum points (turning points).

5. $q(x) = 54 - 45x + 12x^2 - x^3$

This polynomial can not be factored using the Factor by Grouping technique. In Lesson 7, we will learn a technique for factoring this polynomial. At that point we will discover that

$$54 - 45x + 12x^2 - x^3 = (6 - x)(x - 3)^2$$

Zeros (Roots) of q: $q(x) = 0 \Rightarrow 54 - 45x + 12x^2 - x^3 = 0 \Rightarrow$

$(6 - x)(x - 3)^2 = 0 \Rightarrow x = 6, \; x = 3$

Since the factor $6 - x$ produces the zero (root) of 6, its multiplicity is one. Since the factor $(x - 3)^2$ produces the zero (root) of 3, its multiplicity is two.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
3	2	Touches the *x*-axis at $(3, 0)$
6	1	Crosses the *x*-axis at $(6, 0)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

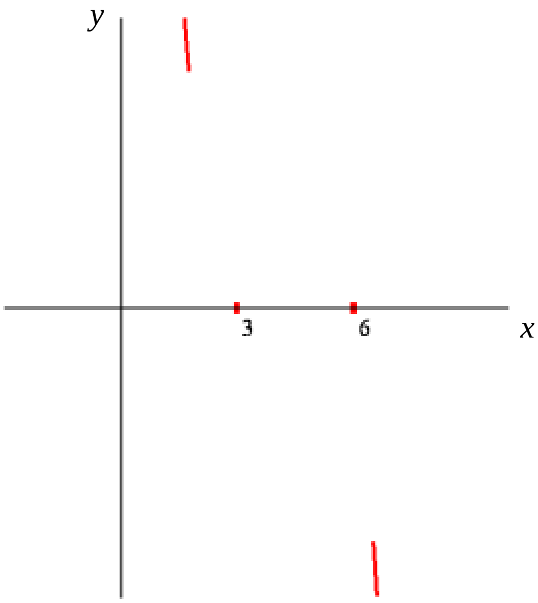
$x \to \infty$:
Use the Leading Coefficient Theorem. Since the leading coefficient of $q(x) = 54 - 45x + 12x^2 - x^3$ is $-1 < 0$, then $q(x) \to -\infty$ as $x \to \infty$.

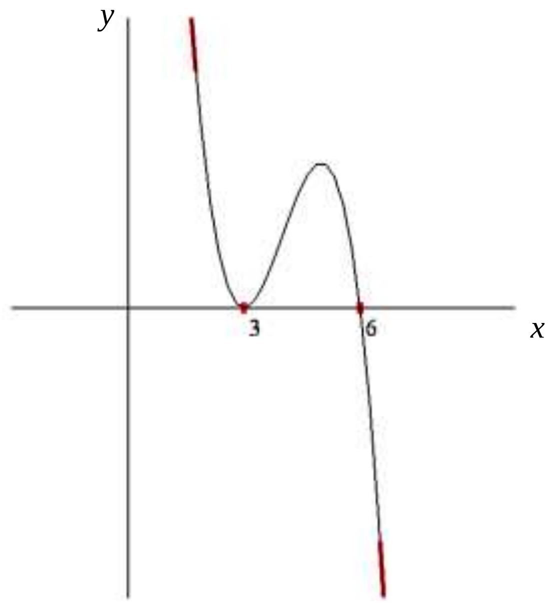
$x \to -\infty$:

The degree of $q(x) = 54 - 45x + 12x^2 - x^3$ is 3, which is odd. Since $q(x) \to -\infty$ as $x \to \infty$, then $q(x) \to \infty$ as $x \to -\infty$.

The polynomial is neither even nor odd. Thus, the graph of the polynomial is not symmetry with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

Since the graph touches at the point $(3, 0)$, then there is a local extremum point (turning point) at this point. Since the graph of the continuous polynomial must cross the *x*-axis at the point $(6, 0)$, then there is another local extremum point (turning point) whose *x*-coordinate is between 3 and 6. We would need calculus in order to obtain information about this *x*-coordinate.

Thus, the graph of the polynomial has two local extremum points (turning points).

6. $q(t) = t^3 + 16t^2 + 64t$

For this problem, it will be helpful to recall the following special factoring formula:

$$a^2 + 2ab + b^2 = (a + b)^2$$

Of course, we also have that $a^2 - 2ab + b^2 = (a - b)^2$

Zeros (Roots) of q: $q(t) = 0 \Rightarrow t^3 + 16t^2 + 64t = 0 \Rightarrow$

$t(t^2 + 16t + 64) = 0 \Rightarrow t(t + 8)^2 = 0 \Rightarrow t = 0, \; t = -8$

Since the factor t produces the zero (root) of 0, its multiplicity is one. Since the factor $(t + 8)^2$ produces the zero (root) of -8, its multiplicity is two.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	1	Crosses the *t*-axis at $(0, 0)$
-8	2	Touches the *t*-axis at $(-8, 0)$

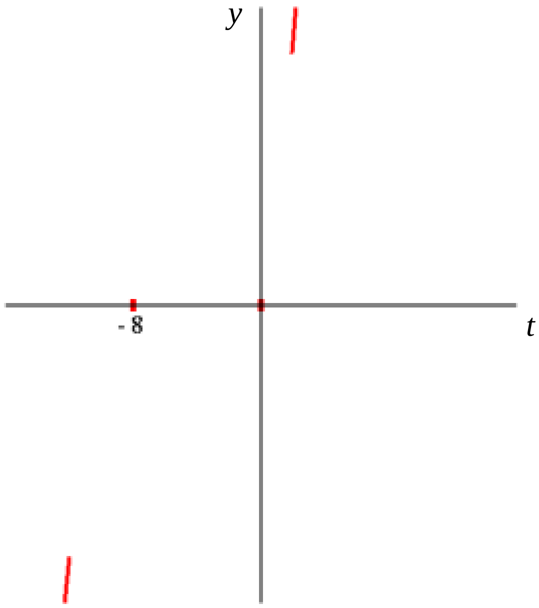
Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *t*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *t*-axis.

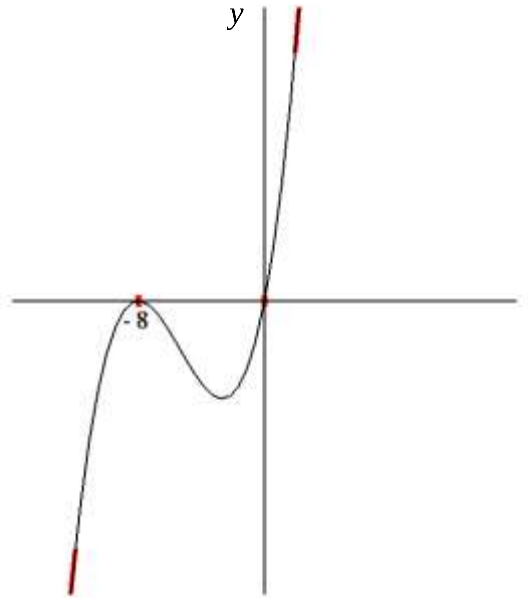
$t \to \infty$:
Use the Leading Coefficient Theorem. Since the leading coefficient of $q(t) = t^3 + 16t^2 + 64t$ is $1 > 0$, then $q(t) \to \infty$ as $t \to \infty$.

$t \to -\infty$:
The degree of $q(t) = t^3 + 16t^2 + 64t$ is 3, which is odd. Since $q(t) \to \infty$ as $t \to \infty$, then $q(t) \to -\infty$ as $t \to -\infty$.

The polynomial is neither even nor odd. Thus, the graph of the polynomial is not symmetry with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $t \to \infty$ and as $t \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

Since the graph touches at the point $(-8, 0)$, then there is a local extremum point (turning point) at this point. Since the graph of the continuous polynomial must cross the *t*-axis at the origin, then there is another local extremum point (turning point) whose *t*-coordinate is between -8 and 0. We would need calculus in order to obtain information about this *t*-coordinate.

Thus, the graph of the polynomial has two local extremum points (turning points).

7. $f(x) = x^3 - 2x^2 + 9x - 18$

NOTE: The expression $x^3 - 2x^2 + 9x - 18$ can be factored by grouping.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow x^3 - 2x^2 + 9x - 18 = 0 \Rightarrow$

$$x^2(x-2)+9(x-2)=0 \Rightarrow (x-2)(x^2+9)=0 \Rightarrow$$

$$x=2,\ x=\pm 3i$$

Since the factor $x-2$ produces the zero (root) of 2, its multiplicity is one. Since the factor x^2+9 produces the zeros (roots) of $\pm 3i$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
2	1	Crosses the *x*-axis at $(2, 0)$
$-3i$	1	No implication
$3i$	1	No implication

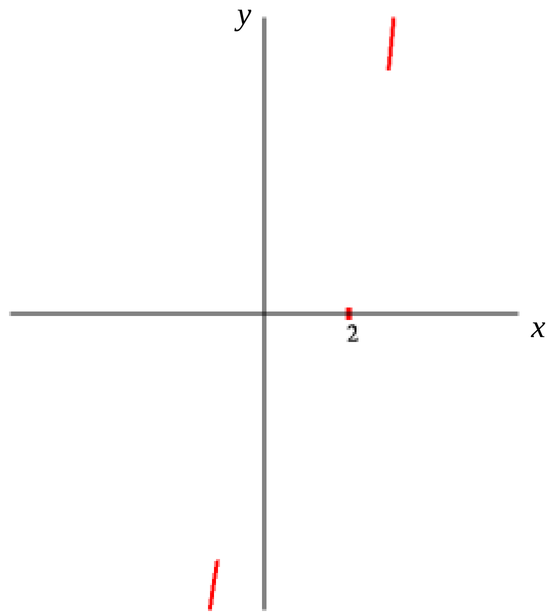
Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

$x\to\infty$:
Use the Leading Coefficient Theorem. Since the leading coefficient of $f(x)=x^3-2x^2+9x-18$ is $1>0$, then $f(x)\to\infty$ as $x\to\infty$.

$x\to-\infty$:
The degree of $f(x)=x^3-2x^2+9x-18$ is 3, which is odd. Since $f(x)\to\infty$ as $x\to\infty$, then $h(x)\to-\infty$ as $x\to-\infty$.

The polynomial is neither even nor odd. Thus, the graph of the polynomial is not symmetry with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x\to\infty$ and as $x\to-\infty$.



Since the degree of the polynomial is 3, then the graph of the polynomial can have at most 2 local extremum points (turning points).

The graph of the continuous polynomial could have no local extremum points (turning points) or it could have two local extremum points (turning points). There is not enough information from the zeros (roots) and the graph is not symmetry to the *y*-axis nor the origin. We need calculus in order to get information about the local extremum point(s).

8. $g(x) = 3x^4 + 5x^3 - 12x^2$

Zeros (Roots) of g: $g(x) = 0 \Rightarrow 3x^4 + 5x^3 - 12x^2 = 0 \Rightarrow$

$x^2(3x^2 + 5x - 12) = 0 \Rightarrow x^2(x + 3)(3x - 4) = 0 \Rightarrow$

$x = 0, \; x = -3 \;\; x = \dfrac{4}{3}$

Since the factor x^2 produces the zero (root) of 0, its multiplicity is two. Since the factor $x + 3$ produces the zero (root) of -3, its multiplicity is one. Since the factor $3x - 4$ produces the zero (root) of $\frac{4}{3}$, its multiplicity is one.

Zero (Root)	Multiplicity	Implication for the Graph of the Polynomial
-3	1	Crosses the *x*-axis at $(-3, 0)$
0	2	Touches the *x*-axis at $(0, 0)$
$\frac{4}{3}$	1	Crosses the *x*-axis at $\left(\frac{4}{3}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

$x \to \infty$:

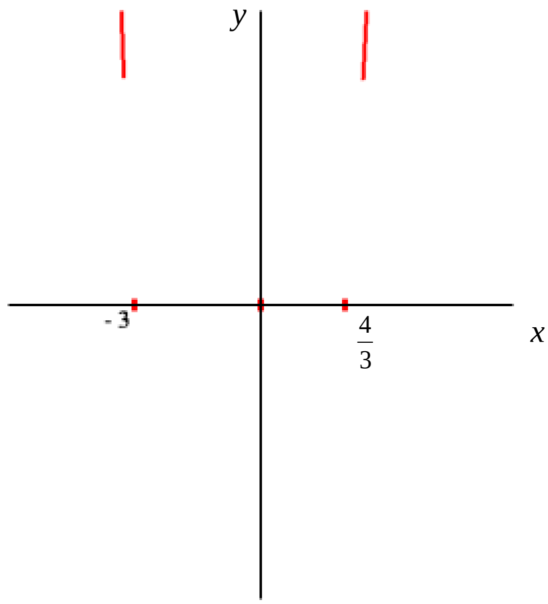
Use the Leading Coefficient Theorem. Since the leading coefficient of $g(x) = 3x^4 + 5x^3 - 12x^2$ is $3 > 0$, then $g(x) \to \infty$ as $x \to \infty$.

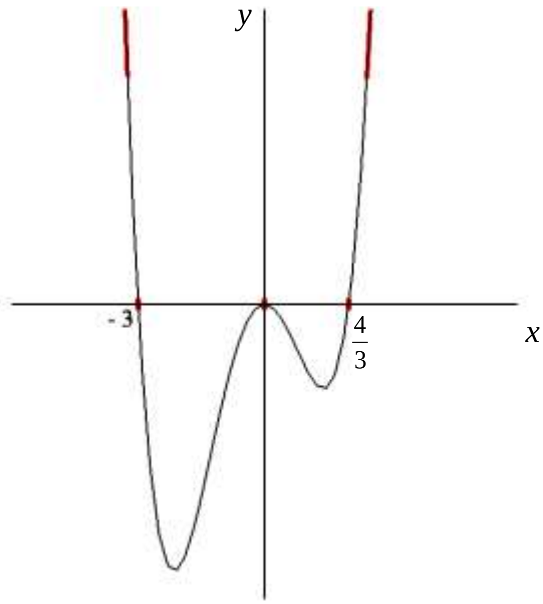
$x \to -\infty$:

The degree of $g(x) = 3x^4 + 5x^3 - 12x^2$ is 4, which is even. Since $g(x) \to \infty$ as $x \to \infty$, then $g(x) \to \infty$ as $x \to -\infty$.

The polynomial is neither even nor odd. Thus, the graph of the polynomial is not symmetry with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the point $(-3, 0)$. Then the graph must touch the *x*-axis at the origin. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -3 and 0. Since the graph touches at the origin, then there is a local extremum point (turning point) at the origin. Then the graph of the polynomial must cross the *x*-axis at the point $\left(\frac{4}{3}, 0\right)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between 0 and $\frac{4}{3}$. We would need calculus in order to obtain information about the *x*-coordinate of two of these local extremum points (turning points).

Thus, the graph of the polynomial has three local extremum points (turning points).

9. $h(x) = x^4 - 29x^2 + 100$

NOTE: The expression $x^4 - 29x^2 + 100$ is quadratic in x^2. Thus, it factors like $a^2 - 29a + 100$, where $a = x^2$. Since $a^2 - 29a + 100 =$

$(a-4)(a-25)$, then $x^4 - 29x^2 + 100 = (x^2-4)(x^2-25)$.

Zeros (Roots) of h: $h(x) = 0 \Rightarrow x^4 - 29x^2 + 100 = 0 \Rightarrow$

$(x^2-4)(x^2-25) = 0 \Rightarrow x = \pm 2, \; x = \pm 5$

Since the factor $x^2 - 4$ produces the zeros (roots) of -2 and 2, the multiplicity of each zero (root) is one. Since the factor $x^2 - 25$ produces the zeros (roots) of -5 and 5, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
-5	1	Crosses the *x*-axis at $(-5, 0)$
-2	1	Crosses the *x*-axis at $(-2, 0)$
2	1	Crosses the *x*-axis at $(2, 0)$
5	1	Crosses the *x*-axis at $(5, 0)$

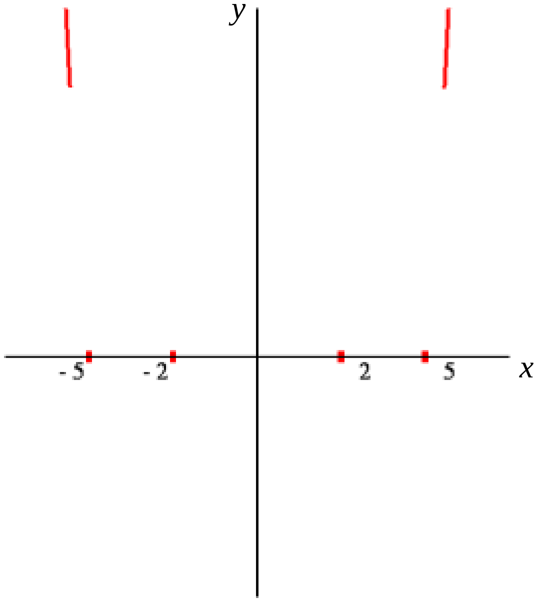
Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

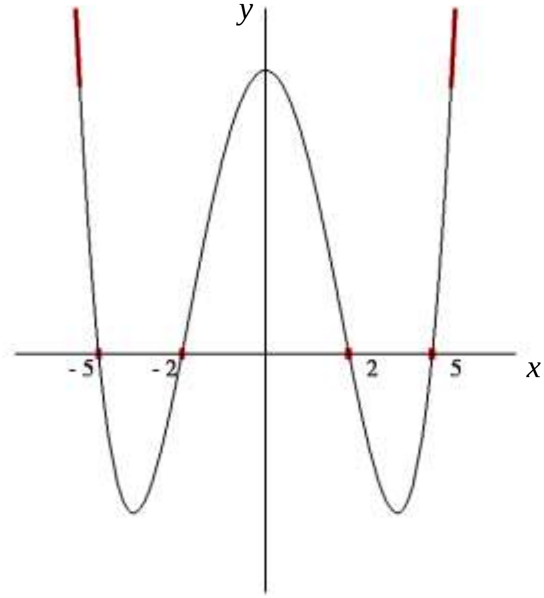
$x \to \infty$:
Use the Leading Coefficient Theorem. Since the leading coefficient of $h(x) = x^4 - 29x^2 + 100$ is $1 > 0$, then $h(x) \to \infty$ as $x \to \infty$.

$x \to -\infty$:
The degree of $h(x) = x^4 - 29x^2 + 100$ is 4, which is even. Since $h(x) \to \infty$ as $x \to \infty$, then $h(x) \to \infty$ as $x \to -\infty$.

The polynomial is even. Thus, the graph of h is symmetry about the *y*-axis.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the point $(-5, 0)$. Then the graph must cross the *x*-axis at the point $(-2, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -5 and -2. Then the graph must cross the *x*-axis at the point $(2, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -2 and 2. Then the graph must cross the *x*-axis at the point $(5, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between 2 and 5. We would need calculus in order to obtain information about the *x*-coordinate of these three local extremum points.

Thus, the graph of the polynomial has three local extremum points (turning points).

10. $f(x) = -16x^4 + 56x^2 - 49$

The expression $-16x^4 + 56x^2 - 49$ is quadratic in x^2. Thus, it factors like $-16u^2 + 56u - 49$, where $u = x^2$. Since $-16u^2 + 56u - 49 = -(16u^2 - 56u + 49) = -(4u - 7)^2$, then $-16x^4 + 56x^2 - 49 = -(4x^2 - 7)^2$

NOTE: We used the special factoring formula $a^2 - 2ab + b^2 = (a - b)^2$ to factor $16u^2 - 56u + 49$ since $16u^2 - 56u + 49 = (4u)^2 - 2(28u) + 7^2$.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow -16x^4 + 56x^2 - 49 = 0 \Rightarrow$

$$-(4x^2 - 7)^2 = 0 \Rightarrow 4x^2 - 7 = 0 \Rightarrow x = \pm \frac{\sqrt{7}}{2}$$

Since the factor $(4x^2 - 7)^2$ produces the zeros (roots) of $-\frac{\sqrt{7}}{2}$ and $\frac{\sqrt{7}}{2}$, the multiplicity of each zero (root) is two.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
$-\frac{\sqrt{7}}{2}$	2	Touches the *x*-axis at $\left(-\frac{\sqrt{7}}{2}, 0\right)$
$\frac{\sqrt{7}}{2}$	2	Touches the *x*-axis at $\left(\frac{\sqrt{7}}{2}, 0\right)$

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

$x \to \infty$:

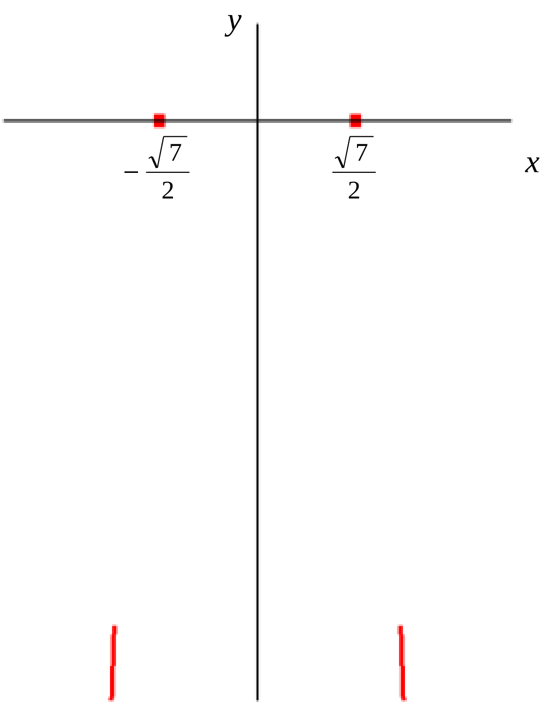
Use the Leading Coefficient Theorem. Since the leading coefficient of $f(x) = -16x^4 + 56x^2 - 49$ is $-16 < 0$, then $f(x) \to -\infty$ as $x \to \infty$.

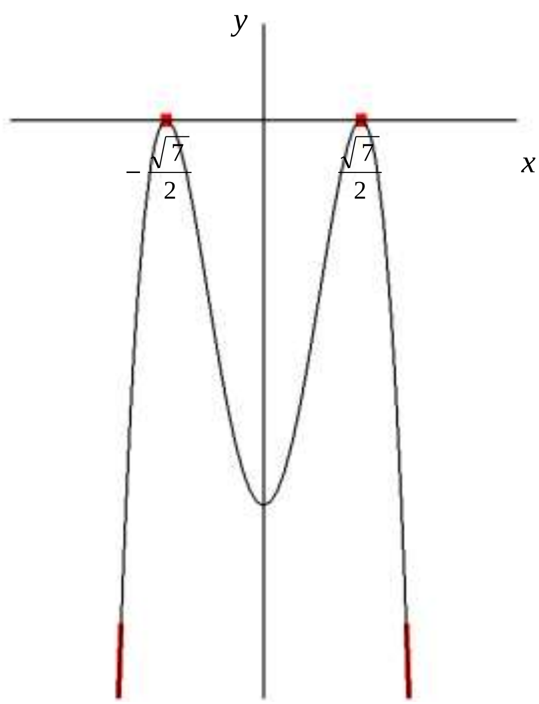
$x \to -\infty$:

The degree of $f(x) = -16x^4 + 56x^2 - 49$ is 4, which is even. Since $f(x) \to -\infty$ as $x \to \infty$, then $f(x) \to -\infty$ as $x \to -\infty$.

The polynomial is even. Thus, the graph of f is symmetry about the *y*-axis.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

Since the graph of the continuous polynomial touches the *x*-axis at the point $\left(-\frac{\sqrt{7}}{2}, 0\right)$, then there is a local extremum point (turning point) at this point. Then the graph must touch the *x*-axis at the point $\left(\frac{\sqrt{7}}{2}, 0\right)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between $-\frac{\sqrt{7}}{2}$ and $\frac{\sqrt{7}}{2}$. Since the graph touches at the point

$\left(\frac{\sqrt{7}}{2}, 0 \right)$, then there is a local extremum point (turning point) at this point. We would need calculus in order to obtain information about the *x*-coordinate of the local extremum points (turning points) whose *x*-coordinate is between $-\frac{\sqrt{7}}{2}$ and $\frac{\sqrt{7}}{2}$.

Thus, the graph of the polynomial has three local extremum points (turning points).

11. $p(x) = 15 - 2x^2 - x^4$

NOTE: The expression $15 - 2x^2 - x^4$ is quadratic in x^2. Thus, it factors like $15 - 2a - a^2$, where $a = x^2$. Since $15 - 2a - a^2 = (5 + a)(3 - a)$, then $15 - 2x^2 - x^4 = (5 + x^2)(3 - x^2)$.

Zeros (Roots) of h: $p(x) = 0 \Rightarrow 15 - 2x^2 - x^4 = 0 \Rightarrow$

$(5 + x^2)(3 - x^2) = 0 \Rightarrow x = \pm i\sqrt{5}, \; x = \pm\sqrt{3}$

Since the factor $5 + x^2$ produces the zeros (roots) of $-i\sqrt{5}$ and $i\sqrt{5}$, the multiplicity of each zero (root) is one. Since the factor $3 - x^2$ produces the zeros (roots) of $-\sqrt{3}$ and $\sqrt{3}$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
$-\sqrt{3}$	1	Crosses the *x*-axis at $(-\sqrt{3}, 0)$
$\sqrt{3}$	1	Crosses the *x*-axis at $(\sqrt{3}, 0)$
$-i\sqrt{5}$	1	No implication
$i\sqrt{5}$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

$x \to \infty$:

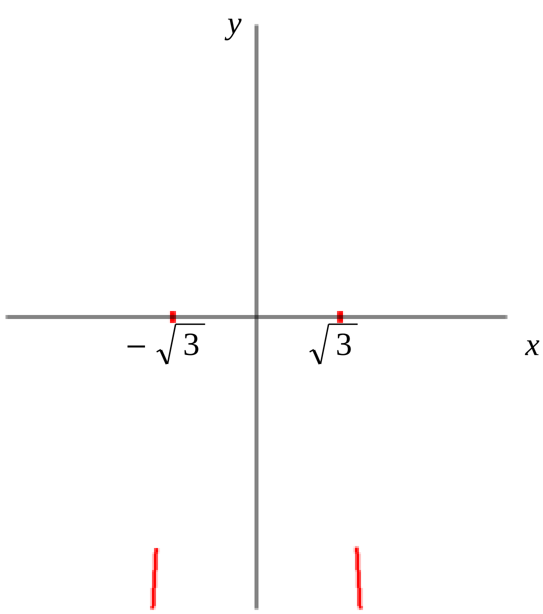
Use the Leading Coefficient Theorem. Since the leading coefficient of $p(x) = 15 - 2x^2 - x^4$ is $-1 < 0$, then $p(x) \to -\infty$ as $x \to \infty$.

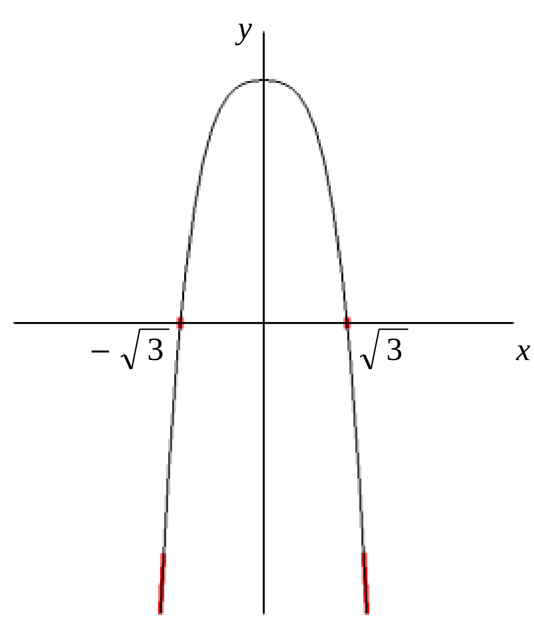
$x \to -\infty$:

The degree of $p(x) = 15 - 2x^2 - x^4$ is 4, which is even. Since $p(x) \to -\infty$ as $x \to \infty$, then $p(x) \to -\infty$ as $x \to -\infty$.

The polynomial is even. Thus, the graph of p is symmetry about the *y*-axis.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

The graph of the continuous polynomial must cross the x-axis at the point $(-\sqrt{3}, 0)$. Then the graph must cross the x-axis at the point $(\sqrt{3}, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose x-coordinate is between $-\sqrt{3}$ and $\sqrt{3}$. Since the graph of the polynomial is symmetry about the y-axis, the x-coordinate of this local extremum point is 0. If the graph had a local extremum point (turning point) for some $x < -\sqrt{3}$, then it would have to have another one in order to pass through the point $(-\sqrt{3}, 0)$. Because the graph is symmetry about the y-axis, the graph would have to have two more local extremum points (turning points) for $x > \sqrt{3}$. Thus, the polynomial would have five local extremum points (turning points). So, there are no local extremum points (turning points) for $x < -\sqrt{3}$ nor for $x > \sqrt{3}$. The same argument shows that there are no local extremum points (turning points) for $-\sqrt{3} < x < 0$ nor for $0 < x < \sqrt{3}$.

Thus, the graph of the polynomial has one local extremum points (turning points).

12. $q(x) = 8x^3 - 14x^4$

Zeros (Roots) of q: $q(x) = 0 \Rightarrow 8x^3 - 14x^4 = 0 \Rightarrow$

$2x^3(4 - 7x) = 0 \Rightarrow x = 0, \; x = \dfrac{4}{7}$

Since the factor x^3 produces the zero (root) of 0, its multiplicity is three. Since the factor $4 - 7x$ produces the zero (root) of $\frac{4}{7}$, its multiplicity is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	3	Crosses the x-axis at $(0, 0)$

$\frac{4}{7}$	1	Crosses the *x*-axis at $\left(\frac{4}{7}, 0\right)$

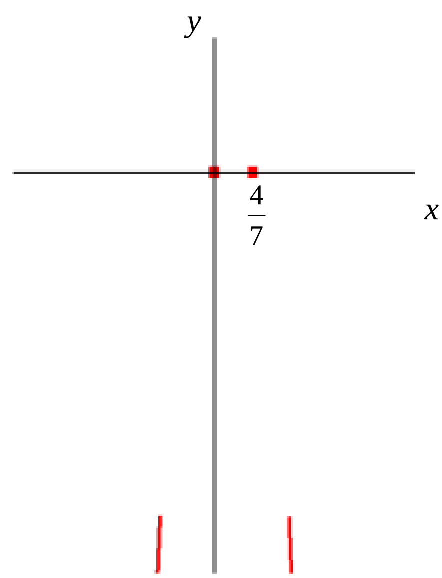
Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

$x \to \infty$:
Use the Leading Coefficient Theorem. Since the leading coefficient of $q(x) = 8x^3 - 14x^4$ is $-14 < 0$, then $q(x) \to -\infty$ as $x \to \infty$.

$x \to -\infty$:
The degree of $q(x) = 8x^3 - 14x^4$ is 4, which is even. Since $q(x) \to -\infty$ as $x \to \infty$, then $q(x) \to -\infty$ as $x \to -\infty$.

The polynomial is neither even nor odd. Thus, the graph of the polynomial is not symmetry with respect to the *y*-axis nor the origin.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.



Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

The graph of the continuous polynomial must cross the x-axis at the origin. Then the graph must cross the x-axis at the point $\left(\frac{4}{7}, 0\right)$. In order for this to happen, there must be a local extremum point (turning point) whose x-coordinate is between 0 and $\frac{4}{7}$. If the graph had a local extremum point (turning point) for some $x < 0$, then it would have to have another local extremum point (turning point) in order to pass through the origin. Since there is no symmetry in graph, this is possible and the graph would have three local extremum points (turning points). The same argument would be true for $0 < x < \frac{4}{7}$ and for $x > \frac{4}{7}$. We need calculus in order to get information about the local extremum point(s).

Thus, the graph of the polynomial has one or three local extremum points (turning points).

13. $f(x) = 54 + 27x - 2x^3 - x^4$

NOTE: The expression $54 + 27x - 2x^3 - x^4$ can be factored by grouping.

Zeros (Roots) of f: $f(x) = 0 \Rightarrow 54 + 27x - 2x^3 - x^4 = 0 \Rightarrow$

$27(2 + x) - x^3(2 + x) = 0 \Rightarrow (x + 2)(27 - x^3) = 0$

Recalling the difference of cubes factoring formula:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Thus, $27 - x^3 = (3 - x)(9 + 3x + x^2) = (3 - x)(x^2 + 3x + 9)$

Thus, $(x + 2)(27 - x^3) = 0 \Rightarrow (x + 2)(3 - x)(x^2 + 3x + 9) = 0 \Rightarrow$

$x = -2, \; x = 3, \; x^2 + 3x + 9 = 0$

Using the Quadratic Formula to solve $x^2 + 3x + 9 = 0$, we have that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 4(1)9}}{2} = \frac{-3 \pm \sqrt{9[1 - 4(1)]}}{2}$$

$$= \frac{-3 \pm \sqrt{9(1 - 4)}}{2} = \frac{-3 \pm 3\sqrt{-3}}{2} = \frac{-3 \pm 3i\sqrt{3}}{2}$$

Since the factor $x + 2$ produces the zero (root) of -2, its multiplicity is one. Since the factor $3 - x$ produces the zero (root) of 3, its multiplicity is one. Since the factor $x^2 + 3x + 9$ produces the zeros (roots) of $\frac{-3 - 3i\sqrt{3}}{2}$ and $\frac{-3 + 3i\sqrt{3}}{2}$, the multiplicity of each zero (root) is one.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
-2	1	Crosses the *x*-axis at $(-2, 0)$
3	1	Crosses the *x*-axis at $(3, 0)$
$\frac{-3 - 3i\sqrt{3}}{2}$	1	No implication
$\frac{-3 + 3i\sqrt{3}}{2}$	1	No implication

Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

$x \to \infty$:

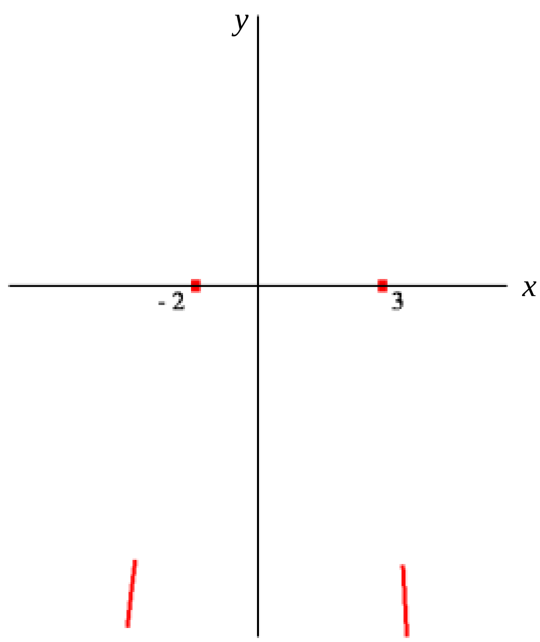
Use the Leading Coefficient Theorem. Since the leading coefficient of $f(x) = 54 + 27x - 2x^3 - x^4$ is $-1 < 0$, then $f(x) \to -\infty$ as $x \to \infty$.

$x \to -\infty$:

The degree of $f(x) = 54 + 27x - 2x^3 - x^4$ is 4, which is even. Since $f(x) \to -\infty$ as $x \to \infty$, then $f(x) \to -\infty$ as $x \to -\infty$.

The polynomial is neither even nor odd.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $x \to \infty$ and as $x \to -\infty$.



Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

The graph of the continuous polynomial must cross the *x*-axis at the point $(-2, 0)$. Then the graph must cross the *x*-axis at the point $(3, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *x*-coordinate is between -2 and 3. If the graph had a local extremum point (turning point) for some $x < -2$, then it would have to have another local extremum point (turning point) in order to pass through the point $(-2, 0)$. Since there is no symmetry in graph, this is possible and the graph would have three local extremum points (turning points). The same argument would be true for $-2 < x < 3$ and for $x > 3$. We need calculus in order to get information about the local extremum point(s).

Thus, the graph of the polynomial has one or three local extremum points (turning points).

14. $g(t) = 3t^4 - 24t^3 + 48t^2$

Zeros (Roots) of g: $g(t) = 0 \Rightarrow 3t^4 - 24t^3 + 48t^2 = 0 \Rightarrow$

$3t^2(t^2 - 8t + 16) = 0 \Rightarrow 3t^2(t - 4)^2 = 0 \Rightarrow t = 0, \; t = 4$

Since the factor t^2 produces the zero (root) of 0, its multiplicity is two. Since the factor $(t - 4)^2$ produces the zero (root) of 4, its multiplicity is two.

Zero (Root)	**Multiplicity**	**Implication for the Graph of the Polynomial**
0	2	Touches the *t*-axis at $(0, 0)$
4	2	Touches the *t*-axis at $(4, 0)$

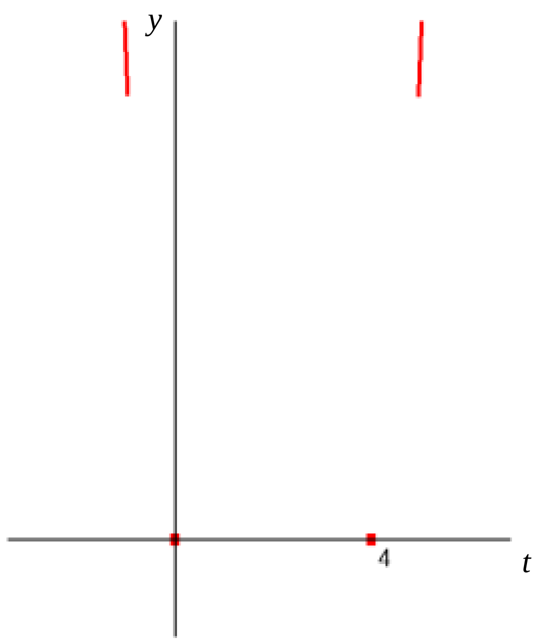
Recall: If the multiplicity of the zero (root) is odd, the graph of the polynomial crosses the *x*-axis, and if the multiplicity of the zero (root) is even, the graph of the polynomial touches the *x*-axis.

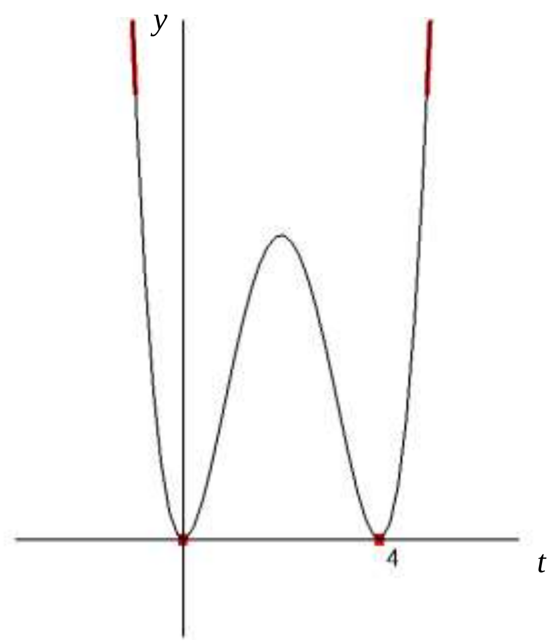
$t \to \infty$:
Use the Leading Coefficient Theorem. Since the leading coefficient of $g(t) = 3t^4 - 24t^3 + 48t^2$ is $3 > 0$, then $g(t) \to \infty$ as $t \to \infty$.

$t \to -\infty$:
The degree of $g(t) = 3t^4 - 24t^3 + 48t^2$ is 4, which is even. Since $g(t) \to \infty$ as $t \to \infty$, then $g(t) \to \infty$ as $t \to -\infty$.

The polynomial is neither even nor odd.

Here is the information about the zeros (roots) of the polynomial and the graph of the polynomial as $t \to \infty$ and as $t \to -\infty$.





The Drawing of this Sketch

Since the degree of the polynomial is 4, then the graph of the polynomial can have at most 3 local extremum points (turning points).

Since the graph of the continuous polynomial touches the *t*-axis at the origin, then there is a local extremum point (turning point) at the origin. Then the graph must touch the *t*-axis at the point $(4, 0)$. In order for this to happen, there must be a local extremum point (turning point) whose *t*-coordinate is between 0 and 4. We would need calculus in order to obtain information about the *t*-coordinate of this local extremum point (turning point). Since the graph touches at the point $(4, 0)$, then there is a local extremum point (turning point) at this point.

Thus, the graph of the polynomial has three local extremum points (turning points).

COMMENT: It appears that the graph of the polynomial *g* is symmetry about the vertical line $t = 2$. If we introduced a new *xy* coordinate system, whose origin is at the point $(2, 0)$ in the *ty* coordinate system, we would have that the point $(2, 0) = (t, y)$ in the original *ty* coordinate plane is the point $(0, 0) = (x, y)$ in the new *xy* coordinate plane. Thus, we have that $x = t - 2$, or $t = x + 2$.

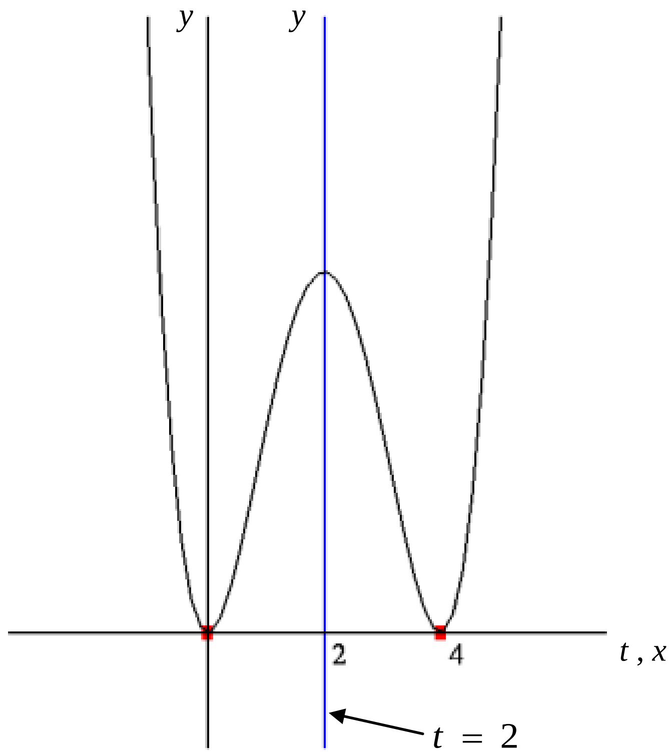
Thus, $g(t) = 3t^4 - 24t^3 + 48t^2 = 3t^2(t - 4)^2 \Rightarrow$

$h(x) = g(x+2) = 3(x+2)^2(x+2-4)^2 = 3(x+2)^2(x-2)^2 =$

$3[(x+2)(x-2)]^2 = 3(x^2-4)^2$

The polynomial h is an even function since $h(-x) = 3[(-x)^2 - 4]^2 =$

$3(x^2-4)^2 = h(x)$ Thus, the graph of the polynomial h is symmetry about the y-axis in the xy-plane.



15. $h(x) = x^4 + 14x^2 + 49$

16. $p(x) = 20x^4 + 13x^3 - 53x^2 + 8x + 12$

17. $q(x) = 27x^5 + 64x^2$

18. $f(x) = 150x^3 - 6x^5$

19. $g(x) = x^5 + 17x^3 + 16x$

20. $h(x) = 162x - 2x^5$

21. $p(t) = t^5 - 18t^3 + 32t$

22. $f(x) = x^5 - 4x^3 - 8x^2 + 32$

23. $g(x) = 9x^4 - 4x^5$

24. $h(x) = 8x^5 + 48x^3$

Examples Find a polynomial p that has the given zeros (roots) and multiplicity.

1.

Zero (Root)	Multiplicity
-4	1
$\frac{2}{3}$	1

In order for -4 to be a zero (root) of multiplicity 1, $x + 4$ must be a factor of p. In order for $\frac{2}{3}$ to be a zero (root) of multiplicity 1, $3x - 2$ must be a factor of p.

Thus, $p(x) = a(x + 4)(3x - 2)$, where a is any nonzero real number.

Since $(x+4)(3x-2) = 3x^2 - 2x + 12x - 8 = 3x^2 + 10x - 8$, then $p(x) = a(3x^2 + 10x - 8)$

Answer: $p(x) = a(3x^2 + 10x - 8)$, where a is any nonzero real number

2.

Zero (Root)	Multiplicity
-3	1
-5	1
5	1

In order for -3 to be a zero (root) of multiplicity 1, $x+3$ must be a factor of p. In order for -5 to be a zero (root) of multiplicity 1, $x+5$ must be a factor of p. In order for 5 to be a zero (root) of multiplicity 1, $x-5$ must be a factor of p.

Thus, $p(x) = a(x+3)(x+5)(x-5)$, where a is any nonzero real number.

Using the special product formula $(a+b)(a-b) = a^2 - b^2$, we have that $(x+5)(x-5) = x^2 - 25$.

Thus, $(x+3)(x+5)(x-5) = (x+3)(x^2 - 25) =$

$x^3 - 25x + 3x^2 - 75 = x^3 + 3x^2 - 25x - 75$

Thus, $p(x) = a(x^3 + 3x^2 - 25x - 75)$

Answer: $p(x) = a(x^3 + 3x^2 - 25x - 75)$, where a is any nonzero real number

3. Zero (Root) Multiplicity

Zero (Root)	Multiplicity
2	1
4	2

In order for 2 to be a zero (root) of multiplicity 1, $x - 2$ must be a factor of p. In order for 4 to be a zero (root) of multiplicity 2, $(x - 4)^2$ must be a factor of p.

Thus, $p(x) = a(x - 2)(x - 4)^2$, where a is any nonzero real number.

Using the special product formula $(a - b)^2 = a^2 - 2ab + b^2$, we have that $(x - 4)^2 = x^2 - 8x + 16$.

Thus, $(x - 2)(x - 4)^2 = (x - 2)(x^2 - 8x + 16) =$

$x^3 - 8x^2 + 16x - 2x^2 + 16x - 32 = x^3 - 10x^2 + 32x - 32$

Thus, $p(x) = a(x^3 - 10x^2 + 32x - 32)$

Answer: $p(x) = a(x^3 - 10x^2 + 32x - 32)$, where a is any nonzero real number

4. Zero (Root) Multiplicity

Zero (Root)	Multiplicity
-1	1
6	1
$-7i$	1
$7i$	1

In order for -1 to be a zero (root) of multiplicity 1, $x + 1$ must be a factor of p. In order for 6 to be a zero (root) of multiplicity 1, $x - 6$ must be a factor of p. In order for $-7i$ to be a zero (root) of multiplicity 1, $x + 7i$ must be a factor of p. In order for $7i$ to be a zero (root) of multiplicity 1, $x - 7i$ must be a factor of p.

Thus, $p(x) = a(x + 1)(x - 6)(x + 7i)(x - 7i)$, where a is any nonzero real number.

We will use the special product formula $(a + b)(a - b) = a^2 - b^2$ for $(x + 7i)(x - 7i)$. Thus, $(x + 7i)(x - 7i) = x^2 - 49i^2$. Since $i = \sqrt{-1}$, then $i^2 = -1$. Thus, $(x + 7i)(x - 7i) = x^2 - 49i^2 = x^2 + 49$.

Since $(x + 1)(x - 6) = x^2 - 6x + x - 6 = x^2 - 5x - 6$, then

$(x + 1)(x - 6)(x + 7i)(x - 7i) = (x^2 - 5x - 6)(x^2 + 49) =$

$(x^2 + 49)(x^2 - 5x - 6) = x^4 - 5x^3 - 6x^2 + 49x^2 - 245x - 294 =$

$x^4 - 5x^3 + 43x^2 - 245x - 294$

Thus, $p(x) = a(x^4 - 5x^3 + 43x^2 - 245x - 294)$

Answer: $p(x) = a(x^4 - 5x^3 + 43x^2 - 245x - 294)$, where a is any nonzero real number

5.

Zero (Root)	Multiplicity
$-\frac{5}{4}$	2
$2 - 3i$	1
$2 + 3i$	1

In order for $-\frac{5}{4}$ to be a zero (root) of multiplicity 2, $(4x + 5)^2$ must be a factor of p. In order for $2 - 3i$ to be a zero (root) of multiplicity 1, $x - (2 - 3i)$ must be a factor of p. In order for $2 + 3i$ to be a zero (root) of multiplicity 1, $x - (2 + 3i)$ must be a factor of p.

Thus, $p(x) = a(4x + 5)^2[x - (2 - 3i)][x - (2 + 3i)]$, where a is any nonzero real number.

Since $[x - (2 - 3i)][x - (2 + 3i)] = [(x - 2) - 3i][(x - 2) + 3i]$, then using the special product formula $(a + b)(a - b) = a^2 - b^2$, we have that $[(x - 2) - 3i][(x - 2) + 3i] = (x - 2)^2 - 9i^2$. Using the special product formula $(a - b)^2 = a^2 - 2ab + b^2$, we have that $(x - 2)^2 = x^2 - 4x + 4$. Thus, $(x - 2)^2 - 9i^2 = x^2 - 4x + 4 + 9$ $= x^2 - 4x + 13$.

Using the special product formula $(a + b)^2 = a^2 + 2ab + b^2$, we have that $(4x + 5)^2 = 16x^2 + 40x + 25$.

Thus, $(4x + 5)^2[x - (2 - 3i)][x - (2 + 3i)] =$

$(16x^2 + 40x + 25)(x^2 - 4x + 13) = 16x^4 - 64x^3 + 208x^2 + 40x^3$

$- 160x^2 + 520x + 25x^2 - 100x + 325 =$

$16x^4 - 24x^3 + 73x^2 + 420x + 325$

Thus, $p(x) = a(16x^4 - 24x^3 + 73x^2 + 420x + 325)$

Answer: $p(x) = a(16x^4 - 24x^3 + 73x^2 + 420x + 325)$, where a is any nonzero real number

6. Zero (Root) Multiplicity

Zero (Root)	Multiplicity
-8	5

$-\frac{1}{2}$	3
$\frac{11}{7}$	4
6	2

In order for -8 to be a zero (root) of multiplicity 5, $(x+8)^5$ must be a factor of p. In order for $-\frac{1}{2}$ to be a zero (root) of multiplicity 3, $(2x+1)^3$ must be a factor of p. In order for $\frac{11}{7}$ to be a zero (root) of multiplicity 4, $(7x-11)^4$ must be a factor of p. In order for 6 to be a zero (root) of multiplicity 2, $(x-6)^2$ must be a factor of p.

Thus, $p(x) = a(x+8)^5(2x+1)^3(7x-11)^4(x-6)^2$, where a is any nonzero real number.

Answer: $p(x) = a(x+8)^5(2x+1)^3(7x-11)^4(x-6)^2$, where a is any nonzero real number

7.

Zero (Root)	Multiplicity
$-\frac{7}{3}$	1
$\frac{7}{3}$	1
$-i\sqrt{5}$	1
$i\sqrt{5}$	1

In order for $-\frac{7}{3}$ to be a zero (root) of multiplicity 1, $3x+7$ must be a factor of p. In order for $\frac{7}{3}$ to be a zero (root) of multiplicity 1, $3x-7$

must be a factor of p. In order for $-i\sqrt{5}$ to be a zero (root) of multiplicity 1, $x + i\sqrt{5}$ must be a factor of p. In order for $i\sqrt{5}$ to be a zero (root) of multiplicity 1, $x - i\sqrt{5}$ must be a factor of p.

Thus, $p(x) = a(3x + 7)(3x - 7)(x + i\sqrt{5})(x - i\sqrt{5})$, where a is any nonzero real number.

We will use the special product formula $(a + b)(a - b) = a^2 - b^2$ for $(3x + 7)(3x - 7)$. Thus, $(3x + 7)(3x - 7) = 9x^2 - 49$.

We will use the special product formula $(a + b)(a - b) = a^2 - b^2$ for $(x + i\sqrt{5})(x - i\sqrt{5})$. Thus, $(x + i\sqrt{5})(x - i\sqrt{5}) = x^2 - 5i^2$. Thus, $(x + i\sqrt{5})(x - i\sqrt{5}) = x^2 - 5i^2 = x^2 + 5$.

Thus, $(3x + 7)(3x - 7)(x + i\sqrt{5})(x - i\sqrt{5}) = (9x^2 - 49)(x^2 + 5) =$

$9x^4 + 45x^2 - 49x^2 - 245 = 9x^4 - 4x^2 - 245$

Thus, $p(x) = a(9x^4 - 4x^2 - 245)$

Answer: $p(x) = a(9x^4 - 4x^2 - 245)$, where a is any nonzero real number

8.

Zero (Root)	Multiplicity
-3	2
3	2
$-i$	2
i	2

In order for -3 to be a zero (root) of multiplicity 2, $(x+3)^2$ must be a factor of p. In order for 3 to be a zero (root) of multiplicity 2, $(x-3)^2$ must be a factor of p. In order for $-i$ to be a zero (root) of multiplicity 2, $(x+i)^2$ must be a factor of p. In order for i to be a zero (root) of multiplicity 2, $(x-i)^2$ must be a factor of p.

Thus, $p(x) = a(x+3)^2(x-3)^2(x+i)^2(x-i)^2$, where a is any nonzero real number.

Since $(ab)^n = a^n b^n$, then $(x+3)^2(x-3)^2 = [(x+3)(x-3)]^2 =$

$(x^2-9)^2 = x^4 - 18x^2 + 81$ and $(x+i)^2(x-i)^2 =$

$[(x+i)(x-i)]^2 = (x^2+1)^2 = x^4 + 2x^2 + 1$

Thus, $(x+3)^2(x-3)^2(x+i)^2(x-i)^2 =$

$(x^4 - 18x^2 + 81)(x^4 + 2x^2 + 1) = x^8 + 2x^6 + x^4 - 18x^6 - 36x^4$

$- 18x^2 + 81x^4 + 162x^2 + 81 = x^8 - 16x^6 + 46x^4 + 144x^2 + 81$

Thus, $p(x) = a(x^8 - 16x^6 + 46x^4 + 144x^2 + 81)$

Answer: $p(x) = a(x^8 - 16x^6 + 46x^4 + 144x^2 + 81)$, where a is any nonzero real number