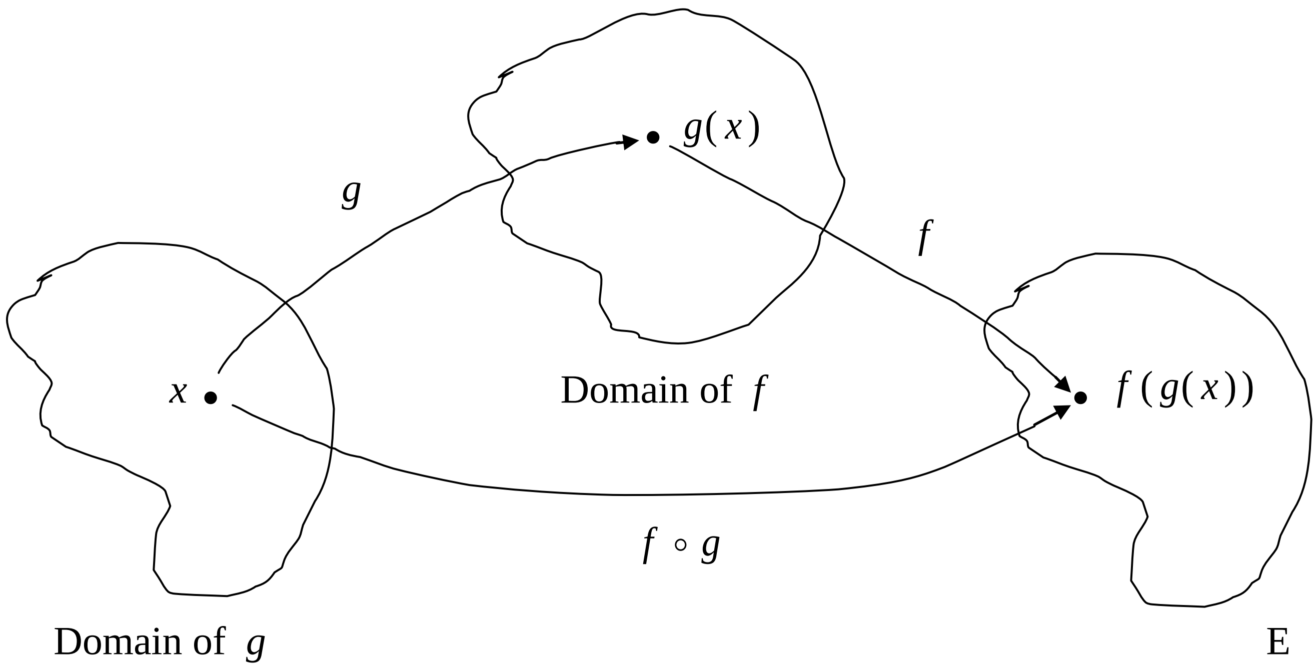
LESSON 4 COMPOSITION FUNCTIONS AND INVERSE FUNCTIONS

1. COMPOSTION FUNCTIONS

<u>Definiton</u> Let f and g be two functions. The composite function $f \circ g$ is the function defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.



<u>Example</u> In Lesson 3, we consider the horizontal shift of the function h given by $h(x) = \sqrt{x - 4}$. This function is a composite function. Let $f(x) = \sqrt{x}$ and $g(x) = x - 4$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(x - 4) = \sqrt{x - 4}$$

NOTE: The domain of g is the set of all real numbers. Since the domain of f is $[0, \infty)$, then $g(x) \in [0, \infty) \Leftrightarrow g(x) \geq 0 \Leftrightarrow x - 4 \geq 0 \Leftrightarrow x \geq 4$. Thus, the domain of $h = f \circ g$ is $[4, \infty)$.

Example In Lesson 2, we found the domain of the function h given by $h(x) = \sqrt[4]{\dfrac{5x+3}{x+2}}$. This function is a composite function. Let $f(x) = \sqrt[4]{x}$ and $g(x) = \dfrac{5x+3}{x+2}$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f\left(\frac{5x+3}{x+2}\right) = \sqrt[4]{\frac{5x+3}{x+2}}$$

NOTE: The domain of g is the set of all real numbers x such that $x \neq -2$. Since the domain of f is $[0, \infty)$, then $g(x) \in [0, \infty) \Leftrightarrow g(x) \geq 0 \Leftrightarrow \dfrac{5x+3}{x+2} \geq 0$. We found the solution to this non-linear inequality in Lesson 2 obtaining that $(-\infty, -2) \cup \left[-\dfrac{3}{5}, \infty\right)$. Thus, the domain of $h = f \circ g$ is $(-\infty, -2) \cup \left[-\dfrac{3}{5}, \infty\right)$.

Examples Given the function h, find functions f and g such that $h = f \circ g$.

1. $h(x) = \sqrt{9 - 4x}$ (From Lesson 2)

Let $f(x) = \sqrt{x}$ and $g(x) = 9 - 4x$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(9 - 4x) = \sqrt{9 - 4x}$$

Answer: $f(x) = \sqrt{x}$ and $g(x) = 9 - 4x$

2. $h(x) = \sqrt{2x^2 - 5x - 3}$ (From Lesson 2)

Let $f(x) = \sqrt{x}$ and $g(x) = 2x^2 - 5x - 3$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(2x^2 - 5x - 3) = \sqrt{2x^2 - 5x - 3}$$

Answer: $f(x) = \sqrt{x}$ and $g(x) = 2x^2 - 5x - 3$

3. $h(x) = \sqrt[3]{2x^2 - 5x + 3}$ (From Lesson 2)

Let $f(x) = \sqrt[3]{x}$ and $g(x) = 2x^2 - 5x + 3$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(2x^2 - 5x + 3) = \sqrt[3]{2x^2 - 5x + 3}$$

Answer: $f(x) = \sqrt[3]{x}$ and $g(x) = 2x^2 - 5x + 3$

4. $h(x) = \sqrt[5]{\dfrac{5x + 3}{x + 2}}$ (From Lesson 2)

Let $f(x) = \sqrt[5]{x}$ and $g(x) = \dfrac{5x + 3}{x + 2}$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f\left(\frac{5x + 3}{x + 2}\right) = \sqrt[5]{\frac{5x + 3}{x + 2}}$$

Answer: $f(x) = \sqrt[5]{x}$ and $g(x) = \dfrac{5x + 3}{x + 2}$

5. $h(x) = \dfrac{8}{3x + 16}$ (From Lesson 2)

Let $f(x) = \dfrac{8}{x}$ and $g(x) = 3x + 16$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(3x + 16) = \frac{8}{3x + 16}$$

Answer: $f(x) = \dfrac{8}{x}$ and $g(x) = 3x + 16$

6. $h(x) = \dfrac{1}{2}x^2 - 5$ (From Lesson 3)

Let $f(x) = x - 5$ and $g(x) = \dfrac{1}{2}x^2$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f\left(\frac{1}{2}x^2\right) = \frac{1}{2}x^2 - 5$$

Answer: $f(x) = x - 5$ and $g(x) = \dfrac{1}{2}x^2$

7. $h(x) = 2(x - 4)^2 - 3$ (From Lesson 3)

Let $f(x) = 2x^2 - 3$ and $g(x) = x - 4$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(x - 4) = 2(x - 4)^2 - 3$$

Answer: $f(x) = 2x^2 - 3$ and $g(x) = x - 4$

8. $h(x) = -5(x + 12)^2$ (From Lesson 3)

Let $f(x) = -5x^2$ and $g(x) = x + 12$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(x + 12) = -5(x + 12)^2$$

Answer: $f(x) = -5x^2$ and $g(x) = x + 12$

9. $h(x) = 2\sqrt{x} - 4$ (From Lesson 3)

Let $f(x) = 2x - 4$ and $g(x) = \sqrt{x}$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 2\sqrt{x} - 4$$

Answer: $f(x) = 2x - 4$ and $g(x) = \sqrt{x}$

10. $h(t) = \sqrt{-5t - 30} - 11$ (From Lesson 3)

Let $f(t) = \sqrt{t} - 11$ and $g(t) = -5t - 30$. Then $h = f \circ g$.

$$h(t) = (f \circ g)(t) = f(g(t)) = f(-5t - 30) = \sqrt{-5t - 30} - 11$$

OR

Let $f(t) = t - 11$ and $g(t) = \sqrt{-5t - 30}$. Then $h = f \circ g$.

$$h(t) = (f \circ g)(t) = f(g(t)) = f(\sqrt{-5t - 30}) = \sqrt{-5t - 30} - 11$$

Answer: $f(t) = \sqrt{t} - 11$ and $g(t) = -5t - 30$

$f(t) = t - 11$ and $g(t) = \sqrt{-5t - 30}$

11. $h(x) = -\dfrac{11}{6}\sqrt{3x - 7} + 2$ (From Lesson 3)

Let $f(x) = -\frac{11}{6}\sqrt{x} + 2$ and $g(x) = 3x - 7$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(3x - 7) = -\frac{11}{6}\sqrt{3x - 7} + 2$$

OR

Let $f(x) = -\frac{11}{6}x + 2$ and $g(x) = \sqrt{3x - 7}$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(\sqrt{3x - 7}) = -\frac{11}{6}\sqrt{3x - 7} + 2$$

Answer: $f(x) = -\frac{11}{6}\sqrt{x} + 2$ and $g(x) = 3x - 7$

$f(x) = -\frac{11}{6}x + 2$ and $g(x) = \sqrt{3x - 7}$

12. $h(x) = |6x - 18|$ (From Lesson 3)

Let $f(x) = |x|$ and $g(x) = 6x - 18$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(6x - 18) = |6x - 18|$$

Answer: $f(x) = |x|$ and $g(x) = 6x - 18$

13. $h(x) = -|4x + 9| + 7$ (From Lesson 3)

Let $f(x) = -|x| + 7$ and $g(x) = 4x + 9$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(4x + 9) = -|4x + 9| + 7$$

OR

Let $f(x) = -x + 7$ and $g(x) = |4x + 9|$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(|4x + 9|) = -|4x + 9| + 7$$

Answer: $f(x) = -|x| + 7$ and $g(x) = 4x + 9$

$f(x) = -x + 7$ and $g(x) = |4x + 9|$

14. $h(x) = -\dfrac{14}{2x + 5} - 6$ (From Lesson 3)

Let $f(x) = -\dfrac{14}{x} - 6$ and $g(x) = 2x + 5$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(2x + 5) = -\frac{14}{2x + 5} - 6$$

Answer: $f(x) = -\dfrac{14}{x} - 6$ and $g(x) = 2x + 5$

15. $h(x) = (4x^3 + 2x^2 - x - 3)^3$ (From Lesson 10 of MATH-1850)

Let $f(x) = x^3$ and $g(x) = 4x^3 + 2x^2 - x - 3$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(4x^3 + 2x^2 - x - 3) =$$

$$(4x^3 + 2x^2 - x - 3)^3$$

Answer: $f(x) = x^3$ and $g(x) = 4x^3 + 2x^2 - x - 3$

16. $h(x) = 3(8x - 17)^{-5}$ (From Lesson 10 of MATH-1850)

Let $f(x) = 3x^{-5}$ and $g(x) = 8x - 17$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(8x - 17) = 3(8x - 17)^{-5}$$

Answer: $f(x) = 3x^{-5}$ and $g(x) = 8x - 17$

17. $h(w) = (3w^2 - 2w)^{3/5}$ (From Lesson 10 of MATH-1850)

Let $f(w) = w^{3/5}$ and $g(w) = 3w^2 - 2w$. Then $h = f \circ g$.

$$h(w) = (f \circ g)(w) = f(g(w)) = f(3w^2 - 2w) = (3w^2 - 2w)^{3/5}$$

Answer: $f(w) = w^{3/5}$ and $g(w) = 3w^2 - 2w$

18. $h(t) = \dfrac{5}{7(4t^5 - 3t^3 + t)^4}$ (From Lesson 10 of MATH-1850)

Let $f(t) = \dfrac{5}{7t^4}$ and $g(t) = 4t^5 - 3t^3 + t$. Then $h = f \circ g$.

$$h(t) = (f \circ g)(t) = f(g(t)) = f(4t^5 - 3t^3 + t) = \frac{5}{7(4t^5 - 3t^3 + t)^4}$$

Answer: $f(t) = \dfrac{5}{7t^4}$ and $g(t) = 4t^5 - 3t^3 + t$

19. $h(x) = \dfrac{-3}{x + 15}$ (From Lesson 10 of MATH-1850)

Let $f(x) = \dfrac{-3}{x}$ and $g(x) = x + 15$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(x + 15) = \frac{-3}{x + 15}$$

Answer: $f(x) = \dfrac{-3}{x}$ and $g(x) = x + 15$

20. $h(x) = \dfrac{21}{9x - 17}$ (From Lesson 10 of MATH-1850)

Let $f(x) = \dfrac{21}{x}$ and $g(x) = 9x - 17$. Then $h = f \circ g$.

$$h(x) = (f \circ g)(x) = f(g(x)) = f(9x - 17) = \frac{21}{9x - 17}$$

Answer: $f(x) = \dfrac{21}{x}$ and $g(x) = 9x - 17$

21. $y = \dfrac{6}{4t^2 + 3t - 22}$ (From Lesson 10 of MATH-1850)

Let $f(t) = \dfrac{6}{t}$ and $g(t) = 4t^2 + 3t - 22$. Then $h = f \circ g$.

$$h(t) = (f \circ g)(t) = f(g(t)) = f(4t^2 + 3t - 22) = \frac{6}{4t^2 + 3t - 22}$$

Answer: $f(t) = \dfrac{6}{t}$ and $g(t) = 4t^2 + 3t - 22$

22. $h(z) = 10\sqrt[3]{(z^2 + 3z - 18)^2}$ (From Lesson 10 of MATH-1850)

Let $f(z) = 10\sqrt[3]{z^2}$ and $g(z) = z^2 + 3z - 18$. Then $h = f \circ g$.

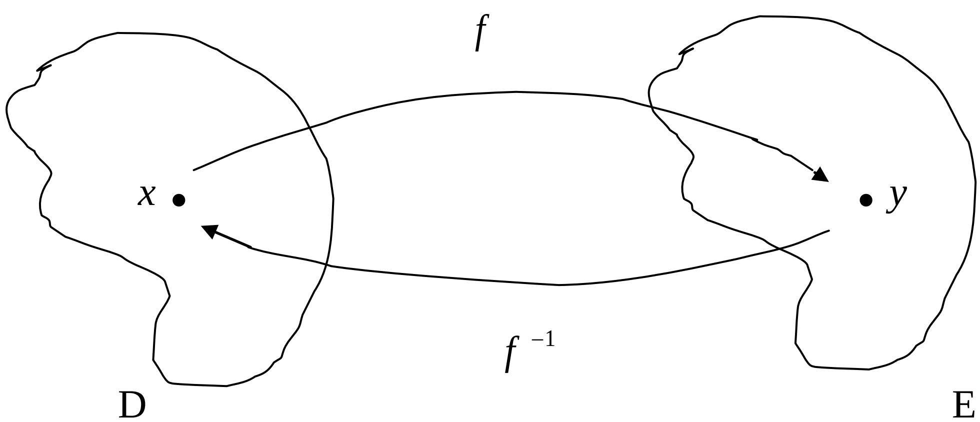
$$h(z) = (f \circ g)(z) = f(g(z)) = f(z^2 + 3z - 18) =$$

$$10\sqrt[3]{(z^2 + 3z - 18)^2}$$

Answer: $f(z) = 10\sqrt[3]{z^2}$ and $g(z) = z^2 + 3z - 18$

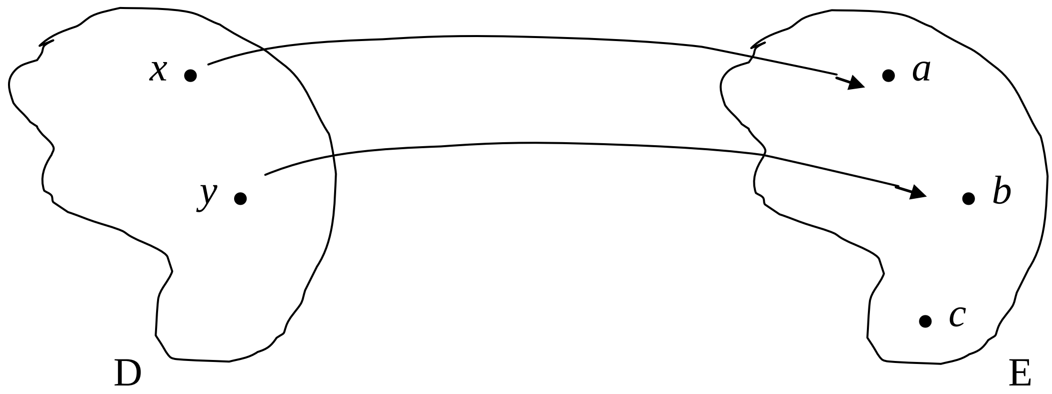
2. INVERSE FUNCTIONS

Example Consider the following function f and its inverse f^{-1}.



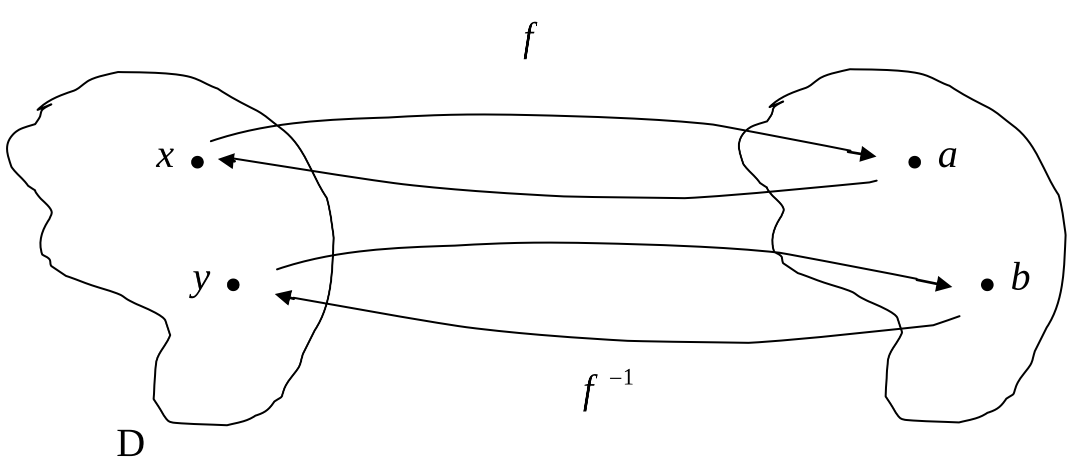
NOTE: The function f maps x in the set D to y in the set E and f^{-1} maps y back to x. Of course, this is what an inverse function is suppose to do.

Example Consider the following function f. Does f have an inverse?



The function f maps x in the set D to a in the set E. So, the inverse function would map a back to x. The function f maps y in the set D to b in the set E. So, the inverse function would map b back to y. However, where does the inverse function map c. Note that the domain of the function f is the set $\{x, y\}$ and the range of the function f is the set $\{a, b\}$.

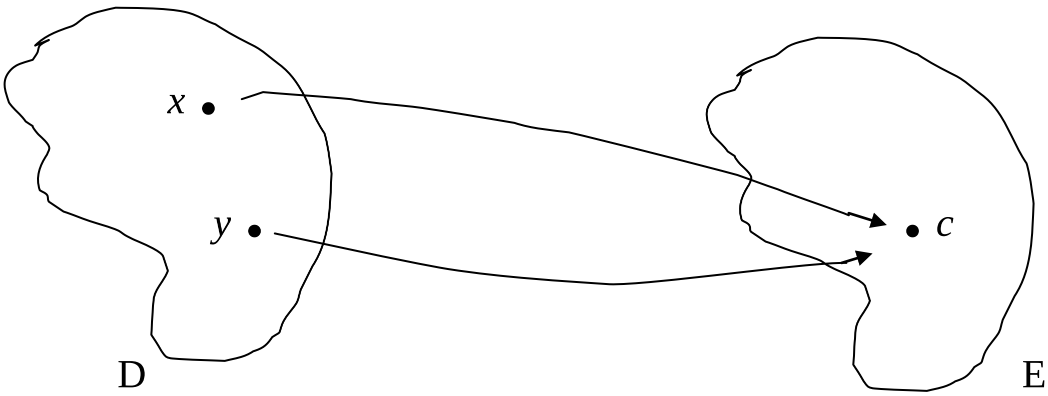
The function f is not an onto function. In order for a function to be an onto function, every element in the set E must be used. For the given function f, the element c in the set E was not used. **In order for a function to have an inverse, it must be an onto function.** If a function is not an onto function, then the lack of this needed condition is easy to fix. To fix the lack of the onto condition, replace the set E by the range of the function.



Range of f = $\{a, b\}$

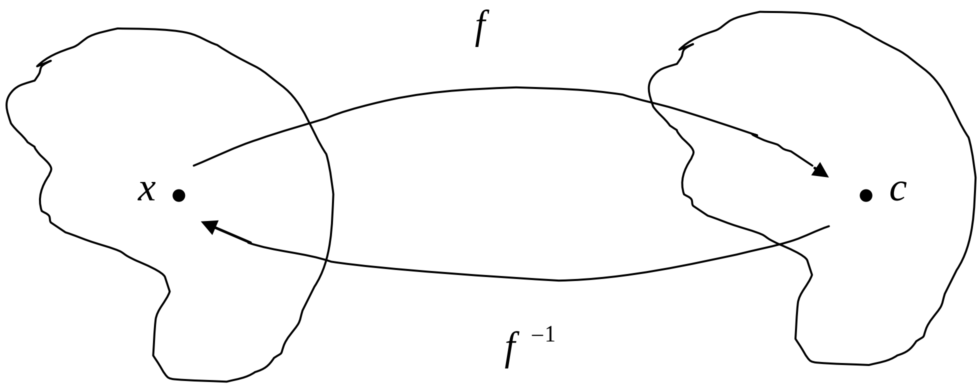
Now, this function has an inverse function. Notice the following relationship above: The domain of the inverse function f^{-1} is equal to the range of the function f.

Example Consider the following function f. Does f have an inverse?



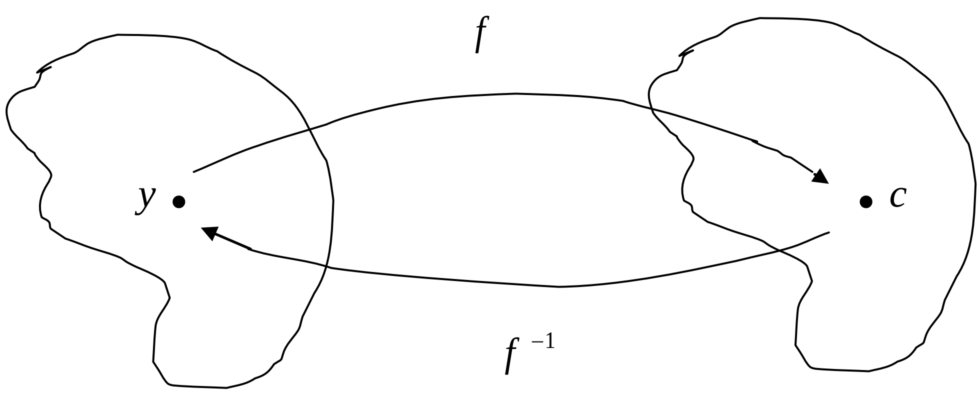
Note that the domain of the function f is the set D, which is the set $\{x, y\}$ and the range of the function f is the set $\{c\}$. Also, note that the function f is an onto function. The set E is the range of the function f. The function f maps *x* in the set D to *c* in the set E. The function f also maps *y* in the set D to *c* in the set E. So, the inverse function would map *c* back to either *x* or *y*. Which one do you use? The problem here is that the function f is not a one-to-one function. In order for a function to be a one-to-one function, you may only use each element in the set E once. **In order for a function to have an inverse, it must be a one-to-one function.** If a function is not a one-to-one function, then the lack of this needed condition is not as easy to fix as the lack of the onto condition. In order to fix the lack of the one-to-one condition, you must put a restriction on the domain of the function. In other words, you must eliminate elements from the set D. What elements in the set D are you going to chose to eliminate? This is the reason that fixing the lack of the one-to-one condition is harder. For the function f, the domain is the set D = $\{x, y\}$. Thus, we will either eliminate *x* or *y*. Each restricted domain will produce an inverse function. Thus, these two choices for the restricted domain will produce two inverse functions.

If we eliminate *y*, then we get the following inverse function:



Restricted domain of $f = \{x\}$ Range of $f = \{c\}$

If we eliminate x, then we get the following inverse function:



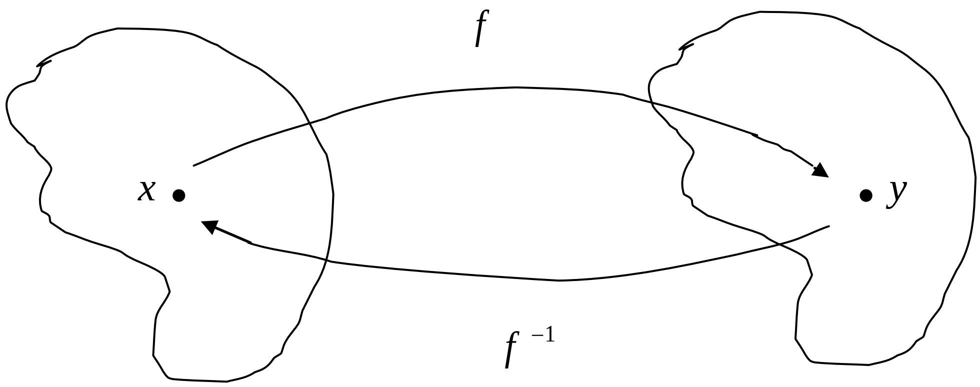
Restricted domain of $f = \{y\}$ Range of $f = \{c\}$

The function f has two possible inverse functions depending on the restricted domain that is chosen.

Notice the following relationship for both inverse functions above: The restricted domain of the function f is equal to the range of the function f^{-1}.

Theorem A function f has an inverse function, denoted by f^{-1} if and only if the function f is one-to-one and onto.

We have the following relationships:



Restricted domain of f || Range of f^{-1}

Range of f || Domain of f^{-1}

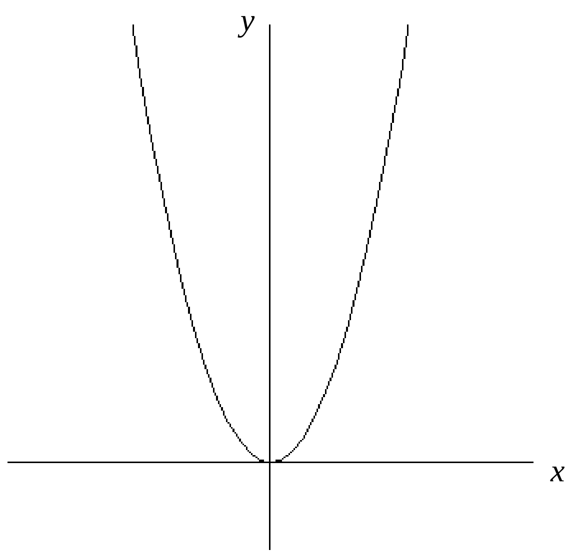
We also have the following two relationships between a function and its inverse function:

1. $f^{-1}(f(x)) = x$ for all x in the **restricted domain of** f

2. $f(f^{-1}(y)) = y$ for all y in the **domain of** f^{-1}

<u>Example</u> Find the inverse function of the function $f(x) = x^2$.

First, consider the graph of $f(x) = x^2$.



Information about the domain of the function f can be determined by the *x*-coordinate of the points on the graph. Since the value of the *x*-coordinates range in value from negative infinity to positive infinity, then the domain of f is all real numbers. Information about the range of the function f can be determined by the *y*-coordinate of the points on the graph. Since the value of the *y*-coordinates range in value from zero to positive infinity, then the range of f is all real numbers greater than or equal to zero. In interval notation, we have the following:

$$\text{Domain of } f = (-\infty, \infty) \text{ and Range of } f = [0, \infty)$$

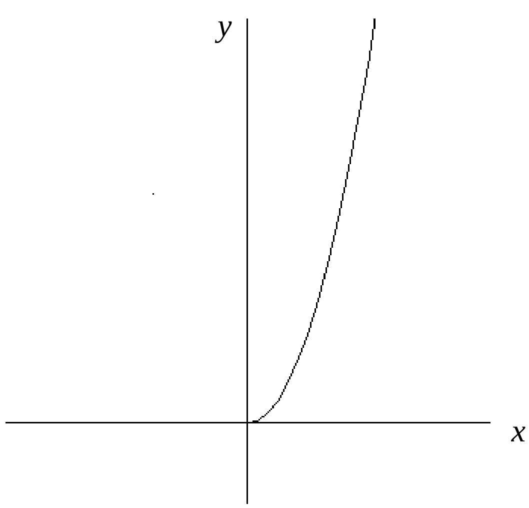
Since the range of f is $[0, \infty)$, then by the discussion above, the domain of f^{-1} is $[0, \infty)$.

Recall the following test for checking the graph of a function for being one-to-one:

The Horizontal Line Test: If a horizontal line intersects the graph of a function in more than one place, then the function is not one-to-one.

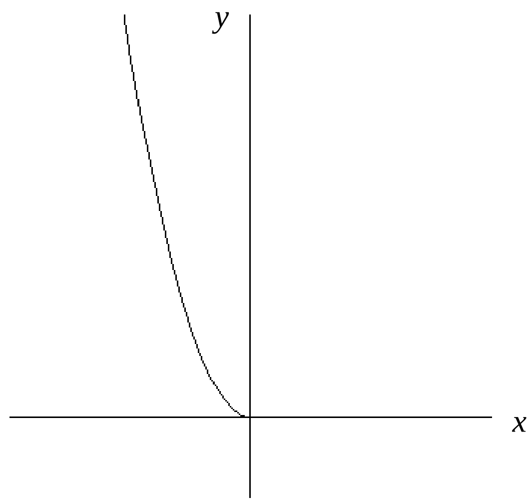
By the horizontal line test, the function f is not one-to-one. We will have to put a restriction on the domain of the function in order to fix this. We have the following two choices for the restricted domain: 1) the interval of numbers $[0, \infty)$; that is, the set of numbers greater than or equal to zero **or** 2) the interval of numbers $(-\infty, 0]$; that is, the set of numbers less than or equal to zero. Each restricted domain will produce an inverse function. Thus, these two choices for the restricted domain will produce two inverse functions.

For the first choice of $[0, \infty)$, our restricted domain is $[0, \infty)$ and the graph of the function f on the restricted domain looks like the following:



Since the restricted domain of f is $[0, \infty)$, then by the discussion above, the range of f^{-1} is $[0, \infty)$.

For the second choice of $(-\infty, 0]$, our restricted domain is $(-\infty, 0]$ and the graph of the function f on the restricted domain looks like the following:



Since the restricted domain of f is $(-\infty, 0]$, then by the discussion above, the range of f^{-1} is $(-\infty, 0]$.

Now, let's find the inverse function(s) algebraically.

Set $f(x) = y$: $\quad y = x^2$

Solve for *x*:

$$y = x^2 \Rightarrow x^2 = y \Rightarrow \sqrt{x^2} = \sqrt{y} \Rightarrow |x| = \sqrt{y} \Rightarrow x = \pm\sqrt{y}$$

Thus, either $f^{-1}(y) = \sqrt{y}$ if the restricted domain of f is $[0, \infty)$

or $f^{-1}(y) = -\sqrt{y}$ if the restricted domain of f is $(-\infty, 0]$.

Notice, as predicted above the domain of f^{-1} is $[0, \infty)$ and the range of f^{-1} is $[0, \infty)$ for the first inverse function and the range of f^{-1} is $(-\infty, 0]$ for the second inverse function.

Now, let's verify the two relationships for this function and its two inverse functions are true.

1. For the first restricted of domain of $[0, \infty)$ for the function $f(x) = x^2$, we have that $f^{-1}(y) = \sqrt{y}$:

   a. $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x| = x$. Note that since x belongs to the restricted domain of $[0, \infty)$, then x is a positive number. The absolute value of a positive number is itself.

   b. $f(f^{-1}(y)) = f(\sqrt{y}) = (\sqrt{y})^2 = y$. Note that since y is in the domain of f^{-1} and the domain of f^{-1} is $[0, \infty)$, then y is greater than or equal to zero. Thus, the square root of x is defined as a real number.

2. For the second restricted of domain of $(-\infty, 0]$ for the function $f(x) = x^2$, we have that $f^{-1}(y) = -\sqrt{y}$:

   a. $f^{-1}(f(x)) = f^{-1}(x^2) = -\sqrt{x^2} = -|x| = -(-x) = x$. Note that since x belongs to the restricted domain of $(-\infty, 0]$, then x is a negative number. The absolute value of a negative number is the negative of itself.

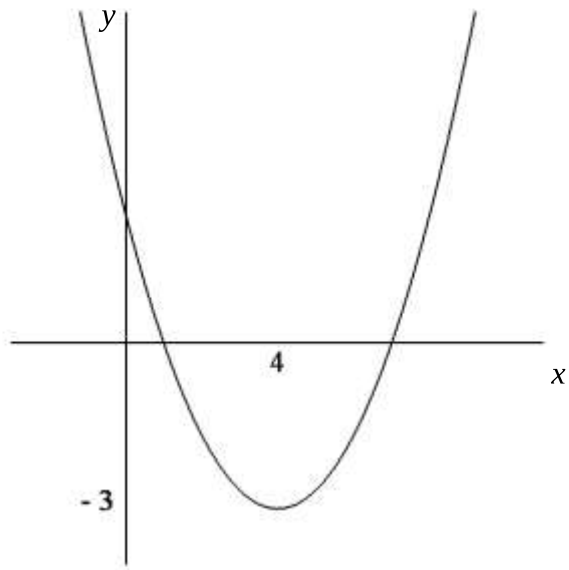
   b. $f(f^{-1}(y)) = f(-\sqrt{y}) = (-\sqrt{y})^2 = y$. Note that since y is in the domain of f^{-1} and the domain of f^{-1} is $[0, \infty)$, then y is greater than or equal to zero. Thus, the square root of x is defined as a real number.

Thus, the two relationships for this function and its two inverse functions hold.

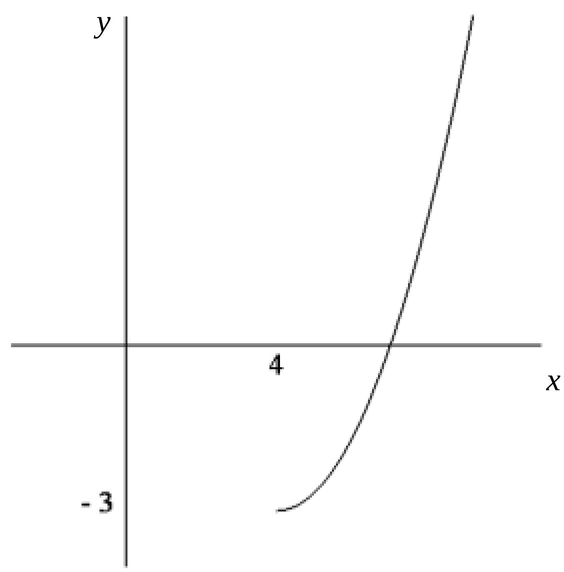
Examples Find the inverse function for the following functions. State the domain and range of both the function and its inverse.

1. $f(x) = 2(x - 4)^2 - 3, \ x \geq 4$

The graph of the function $f(x) = 2(x - 4)^2 - 3$ was sketched in Lesson 3.



You can see from this sketch that the graph fails the Horizontal Line Test. Thus, the function $f(x) = 2(x - 4)^2 - 3$ is not one-to-one. Here is the sketch of the graph of the function for $x \geq 4$.



This graph does pass the Horizontal Line Test. Thus, the function $f(x) = 2(x - 4)^2 - 3$ for $x \geq 4$ is one-to-one.

Restricted Domain of $f = [4, \infty) =$ Range of f^{-1}

Range of $f = [-3, \infty) =$ Domain of f^{-1}

Now, we will find f^{-1}.

Setting $f(x) = y$, we have that $y = 2(x-4)^2 - 3$. Now solving for x, we have that

$$y = 2(x-4)^2 - 3 \Rightarrow y + 3 = 2(x-4)^2 \Rightarrow (x-4)^2 = \frac{y+3}{2} \Rightarrow$$

$x - 4 = \pm\sqrt{\dfrac{y+3}{2}}$. Since $x \geq 4$, then $x - 4 \geq 0$. That is, $x - 4$ is nonnegative. Thus, $x - 4 = \sqrt{\dfrac{y+3}{2}} \Rightarrow x - 4 = \sqrt{\dfrac{2(y+3)}{4}} \Rightarrow$

$$x - 4 = \frac{\sqrt{2(y+3)}}{2} \Rightarrow x = \frac{8}{2} + \frac{\sqrt{2(y+3)}}{2} \Rightarrow$$

$x = \dfrac{8 + \sqrt{2(y+3)}}{2}$. Thus, $f^{-1}(y) = \dfrac{8 + \sqrt{2(y+3)}}{2}$.

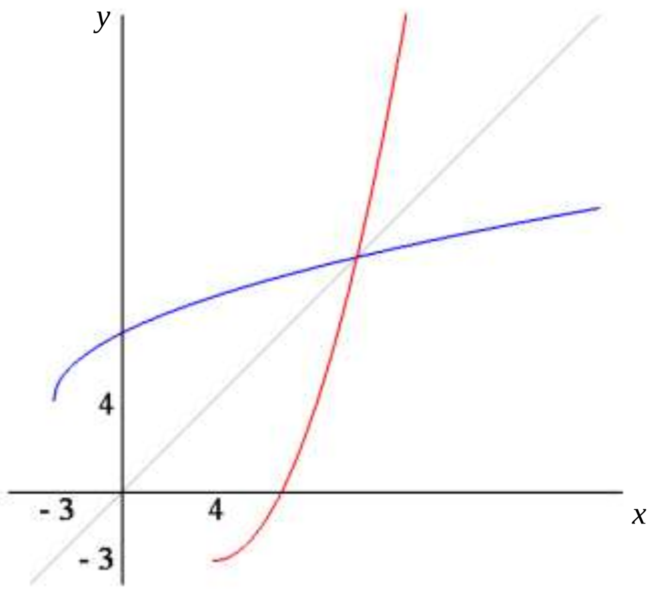
Since the domain of f^{-1} is the range of f, then find the range of f by finding the domain of f^{-1}.

For the domain of f^{-1}, we want that $y + 3 \geq 0$. Thus, $y + 3 \geq 0 \Rightarrow$ $y \geq -3$. The domain of f^{-1} is $[-3, \infty)$. Thus, the range of f is $[-3, \infty)$. This is the same answer that we obtained in Lesson 3 using the sketch of the graph of the function f.

Answer: $f^{-1}(y) = \dfrac{8 + \sqrt{2(y+3)}}{2}$ or $f^{-1}(y) = 4 + \dfrac{1}{2}\sqrt{2(y+3)}$

or $f^{-1}(x) = \dfrac{8 + \sqrt{2(x+3)}}{2}$ or $f^{-1}(x) = 4 + \dfrac{1}{2}\sqrt{2(x+3)}$

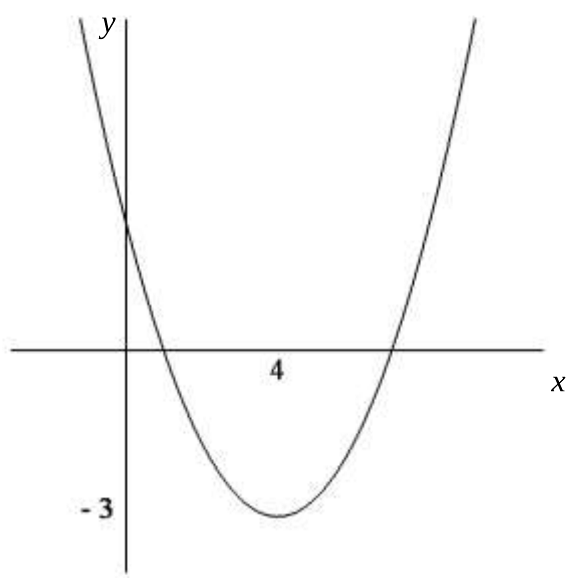
The graphs of the functions $y = f(x) = 2(x - 4)^2 - 3$ for $x \ge 4$ and $y = f^{-1}(x) = \frac{1}{2}\sqrt{2(x + 3)} + 4$ are given below. The graph of $y = f(x)$ is red and the graph of $y = f^{-1}(x)$ is blue:



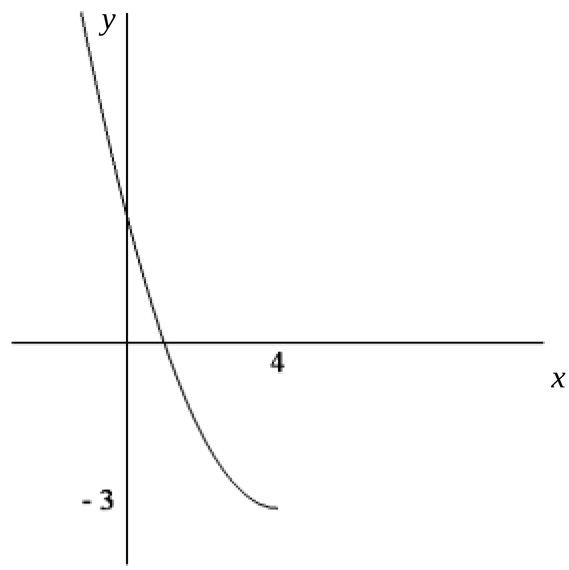
Notice that the graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ through the line $y = x$, which is gray.

2. $f(x) = 2(x - 4)^2 - 3, \; x \le 4$

The graph of the function $f(x) = 2(x - 4)^2 - 3$ was sketched in Lesson 3.



You can see from this sketch that the graph fails the Horizontal Line Test. Thus, the function $f(x) = 2(x - 4)^2 - 3$ is not one-to-one. Here is the sketch of the graph of the function for $x \leq 4$.



This graph does pass the Horizontal Line Test. Thus, the function $f(x) = 2(x - 4)^2 - 3$ for $x \leq 4$ is one-to-one.

Restricted Domain of $f = (-\infty, 4] =$ Range of f^{-1}

Range of $f = [-3, \infty) =$ Domain of f^{-1}

Now, we will find f^{-1}.

Setting $f(x) = y$, we have that $y = 2(x - 4)^2 - 3$. Now solving for x, we have that

$$y = 2(x - 4)^2 - 3 \Rightarrow y + 3 = 2(x - 4)^2 \Rightarrow (x - 4)^2 = \frac{y + 3}{2} \Rightarrow$$

$x - 4 = \pm\sqrt{\dfrac{y + 3}{2}}$. Since $x \leq 4$, then $x - 4 \leq 0$. That is, $x - 4$ is not positive. Thus, $x - 4 = -\sqrt{\dfrac{y + 3}{2}} \Rightarrow x - 4 = -\sqrt{\dfrac{2(y + 3)}{4}} \Rightarrow$

$$x - 4 = -\frac{\sqrt{2(y+3)}}{2} \Rightarrow x = \frac{8}{2} - \frac{\sqrt{2(y+3)}}{2} \Rightarrow$$

$$x = \frac{8 - \sqrt{2(y+3)}}{2}. \text{ Thus, } f^{-1}(y) = \frac{8 - \sqrt{2(y+3)}}{2}.$$

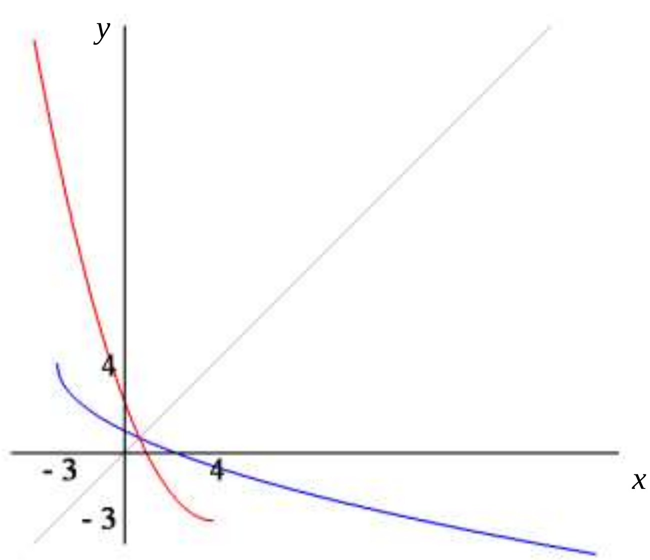
Since the domain of f^{-1} is the range of f, then find the range of f by finding the domain of f^{-1}.

For the domain of f^{-1}, we want that $y + 3 \geq 0$. Thus, $y + 3 \geq 0 \Rightarrow$ $y \geq -3$. The domain of f^{-1} is $[-3, \infty)$. Thus, the range of f is $[-3, \infty)$. This is the same answer that we obtained in Lesson 3 using the sketch of the graph of the function f.

Answer: $f^{-1}(y) = \dfrac{8 - \sqrt{2(y+3)}}{2}$ or $f^{-1}(y) = 4 - \dfrac{1}{2}\sqrt{2(y+3)}$

or $f^{-1}(x) = \dfrac{8 - \sqrt{2(x+3)}}{2}$ or $f^{-1}(x) = 4 - \dfrac{1}{2}\sqrt{2(x+3)}$

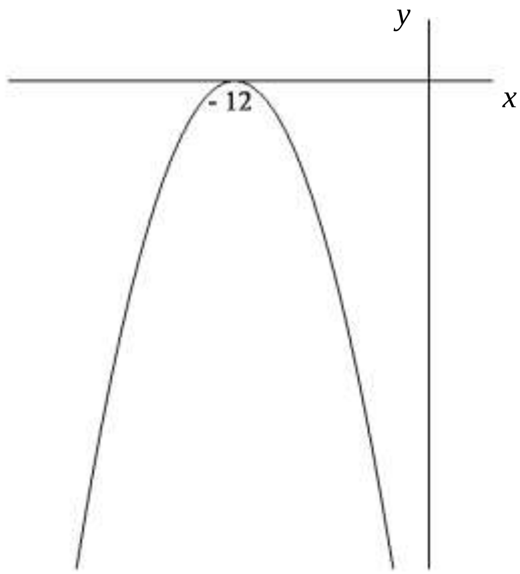
The graphs of the functions $y = f(x) = 2(x-4)^2 - 3$ for $x \leq 4$ and $y = f^{-1}(x) = -\dfrac{1}{2}\sqrt{2(x+3)} + 4$ are given below. The graph of $y = f(x)$ is red and the graph of $y = f^{-1}(x)$ is blue:



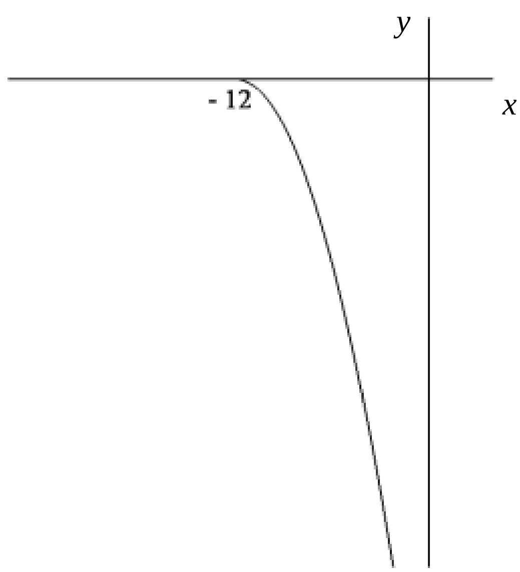
Notice that the graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ through the line $y = x$, which is gray.

3. $g(x) = -5(x + 12)^2, \ x \geq -12$

The graph of the function $g(x) = -5(x + 12)^2$ was sketched in Lesson 3.



You can see from this sketch that the graph fails the Horizontal Line Test. Thus, the function $g(x) = -5(x + 12)^2$ is not one-to-one. Here is the sketch of the graph of the function for $x \geq -12$.



This graph does pass the Horizontal Line Test. Thus, the function

$g(x) = -5(x+12)^2$ for $x \geq -12$ is one-to-one.

Restricted Domain of $g = [-12, \infty) =$ Range of g^{-1}

Range of $g = (-\infty, 0] =$ Domain of g^{-1}

Now, we will find g^{-1}.

Setting $g(x) = y$, we have that $y = -5(x+12)^2$. Now solving for x, we have that

$$y = -5(x+12)^2 \Rightarrow (x+12)^2 = -\frac{y}{5} \Rightarrow x + 12 = \pm\sqrt{-\frac{y}{5}}.$$

Since $x \geq -12$, then $x + 12 \geq 0$. That is, $x + 12$ is

nonnegative. Thus, $x + 12 = \sqrt{-\frac{y}{5}} \Rightarrow x + 12 = \sqrt{-\frac{5y}{25}} \Rightarrow$

$$x + 12 = \frac{\sqrt{-5y}}{5} \Rightarrow x = -\frac{60}{5} + \frac{\sqrt{-5y}}{5} \Rightarrow x = \frac{\sqrt{-5y} - 60}{5}.$$

Thus, $g^{-1}(y) = \dfrac{\sqrt{-5y} - 60}{5}$.

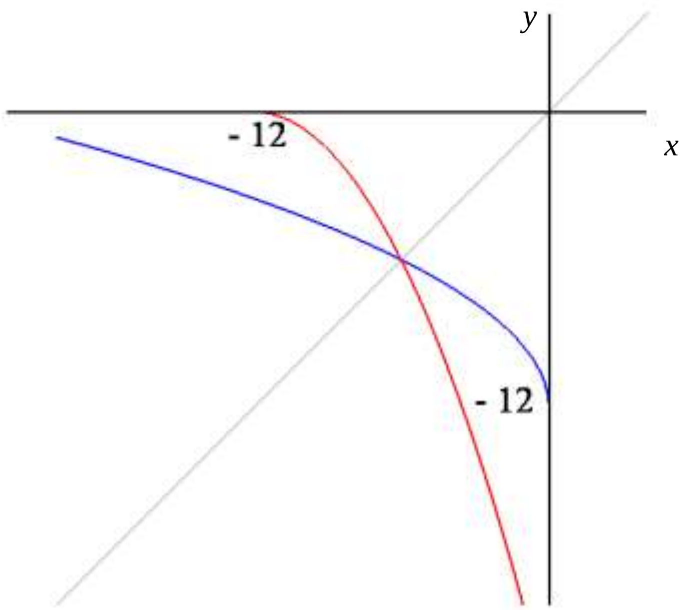
Since the domain of g^{-1} is the range of g, then find the range of g by finding the domain of g^{-1}.

For the domain of g^{-1}, we want that $-5y \geq 0$. Thus, $-5y \geq 0 \Rightarrow$ $y \leq 0$. The domain of g^{-1} is $(-\infty, 0]$. Thus, the range of g is $(-\infty, 0]$. This is the same answer that we obtained in Lesson 3 using the sketch of the graph of the function g.

Answer: $g^{-1}(y) = \dfrac{\sqrt{-5y} - 60}{5}$ or $g^{-1}(y) = \dfrac{1}{5}\sqrt{-5y} - 12$

or $g^{-1}(x) = \dfrac{\sqrt{-5x} - 60}{5}$ or $g^{-1}(x) = \dfrac{1}{5}\sqrt{-5x} - 12$

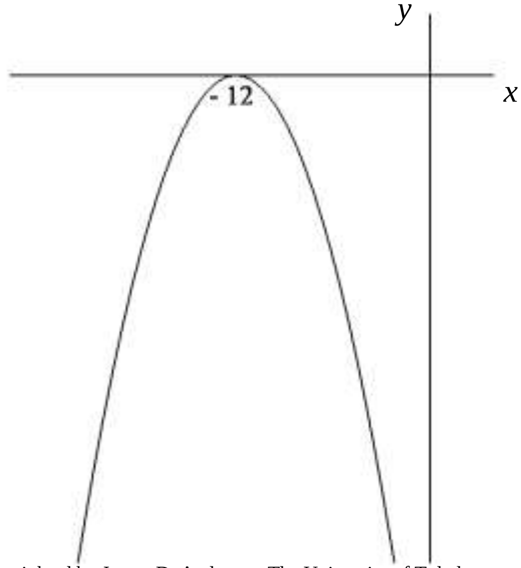
The graphs of the functions $y = g(x) = -5(x+12)^2$ for $x \geq -12$ and $y = g^{-1}(x) = \frac{1}{5}\sqrt{-5x} - 12$ are given below. The graph of $y = g(x)$ is red and the graph of $y = g^{-1}(x)$ is blue:



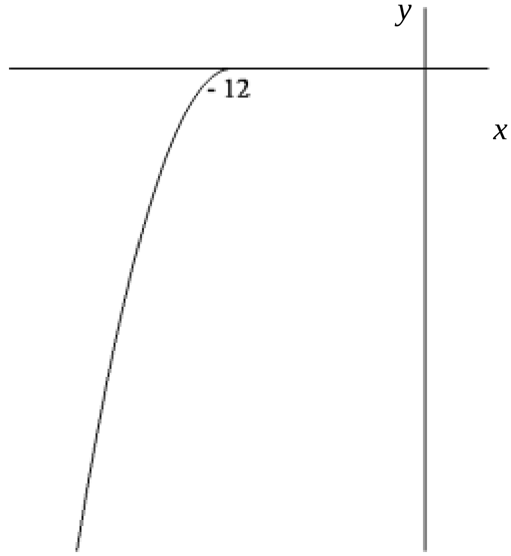
Notice that the graph of $y = g^{-1}(x)$ is a reflection of the graph of $y = g(x)$ through the line $y = x$, which is gray.

4. $g(x) = -5(x+12)^2, \ x \leq -12$

The graph of the function $g(x) = -5(x+12)^2$ was sketched in Lesson 3.



You can see from this sketch that the graph fails the Horizontal Line Test. Thus, the function $g(x) = -5(x + 12)^2$ is not one-to-one. Here is the sketch of the graph of the function for $x \le -12$.



This graph does pass the Horizontal Line Test. Thus, the function $g(x) = -5(x + 12)^2$ for $x \le -12$ is one-to-one.

Restricted Domain of $g = (-\infty, -12] =$ Range of g^{-1}

Range of $g = (-\infty, 0] =$ Domain of g^{-1}

Now, we will find g^{-1}.

Setting $g(x) = y$, we have that $y = -5(x + 12)^2$. Now solving for x, we have that

$$y = -5(x + 12)^2 \Rightarrow (x + 12)^2 = -\frac{y}{5} \Rightarrow x + 12 = \pm\sqrt{-\frac{y}{5}}.$$

Since $x \le -12$, then $x + 12 \le 0$. That is, $x + 12$ is

not positive. Thus, $x + 12 = -\sqrt{-\frac{y}{5}} \Rightarrow x + 12 = -\sqrt{-\frac{5y}{25}} \Rightarrow$

$$x + 12 = -\frac{\sqrt{-5y}}{5} \Rightarrow x = -\frac{60}{5} - \frac{\sqrt{-5y}}{5} \Rightarrow x = \frac{-60 - \sqrt{-5y}}{5}$$

$$\Rightarrow x = -\frac{60 + \sqrt{-5y}}{5}$$

Thus, $g^{-1}(y) = -\dfrac{60 + \sqrt{-5y}}{5}$.

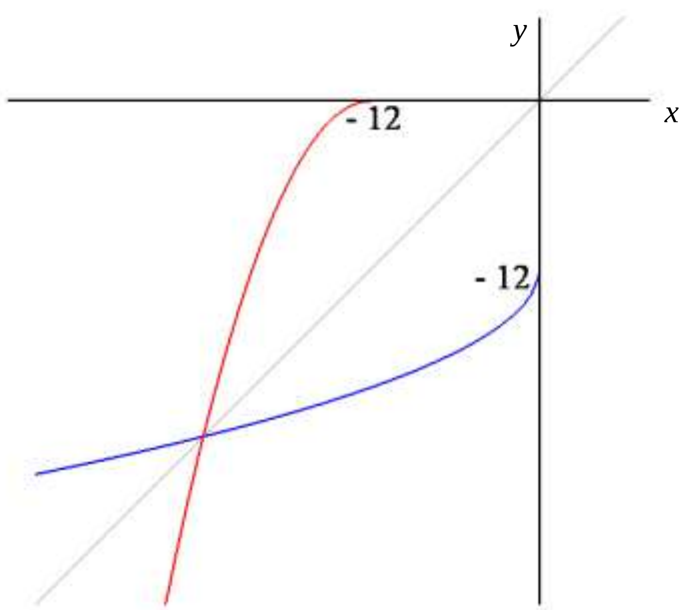
Since the domain of g^{-1} is the range of g, then find the range of g by finding the domain of g^{-1}.

For the domain of g^{-1}, we want that $-5y \geq 0$. Thus, $-5y \geq 0 \Rightarrow y \leq 0$. The domain of g^{-1} is $(-\infty, 0]$. Thus, the range of g is $(-\infty, 0]$. This is the same answer that we obtained in Lesson 3 using the sketch of the graph of the function g.

Answer: $g^{-1}(y) = -\dfrac{60 + \sqrt{-5y}}{5}$ or $g^{-1}(y) = -\dfrac{1}{5}\sqrt{-5y} - 12$

or $g^{-1}(x) = -\dfrac{60 + \sqrt{-5x}}{5}$ or $g^{-1}(x) = -\dfrac{1}{5}\sqrt{-5x} - 12$

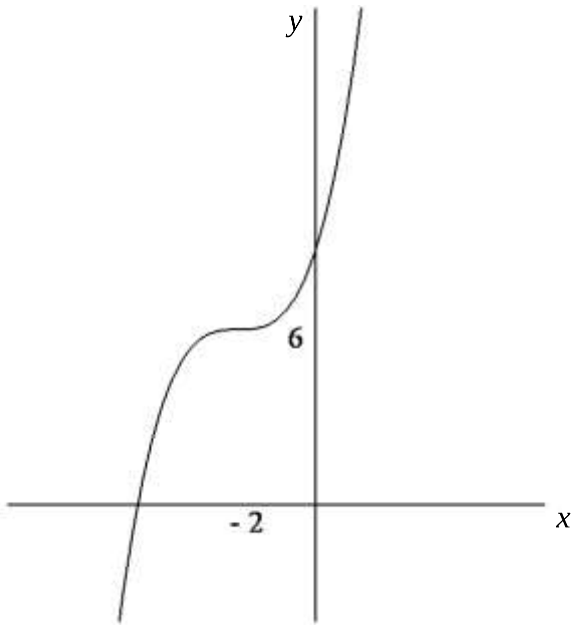
The graphs of the functions $y = g(x) = -5(x + 12)^2$ for $x \leq -12$ and $y = g^{-1}(x) = -\dfrac{1}{5}\sqrt{-5x} - 12$ are given below. The graph of $y = g(x)$ is red and the graph of $y = g^{-1}(x)$ is blue:



Notice that the graph of $y = g^{-1}(x)$ is a reflection of the graph of $y = g(x)$ through the line $y = x$, which is gray.

5. $h(x) = \dfrac{3}{4}(x + 2)^3 + 6$

The graph of this function was sketched in Lesson 3.



This graph passes the Horizontal Line Test. Thus, the function h is one-to-one.

Domain of $h = (-\infty, \infty) =$ Range of h^{-1}

Range of $h = (-\infty, \infty) =$ Domain of h^{-1}

Now, we will find h^{-1}.

Setting $h(x) = y$, we have that $y = \dfrac{3}{4}(x + 2)^3 + 6$. Now solving for x, we have that

$$y = \frac{3}{4}(x + 2)^3 + 6 \Rightarrow y - 6 = \frac{3}{4}(x + 2)^3 \Rightarrow$$

$$(x+2)^3 = \frac{4}{3}(y-6) \quad \Rightarrow \quad (x+2)^3 = \frac{4(y-6)}{3} \quad \Rightarrow$$

$$x + 2 = \sqrt[3]{\frac{4(y-6)}{3}} \quad \Rightarrow \quad x + 2 = \sqrt[3]{\frac{4(y-6)\cdot 3^2}{3\cdot 3^2}} \quad \Rightarrow$$

$$x + 2 = \sqrt[3]{\frac{36(y-6)}{3^3}} \quad \Rightarrow \quad x + 2 = \frac{\sqrt[3]{36(y-6)}}{3} \quad \Rightarrow$$

$$x = -\frac{6}{3} + \frac{\sqrt[3]{36(y-6)}}{3} \quad \Rightarrow \quad x = \frac{\sqrt[3]{36(y-6)} - 6}{3}$$

Thus, $h^{-1}(y) = \dfrac{\sqrt[3]{36(y-6)} - 6}{3}$.

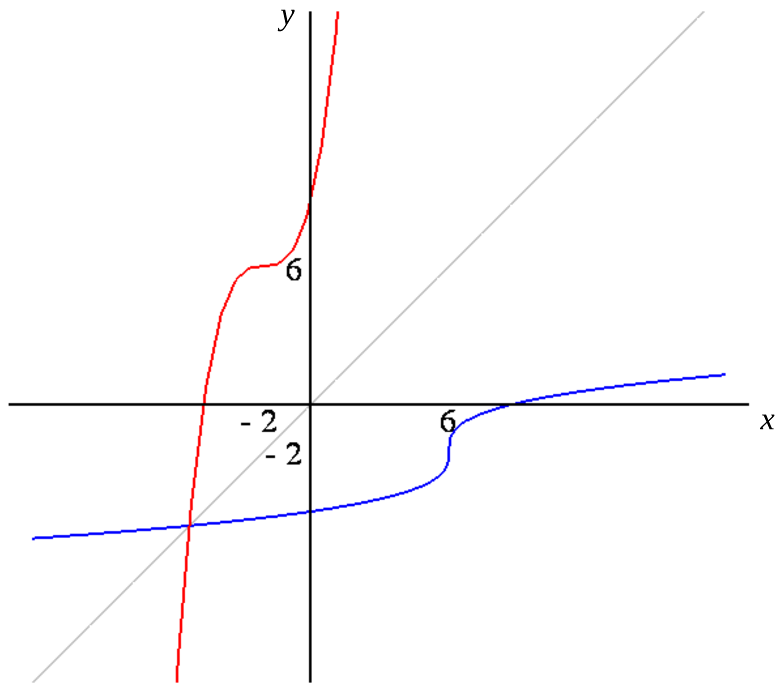
Since the domain of h^{-1} is the range of h, then find the range of h by finding the domain of h^{-1}.

The domain of h^{-1} is the set of all real numbers. Thus, the range of h is the set of all real numbers. This is the same answer that we obtained in Lesson 3 using the sketch of the graph of the function h.

Answer: $h^{-1}(y) = \dfrac{\sqrt[3]{36(y-6)} - 6}{3}$ or $h^{-1}(y) = \dfrac{1}{3}\sqrt[3]{36(y-6)} - 2$

or $h^{-1}(x) = \dfrac{\sqrt[3]{36(x-6)} - 6}{3}$ or $h^{-1}(x) = \dfrac{1}{3}\sqrt[3]{36(x-6)} - 2$

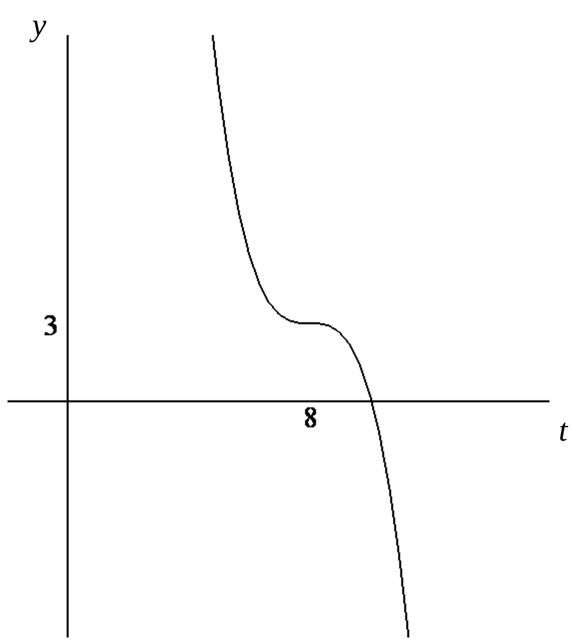
The graphs of the functions $y = h(x) = \dfrac{3}{4}(x+2)^3 + 6$ and $y = h^{-1}(x) = \dfrac{1}{3}\sqrt[3]{36(x-6)} - 2$ are given below. The graph of $y = h(x)$ is red and the graph of $y = h^{-1}(x)$ is blue:



Notice that the graph of $y = h^{-1}(x)$ is a reflection of the graph of $y = h(x)$ through the line $y = x$, which is gray.

6. $f(t) = -(t - 8)^3 + 3$

The graph of this function was sketched in Lesson 3.



This graph passed the Horizontal Line Test. Thus, the function f is one-to-one.

Domain of $f = (-\infty, \infty) =$ Range of f^{-1}

Range of $f = (-\infty, \infty) =$ Domain of f^{-1}

Now, we will find f^{-1}.

Setting $f(t) = y$, we have that $y = -(t - 8)^3 + 3$. Now solving for t, we have that

$$y = -(t - 8)^3 + 3 \Rightarrow (t - 8)^3 = 3 - y \Rightarrow$$

$$t - 8 = \sqrt[3]{3 - y} \Rightarrow t = 8 + \sqrt[3]{3 - y}$$

Thus, $f^{-1}(y) = 8 + \sqrt[3]{3 - y}$.

NOTE: $\sqrt[3]{3 - y} = \sqrt[3]{-y + 3} = \sqrt[3]{-(y - 3)} = \sqrt[3]{-1}\sqrt[3]{y - 3} =$

$-\sqrt[3]{y - 3}$

Thus, $f^{-1}(y) = 8 - \sqrt[3]{y - 3}$.

Since the domain of f^{-1} is the range of f, then find the range of f by finding the domain of f^{-1}.

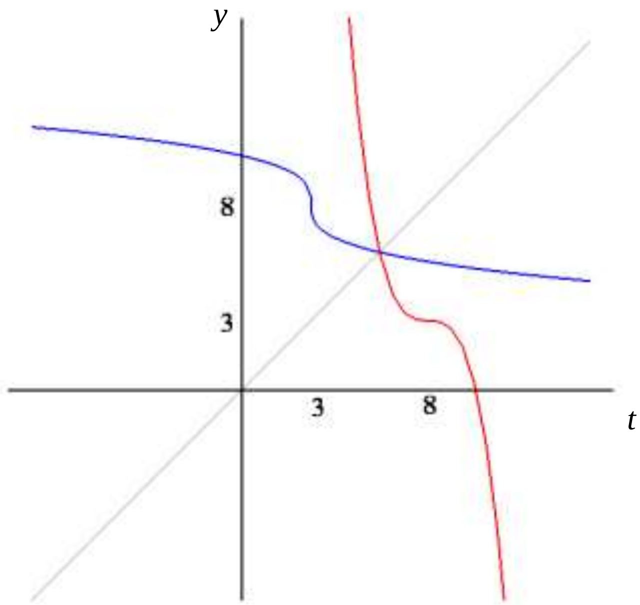
The domain of f^{-1} is the set of all real numbers. Thus, the range of f is the set of all real numbers. This is the same answer that we obtained in Lesson 3 using the sketch of the graph of the function f.

Answer: $f^{-1}(y) = 8 - \sqrt[3]{y - 3}$ or $f^{-1}(y) = -\sqrt[3]{y - 3} + 8$

or $f^{-1}(t) = 8 - \sqrt[3]{t - 3}$ or $f^{-1}(t) = -\sqrt[3]{t - 3} + 8$

The graphs of the functions $y = f(t) = -(t - 8)^3 + 3$ and

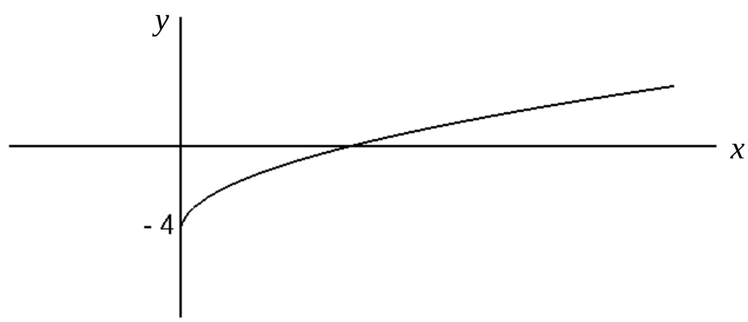
$y = f^{-1}(t) = -\sqrt[3]{t - 3} + 8$ are given below. The graph of $y = f(t)$ is red and the graph of $y = f^{-1}(t)$ is blue:



Notice that the graph of $y = f^{-1}(t)$ is a reflection of the graph of $y = f(t)$ through the line $y = t$, which is gray.

7. $g(x) = 2\sqrt{x} - 4$

The graph of this function was sketched in Lesson 3.



This graph passes the Horizontal Line Test. Thus, the function g is one-to-one.

Domain of $g = [0, \infty) =$ Range of g^{-1}

Range of $g = [-4, \infty) =$ Domain of g^{-1}

Now, we will find g^{-1}.

Setting $g(x) = y$, we have that $y = 2\sqrt{x} - 4$. Now solving for x, we have that

$$y = 2\sqrt{x} - 4 \Rightarrow y + 4 = 2\sqrt{x} \Rightarrow \sqrt{x} = \frac{y+4}{2} \Rightarrow$$

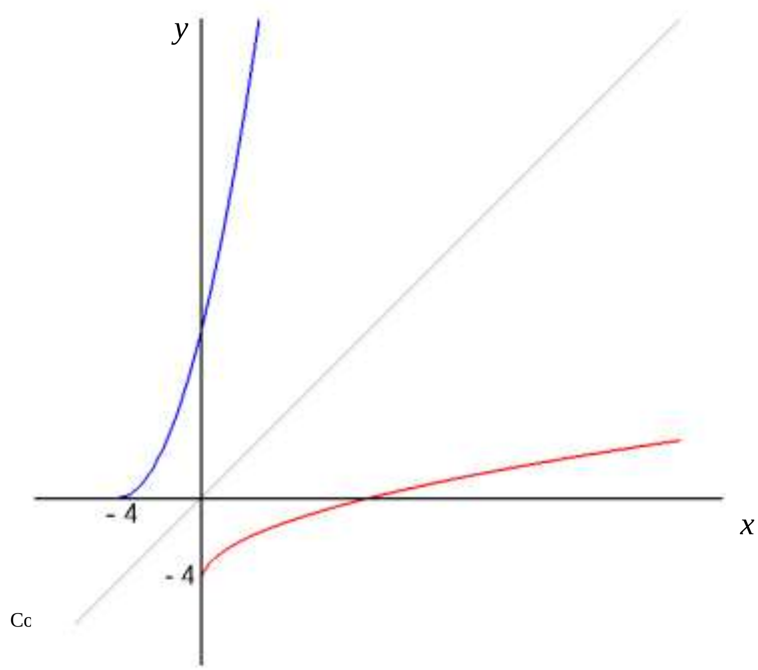
$$x = \frac{(y+4)^2}{4} \Rightarrow x = \frac{1}{4}(y+4)^2.$$

Since the domain of g^{-1} is $[-4, \infty)$, then $g^{-1}(y) = \frac{1}{4}(y+4)^2$, where $y \geq -4$.

Answer: $g^{-1}(y) = \frac{1}{4}(y+4)^2, \ y \geq -4$

or $g^{-1}(x) = \frac{1}{4}(x+4)^2, \ x \geq -4$

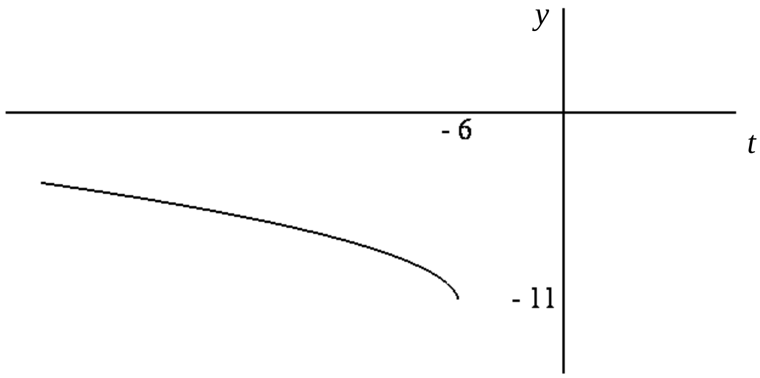
The graphs of the functions $y = g(x) = 2\sqrt{x} - 4$ and $y = g^{-1}(x) = \frac{1}{4}(x+4)^2$ for $x \geq -4$ are given below. The graph of $y = g(x)$ is red and the graph of $y = g^{-1}(x)$ is blue:



Notice that the graph of $y = g^{-1}(x)$ is a reflection of the graph of $y = g(x)$ through the line $y = x$, which is gray.

8. $h(t) = \sqrt{-5t - 30} - 11$

The graph of this function was sketched in Lesson 3.



This graph passes the Horizontal Line Test. Thus, the function h is one-to-one.

Domain of $h = (-\infty, -6] =$ Range of h^{-1}

Range of $h = [-11, \infty) =$ Domain of h^{-1}

Now, we will find h^{-1}.

Setting $h(t) = y$, we have that $y = \sqrt{-5t - 30} - 11$. Now solving for *x*, we have that

$$y = \sqrt{-5t - 30} - 11 \Rightarrow y + 11 = \sqrt{-5t - 30} \Rightarrow$$

$$-5t - 30 = (y + 11)^2 \Rightarrow -5t = (y + 11)^2 + 30 \Rightarrow$$

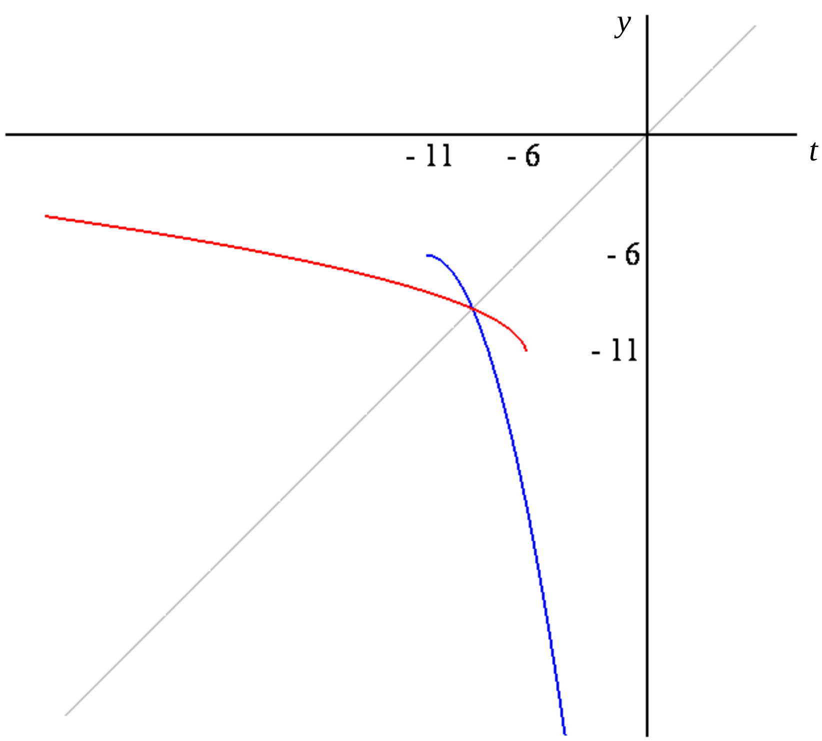
$$t = -\frac{1}{5}(y + 11)^2 - 6$$

Since the domain of h^{-1} is $[-11, \infty)$, then $h^{-1}(y) = -\frac{1}{5}(y + 11)^2 - 6$, where $y \geq -11$.

Answer: $h^{-1}(y) = -\frac{1}{5}(y + 11)^2 - 6, \ y \geq -11$

or $h^{-1}(t) = -\frac{1}{5}(t + 11)^2 - 6, \ t \geq -11$

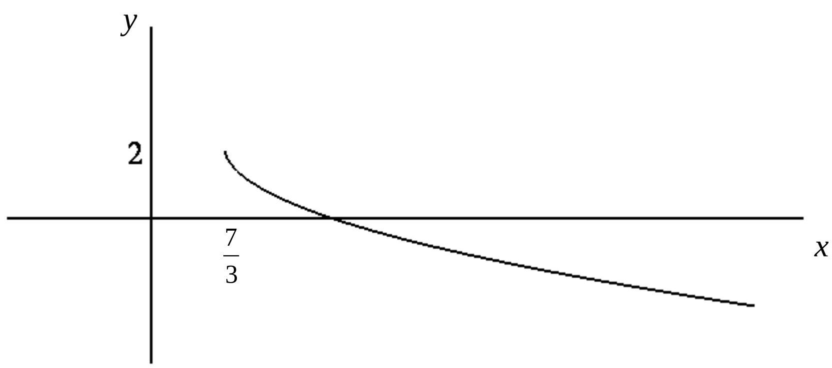
The graphs of the functions $y = h(t) = \sqrt{-5t - 30} - 11$ and $y = h^{-1}(t) = -\frac{1}{5}(t + 11)^2 - 6$ for $t \geq -11$ are given below. The graph of $y = h(t)$ is red and the graph of $y = h^{-1}(t)$ is blue:



Notice that the graph of $y = h^{-1}(t)$ is a reflection of the graph of $y = h(t)$ through the line $y = t$, which is gray.

9. $y = -\frac{11}{6}\sqrt{3x - 7} + 2$

The graph of this function was sketched in Lesson 3.



This graph passes the Horizontal Line Test. Thus, the function h is one-to-one.

Domain of function $= \left[\frac{7}{3}, \infty\right)$ = Range of inverse function

Range of function $= (-\infty, 2]$ = Domain of inverse function

Now, we will find the inverse function.

Solving for x in $y = -\frac{11}{6}\sqrt{3x - 7} + 2$, we have that

$$y = -\frac{11}{6}\sqrt{3x - 7} + 2 \Rightarrow y - 2 = -\frac{11}{6}\sqrt{3x - 7} \Rightarrow$$

$$-\frac{6}{11}(y - 2) = \sqrt{3x - 7} \Rightarrow 3x - 7 = \frac{36}{121}(y - 2)^2 \Rightarrow$$

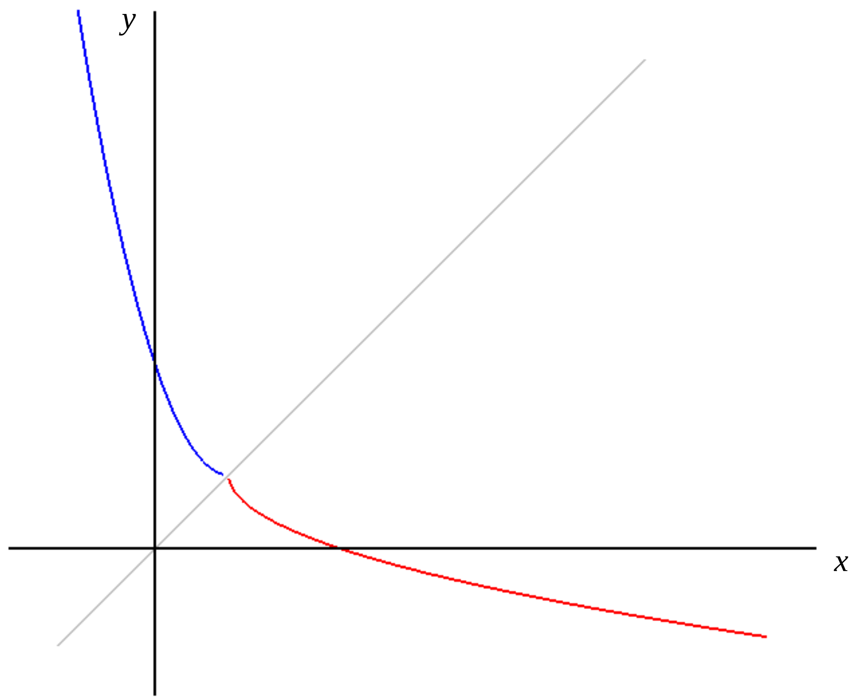
$$3x = \frac{36}{121}(y - 2)^2 + 7 \Rightarrow x = \frac{12}{121}(y - 2)^2 + \frac{7}{3}$$

Since the domain of the inverse function is $(-\infty, 2]$, then
$x = \frac{12}{121}(y - 2)^2 + \frac{7}{3}$, where $y \le 2$.

Answer: $x = \frac{12}{121}(y - 2)^2 + \frac{7}{3}, \ y \le 2$

or $y = \frac{12}{121}(x - 2)^2 + \frac{7}{3}, \; x \le 2$

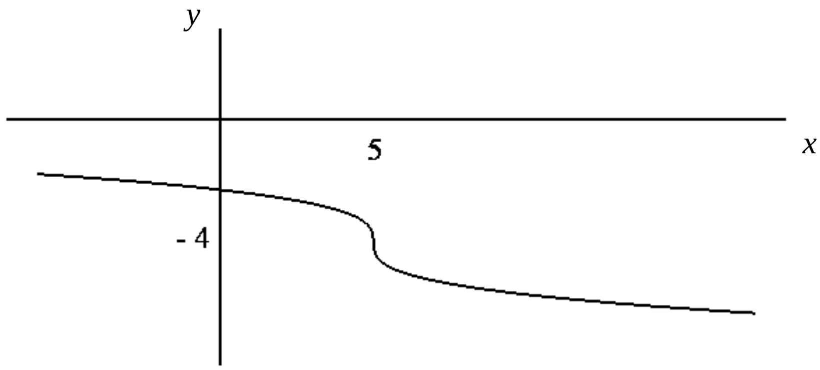
The graphs of the function $y = -\frac{11}{6}\sqrt{3x - 7} + 2$ and its inverse function $y = \frac{12}{121}(x - 2)^2 + \frac{7}{3}$ for $x \le 2$ are given below. The graph of $y = -\frac{11}{6}\sqrt{3x - 7} + 2$ is red and the graph of $y = \frac{12}{121}(x - 2)^2 + \frac{7}{3}$ for $x \le 2$ is blue:



Notice that the graph of the inverse function is a reflection of the graph of the function $y = -\frac{11}{6}\sqrt{3x - 7} + 2$ through the line $y = x$, which is gray.

10. $f(x) = \sqrt[3]{5 - x} - 4$

The graph of this function was sketched in Lesson 3.



This graph passes the Horizontal Line Test. Thus, the function h is one-to-one.

Domain of $f = (-\infty, \infty) =$ Range of f^{-1}

Range of $f = (-\infty, \infty) =$ Domain of f^{-1}

Now, we will find f^{-1}.

Setting $f(x) = y$, we have that $y = \sqrt[3]{5 - x} - 4$. Now solving for x, we have that

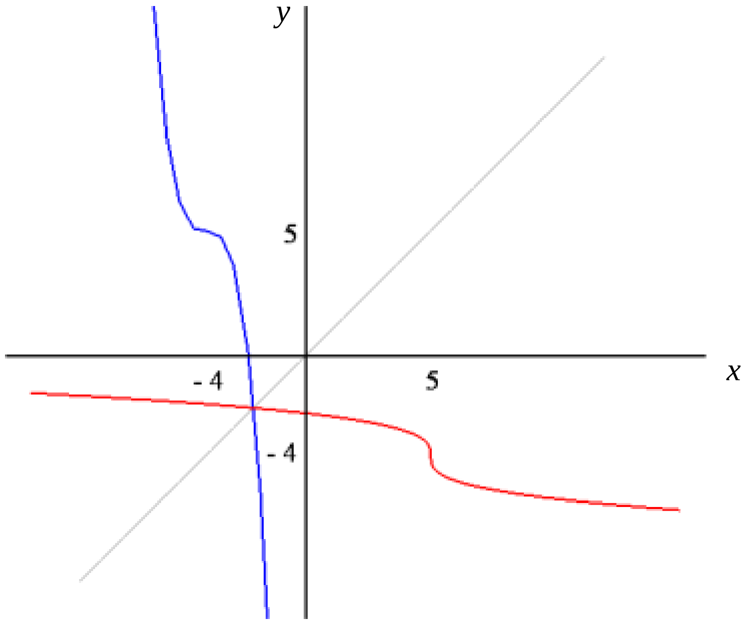
$$y = \sqrt[3]{5 - x} - 4 \Rightarrow y + 4 = \sqrt[3]{5 - x} \Rightarrow$$

$$(y + 4)^3 = 5 - x \Rightarrow x = 5 - (y + 4)^3 \Rightarrow x = -(y + 4)^3 + 5$$

Thus, $f^{-1}(y) = -(y + 4)^3 + 5$.

Answer: $f^{-1}(y) = -(y + 4)^3 + 5$ or $f^{-1}(x) = -(x + 4)^3 + 5$

The graphs of the functions $y = f(x) = \sqrt[3]{5 - x} - 4$ and $y = f^{-1}(x) = -(x + 4)^3 + 5$ are given below. The graph of $y = f(x)$ is red and the graph of $y = f^{-1}(x)$ is blue:



Notice that the graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ through the line $y = x$, which is gray.

11. $f(x) = \dfrac{4}{3x^2} + 8, \quad x > 0$

12. $f(x) = \dfrac{4}{3x^2} + 8, \quad x < 0$

13. $g(x) = -\dfrac{14}{2x + 5} - 6$