LESSON 3 HORIZONTAL AND VERTICAL SHIFTS

Definition The graph of a function f is the set of points in the *xy*-plane of the form (x, y), where $y = f(x)$ and x is in the domain of *f*. Thus, the graph of the function is the collection of the points $(x, f(x))$ in the *xy*-plane.

Example The graph of the function f given by $f(x) = x^2$ is the set $\{(x, x^2) : x \text{ is a real number}\}$. Thus, the points $(3, 9)$, $(-2, 4)$, $(\sqrt{6}, 6)$ and $\left(-\frac{5}{7}, \frac{25}{49}\right)$ are some of the points in this set. Note that the domain of the function is the set of all real numbers.

Example The graph of the function g given by $g(x) = \sqrt{x - 4}$ is the set $\{(x, \sqrt{x-4}) : x \geq 4\}$. Thus, the points $(5, 1)$, $(7, \sqrt{3})$, $(16, 2\sqrt{3})$, $\left(\sqrt{17}, \sqrt{\sqrt{17} - 4}\right)$ and $\left(\frac{41}{9}, \frac{\sqrt{5}}{3}\right)$ are some of the points in this set. Note that the domain of the function is the set of all real numbers given by the interval $[4, \infty)$.

NOTE: $g(16) = \sqrt{12} = 2\sqrt{3}$

$$g\left(\frac{41}{9}\right) = \sqrt{\frac{41}{9} - 4} = \sqrt{\frac{41}{9} - \frac{36}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

COMMENT: A positive real number that is multiplied to the *x* and/or *y* variable(s) in an equation involving x and y will change the shape of the graph of the equation but will not shift the graph.

Example The graph of the functions *f*, *g*, and *h* given by $f(x) = x^2$, $g(x) = 3x^2$, and $h(x) = \frac{1}{2}x^2$ respectively are shown below.

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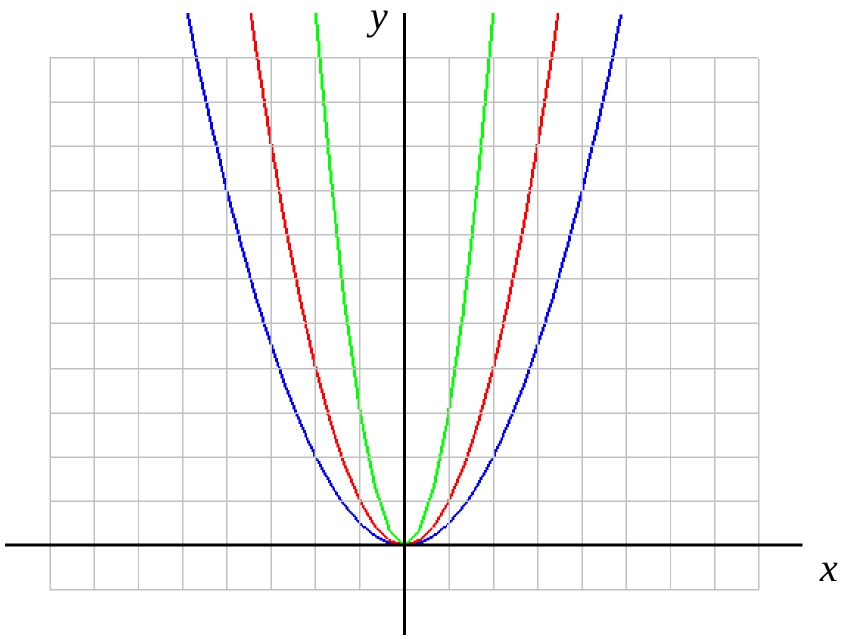
NOTE: In order to graph the function f given by $f(x) = x^2$, we set $f(x) = y$ and graph the equation $y = x^2$. In order to graph the function g given by $g(x) = 3x^2$, we set $g(x) = y$ and graph the equation $y = 3x^2$. In order to graph the function h given by $h(x) = \frac{1}{2}x^2$, we set $h(x) = y$ and graph the equation $y = \frac{1}{2}x^2$.

The graph of f is red, the graph of g is green, and the graph of h is blue.



Definition The sketch of the graph of a function is a graph that is drawn from memory usually plotting only one point. If the graph has a horizontal and/or vertical asymptote(s), then we would draw them as dotted lines.

For example, we know that the graph of the function given by $y = x^2$ is a parabola whose vertex is at the origin and opens upward. The reason that we know this is because at one time we drew the graph of this function by plotting points that we would have obtained by the use of a table with a column labeled as *x* and a column labeled as *y*. In order to create this table, we would have listed some of the numbers in the domain of the function in the column labeled as *x*. We probably used -3, -2, -1, 0, 1, 2, and 3. Then in order to get the numbers in the column labeled as *y*, we would have calculated the value of the function at these numbers

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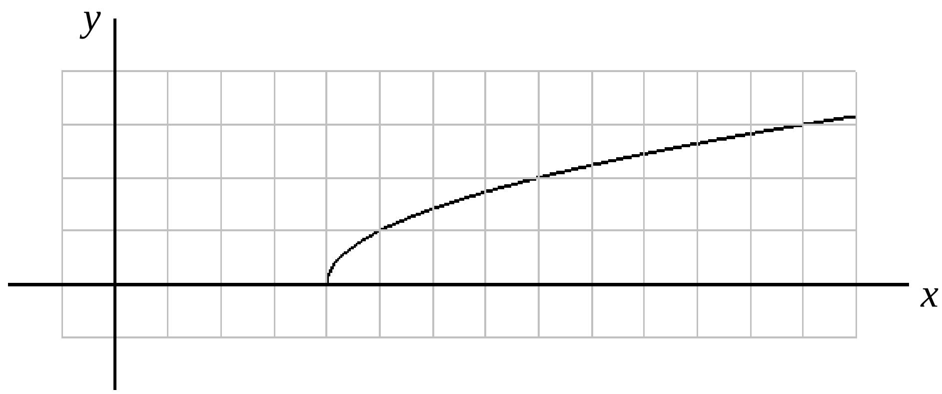
obtaining 9, 4, 1, 0, 1, 4, and 9. Then we would have plotted the points $(-3, 9)$, $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 4)$, and $(3, 9)$ in the *xy*-plane.

COMMENT: Most of the time in mathematics, we are only interested in a sketch of the graph of a function. In this case, we do not have to worry about the true shape of the graph. However, we will have to worry about horizontal and/or vertical shift(s) of the graph of function.

Example We talked about the graph of the function g given by $g(x) = \sqrt{x - 4}$ above. The graph of this function is given below.

NOTE: To graph the function *g*, we set $g(x) = y$ and graph the equation $y = \sqrt{x - 4}$.



The graph of $y = \sqrt{x - 4}$ is the graph of $y = \sqrt{x}$ shifted 4 units to the right.

NOTE: The domain of the function g is $[4, \infty)$ and the range of the function is $[0, \infty)$.

Horizontal Shifts A positive real number that is added or subtracted to the *x* variable in an equation involving x and y will produce a shift with respect to the *x*-axis in the *xy*-plane. Since you can only move to the left or to the right on the *x*-axis, the shift is to the left or to the right and is called a horizontal shift.

Let c be a positive real number. Then

1. The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted c units to the right.

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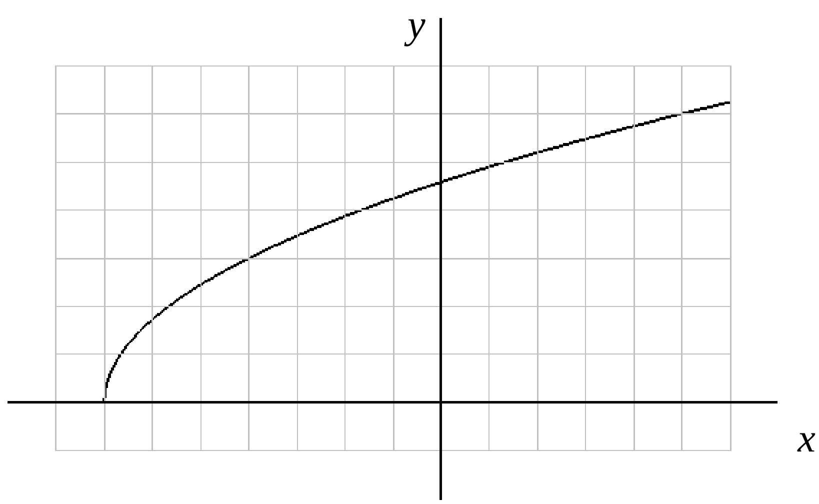
2. The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted c units to the left.

In order to identify the amount and direction of a horizontal shift, the coefficient of the x variable must be one.

Let b and c be nonzero real numbers. Given $y = f(bx + c)$, we can factor out the b, which is the coefficient of x, obtaining that $y = f\left[b\left(x + \frac{c}{b}\right)\right]$. The graph of $y = f\left[b\left(x + \frac{c}{b}\right)\right]$, which is the graph of $y = f(bx + c)$, is the graph of $y = f(bx)$ shifted $\frac{c}{b}$ units to the right if $\frac{c}{b} < 0$ or to the left if $\frac{c}{b} > 0$. The shape of the graph of $y = f(bx)$ is similar to the shape of the graph of $y = f(x)$.

Example Graph the function f given by $f(x) = \sqrt{3x + 21}$.

NOTE: To graph the function f, we set $f(x) = y$ and graph the equation $y = \sqrt{3x + 21}$. Since $\sqrt{3x + 21} = \sqrt{3(x + 7)}$, then we graph the equation $y = \sqrt{3(x + 7)}$. The graph of $y = \sqrt{3(x + 7)}$ is the graph of $y = \sqrt{3x}$ shifted 7 units to the left.



NOTE: The domain of the function f is $[-7, \infty)$ and the range of the function is $[0, \infty)$.

Vertical Shifts A positive real number that is added or subtracted to the *y* variable in an equation involving x and y will produce a shift with respect to the *y*-axis in the *xy*-plane. Since you can only move upward or downward on the *y*-axis, the shift is upward or downward and is called a vertical shift.

Let c be a positive real number. Then

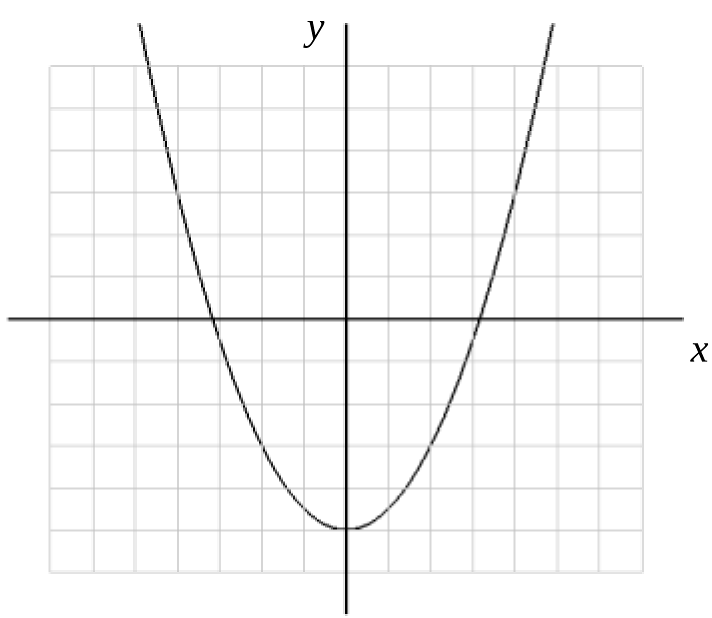
1. The graph of $y - c = f(x)$ is the graph of $y = f(x)$ shifted c units upward.

2. The graph of $y + c = f(x)$ is the graph of $y = f(x)$ shifted c units downward.

In order to identify the amount and direction of a vertical shift, the coefficient of the y variable must be one.

<u>Example</u> Graph the function h given by $h(x) = \dfrac{1}{2}x^2 - 5$.

NOTE: To graph the function h, we set $h(x) = y$ and graph the equation $y = \dfrac{1}{2}x^2 - 5$.



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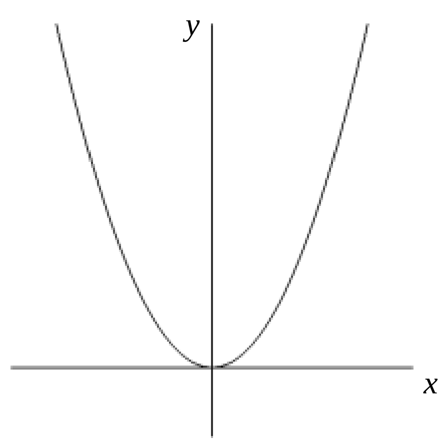
The equation $y = \frac{1}{2}x^2 - 5$ is the same as the equation $y + 5 = \frac{1}{2}x^2$. The graph of $y + 5 = \frac{1}{2}x^2$ is the graph of $y = \frac{1}{2}x^2$ shifted 5 units downward.

NOTE: The domain of the function h is the set of all real numbers and the range of the function is $[-5, \infty)$.

Examples Sketch the graph of the following equations. a is a positive real number.

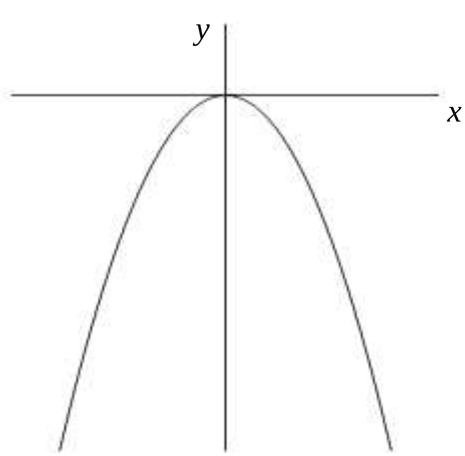
1. $y = a x^2$

NOTE: The graph of this equation is a parabola whose vertex is at the origin and opens upward.

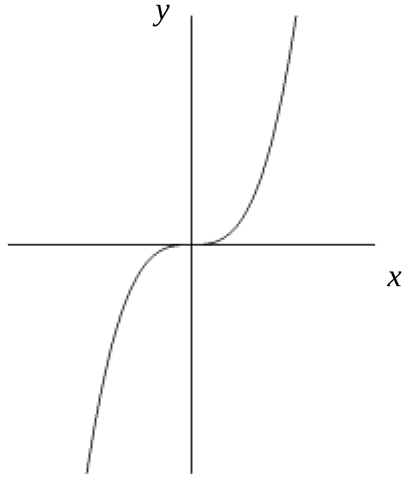


2. $y = -a x^2$

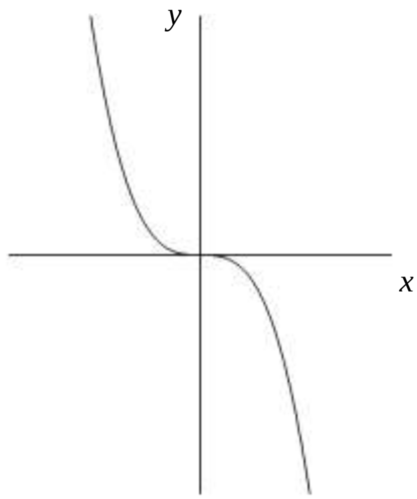
NOTE: The graph of this equation is a parabola whose vertex is at the origin and opens downward.



3. $y = ax^3$

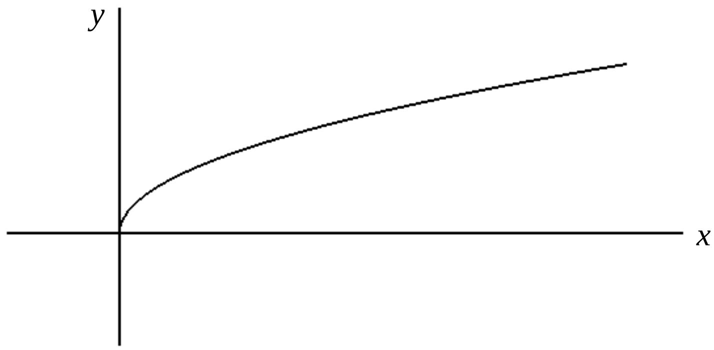


4. $y = -ax^3$



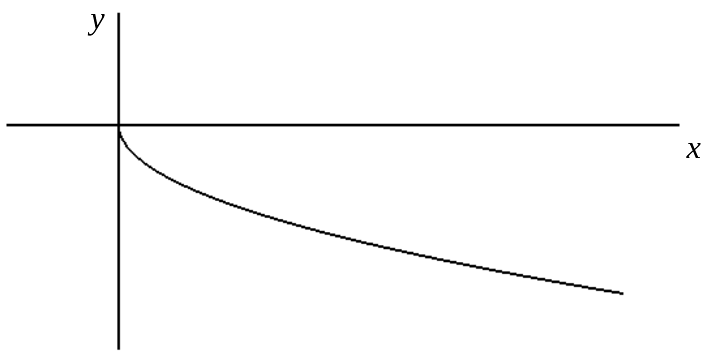
5. $y = \sqrt{ax}$ or $y = a\sqrt{x}$

NOTE: The graph of each equation is the top half of the parabola whose vertex is at the origin and opens to the right. The equation of the parabola whose vertex is at the origin and opens to the right can be given by the equation $x = ay^2$. Solving for y, we have that $x = ay^2 \Rightarrow y^2 = \frac{1}{a}x \Rightarrow y = \pm\sqrt{\frac{1}{a}x} = \pm\frac{1}{\sqrt{a}}\sqrt{x}$. Note, that the graph of the equation $y = -\sqrt{\frac{1}{a}x} = -\frac{1}{\sqrt{a}}\sqrt{x}$ is the bottom half of the parabola. We will show a sketch of these equations in Example 6.



6. $y = -\sqrt{a\,x}$ or $y = -a\sqrt{x}$

NOTE: The graph of each equation is the bottom half of the parabola whose vertex is at the origin and opens to the right.



7. $y = \sqrt{-a\,x}$ or $y = a\sqrt{-x}$

NOTE: The graph of each equation is the top half of the parabola whose vertex is at the origin and opens to the left. The equation of the parabola whose vertex is at the origin and opens to the left can be given by the equation $x = -a\,y^2$. Solving for y, we have that $x = -a\,y^2 \Rightarrow$ $y^2 = -\dfrac{1}{a}\,x \Rightarrow y = \pm\sqrt{-\dfrac{1}{a}\,x} = \pm\dfrac{1}{\sqrt{a}}\sqrt{-x}$. Note, that the graph of the equation $y = -\sqrt{-\dfrac{1}{a}\,x} = -\dfrac{1}{\sqrt{a}}\sqrt{-x}$ is the bottom half of the parabola. We will show a sketch of these equations in Example 8.

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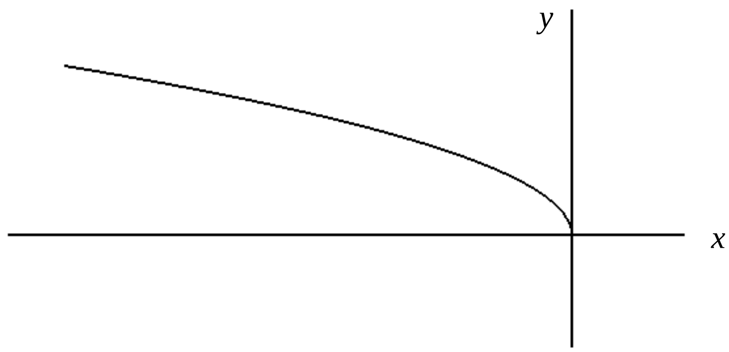
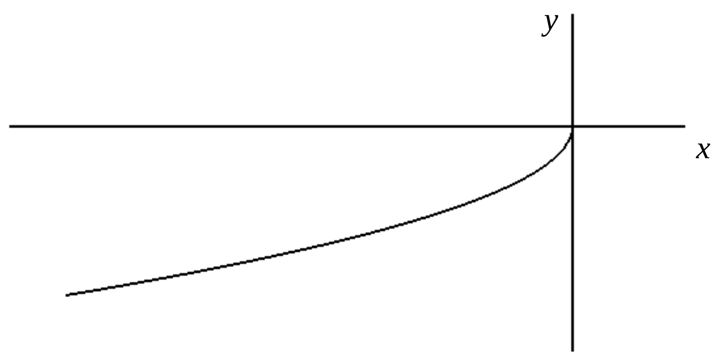
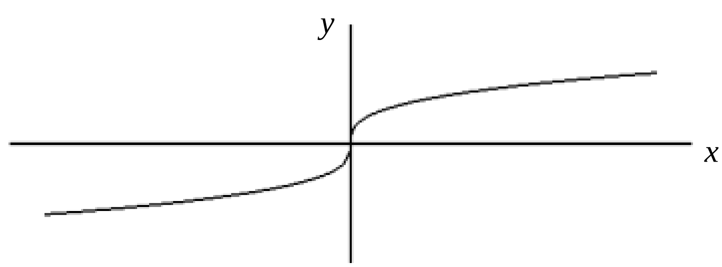


8. $y = -\sqrt{-ax}$ or $y = -a\sqrt{-x}$

NOTE: The graph of each equation is the bottom half of the parabola whose vertex is at the origin and opens to the left.



9. $y = \sqrt[3]{ax}$ or $y = a\sqrt[3]{x}$



10. $y = -\sqrt[3]{ax}$ or $y = -a\sqrt[3]{x}$

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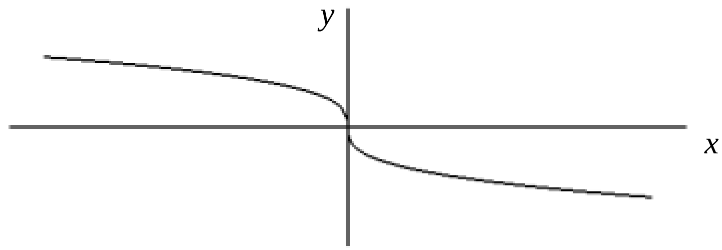
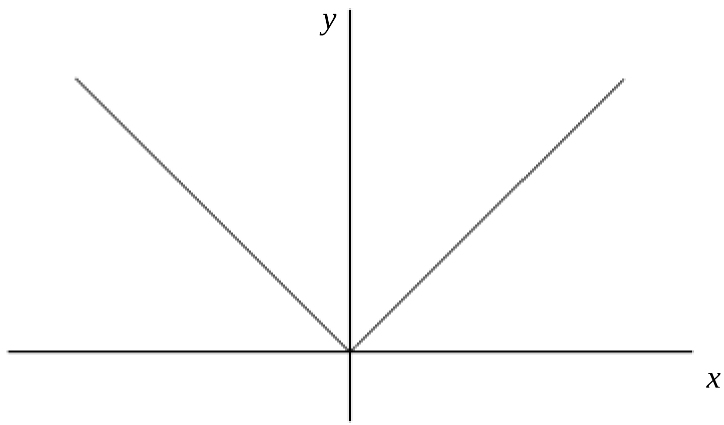
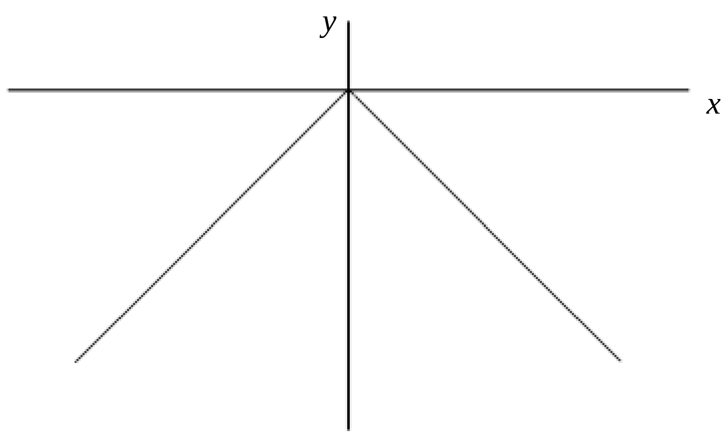


11. $y = a\,|x|$



12. $y = -a\,|x|$



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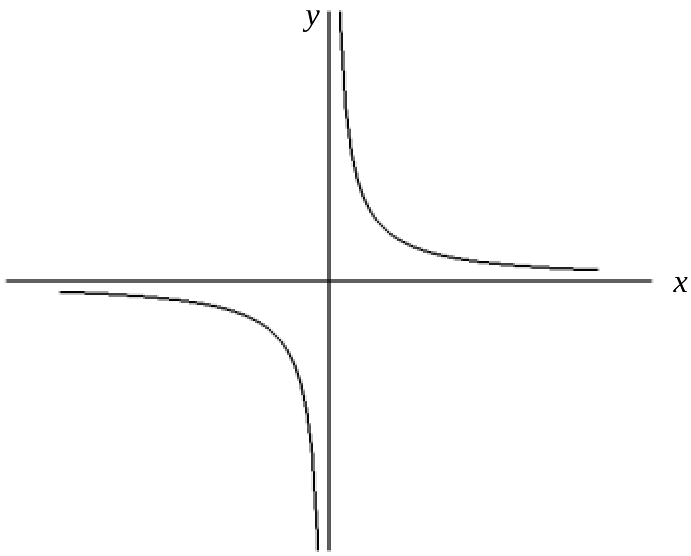
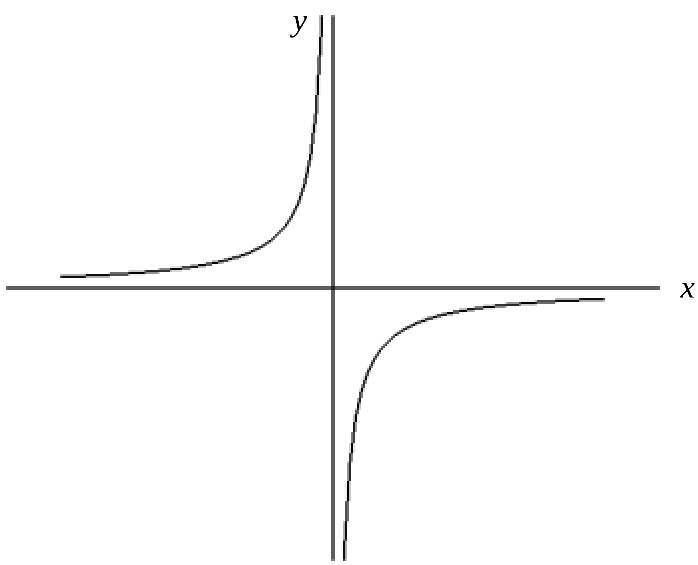
13. $y = \dfrac{a}{x}$ or $y = \dfrac{a}{x^n}$, where n is a positive odd integer

NOTE: The graph of each equation has the x-axis as a horizontal asymptote and the y-axis as a vertical asymptote.



14. $y = -\dfrac{a}{x}$ or $y = -\dfrac{a}{x^n}$, where n is a positive odd integer

NOTE: The graph of each equation has the x-axis as a horizontal asymptote and the y-axis as a vertical asymptote.



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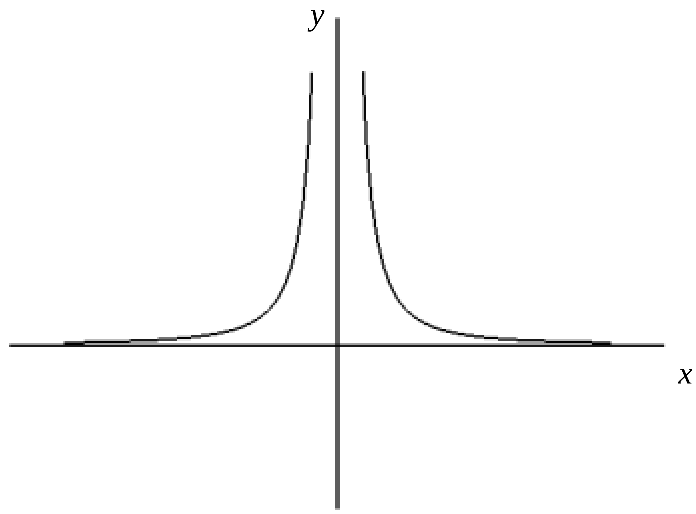
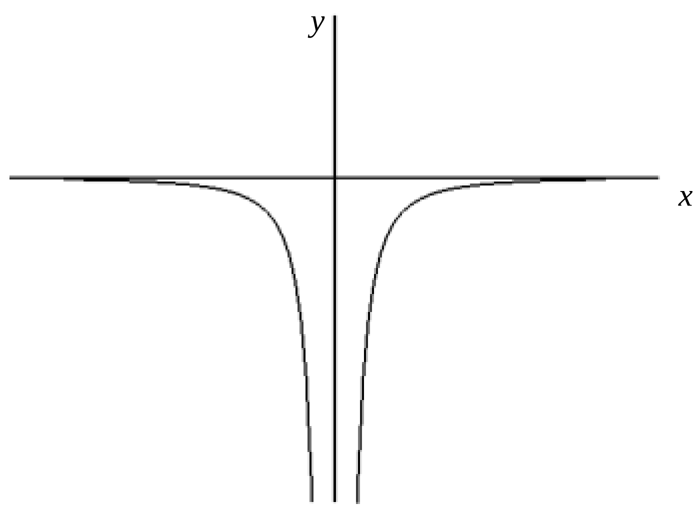
15. $y = \dfrac{a}{x^2}$ or $y = \dfrac{a}{x^n}$, where n is a positive even integer

NOTE: The graph of each equation has the x-axis as a horizontal asymptote and the y-axis as a vertical asymptote.



16. $y = -\dfrac{a}{x^2}$ or $y = -\dfrac{a}{x^n}$, where n is a positive even integer

NOTE: The graph of each equation has the x-axis as a horizontal asymptote and the y-axis as a vertical asymptote.



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Examples Sketch the graph of the following functions. State the domain of the function and use the sketch to state the range of the function.

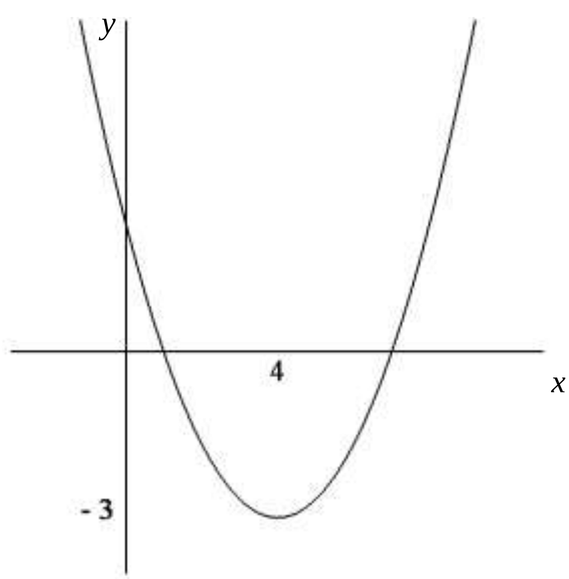
1. $f(x) = 2(x - 4)^2 - 3$

The domain of f is the set of all real numbers.

To graph the function f, we set $f(x) = y$ and graph the equation $y = 2(x - 4)^2 - 3$.

$$y = 2(x - 4)^2 - 3 \Rightarrow y + 3 = 2(x - 4)^2$$

The graph of $y + 3 = 2(x - 4)^2$ is the graph of $y = 2x^2$ shifted 4 units to the right and 3 units downward.



The range of f is $[-3, \infty)$.

NOTE: The horizontal shift of 4 units to the right is determined from the expression $x - 4$ in the equation $y + 3 = 2(x - 4)^2$ and the vertical shift of 3 units downward is determined from the expression $y + 3$ in the equation.

NOTE: To find the y-coordinate of the y-intercept of the graph of the equation $y = 2(x - 4)^2 - 3$, we would set $x = 0$ obtaining that

$y = 2(-4)^2 - 3$. Thus, $y = 32 - 3 = 29$. Thus, the *y*-intercept is the point $(0, 29)$.

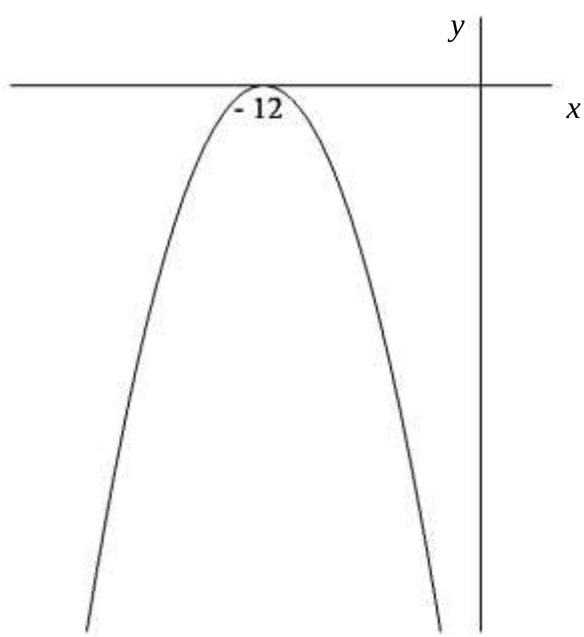
To find the *x*-coordinate of the *x*-intercepts of the graph of the equation $y + 3 = 2(x - 4)^2$, we would set $y = 0$ obtaining that $3 = 2(x - 4)^2$. Solving for *x*, we have that $3 = 2(x - 4)^2 \Rightarrow \frac{3}{2} = (x - 4)^2 \Rightarrow x - 4 = \pm\sqrt{\frac{3}{2}} \Rightarrow x - 4 = \pm\frac{\sqrt{6}}{2} \Rightarrow x = 4 \pm \frac{\sqrt{6}}{2} = \frac{8}{2} \pm \frac{\sqrt{6}}{2} = \frac{8 \pm \sqrt{6}}{2}$. Thus, the *x*-intercepts are the points $\left(\frac{8 - \sqrt{6}}{2}, 0\right)$ and $\left(\frac{8 + \sqrt{6}}{2}, 0\right)$.

2. $g(x) = -5(x + 12)^2$

The domain of g is the set of all real numbers.

To graph the function g, we set $g(x) = y$ and graph the equation $y = -5(x + 12)^2$.

The graph of $y = -5(x + 12)^2$ is the graph of $y = -5x^2$ shifted 12 units to the left. There is no vertical shift.



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The range of g is $(-\infty, 0]$.

NOTE: The horizontal shift of 12 units to the left is determined from the expression $x + 12$ in the equation $y = -5(x + 12)^2$.

NOTE: To find the *y*-coordinate of the *y*-intercept of the graph of the equation $y = -5(x + 12)^2$, we would set $x = 0$ obtaining that $y = -5(12)^2$. Thus, $y = -5(144) = -720$. Thus, the *y*-intercept is the point $(0, -720)$.

To find the *x*-coordinate of the *x*-intercept of the graph of the equation $y = -5(x + 12)^2$, we would set $y = 0$ obtaining that $0 = -5(x + 12)^2$. Solving for *x*, we have that $0 = -5(x + 12)^2 \Rightarrow 0 = (x + 12)^2 \Rightarrow x + 12 = 0 \Rightarrow x = -12$. Thus, the *x*-intercept is the point $(-12, 0)$. Of course, we could have obtain this from the sketch of the graph of the equation $y = -5(x + 12)^2$.

3. $h(x) = \dfrac{3}{4}(x + 2)^3 + 6$

The domain of h is the set of all real numbers.

To graph the function h, we set $h(x) = y$ and graph the equation $y = \dfrac{3}{4}(x + 2)^3 + 6$.

$$y = \frac{3}{4}(x + 2)^3 + 6 \Rightarrow y - 6 = \frac{3}{4}(x + 2)^3$$

The graph of $y - 6 = \dfrac{3}{4}(x + 2)^3$ is the graph of $y = \dfrac{3}{4}x^3$ shifted 2 units to the left and 6 units upward.

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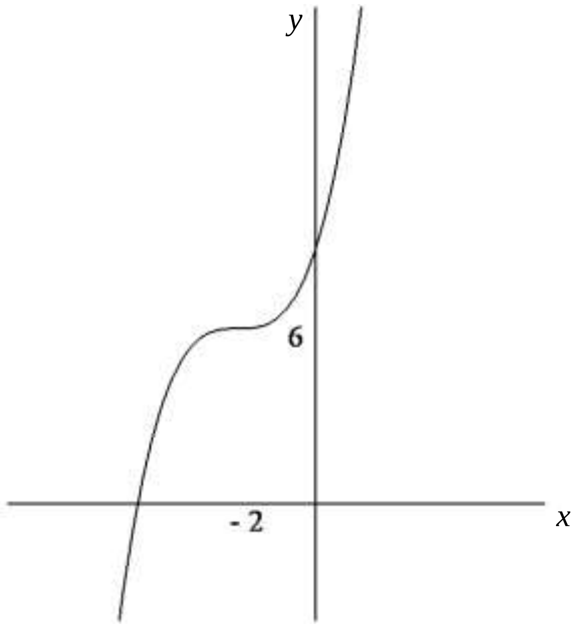

The range of h is the set of all real numbers.

NOTE: The horizontal shift of 2 units to the left is determined from the expression $x + 2$ in the equation $y - 6 = \frac{3}{4}(x + 2)^3$ and the vertical shift of 6 units upward is determined from the expression $y - 6$ in the equation.

NOTE: To find the *y*-coordinate of the *y*-intercept of the graph of the equation $y = \frac{3}{4}(x + 2)^3 + 6$, we would set $x = 0$ obtaining that $y = \frac{3}{4}(2)^3 + 6$. Thus, $y = \frac{3}{4}(8) + 6 = 6 + 6 = 12$. Thus, the *y*-intercept is the point $(0, 12)$.

To find the *x*-coordinate of the *x*-intercept of the graph of the equation $y - 6 = \frac{3}{4}(x + 2)^3$, we would set $y = 0$ obtaining that

$-6 = \frac{3}{4}(x + 2)^3$. Solving for *x*, we have that $-6 = \frac{3}{4}(x + 2)^3 \Rightarrow$
$-24 = 3(x + 2)^3 \Rightarrow -8 = (x + 2)^3 \Rightarrow x + 2 = \sqrt[3]{-8} \Rightarrow$
$x + 2 = -2 \Rightarrow x = -4$. Thus, the *x*-intercept is the point $(-4, 0)$.

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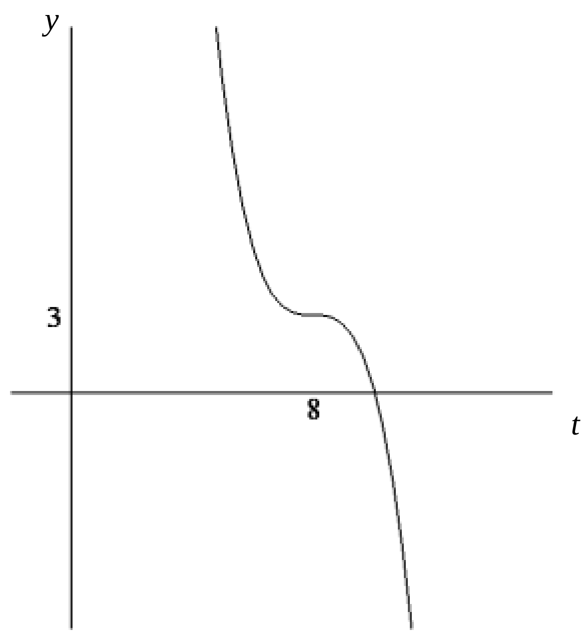
4. $f(t) = -(t - 8)^3 + 3$

The domain of f is the set of all real numbers.

To graph the function f, we set $f(t) = y$ and graph the equation $y = -(t - 8)^3 + 3$.

$$y = -(t - 8)^3 + 3 \Rightarrow y - 3 = -(t - 8)^3$$

The graph of $y - 3 = -(t - 8)^3$ is the graph of $y = -t^3$ shifted 8 units to the right and 3 units upward.



The range of f is the set of all real numbers.

NOTE: The horizontal shift of 8 units to the right is determined from the expression $t - 8$ in the equation $y - 3 = -(t - 8)^3$ and the vertical shift of 3 units upward is determined from the expression $y - 3$ in the equation.

NOTE: To find the *y*-coordinate of the *y*-intercept of the graph of the equation $y = -(t - 8)^3 + 3$, we would set $t = 0$ obtaining that $y = -(-8)^3 + 3$. Thus, $y = -(-512) + 3 = 512 + 3 = 515$. Thus, the *y*-intercept is the point $(0, 515)$.

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To find the t-coordinate of the t-intercept of the graph of the equation $y - 3 = -(t - 8)^3$, we would set $y = 0$ obtaining that $-3 = -(t - 8)^3$. Solving for t, we have that $-3 = -(t - 8)^3 \Rightarrow 3 = (t - 8)^3 \Rightarrow t - 8 = \sqrt[3]{3} \Rightarrow t = 8 + \sqrt[3]{3}$. Thus, the t-intercept is the point $(8 + \sqrt[3]{3}, 0)$.

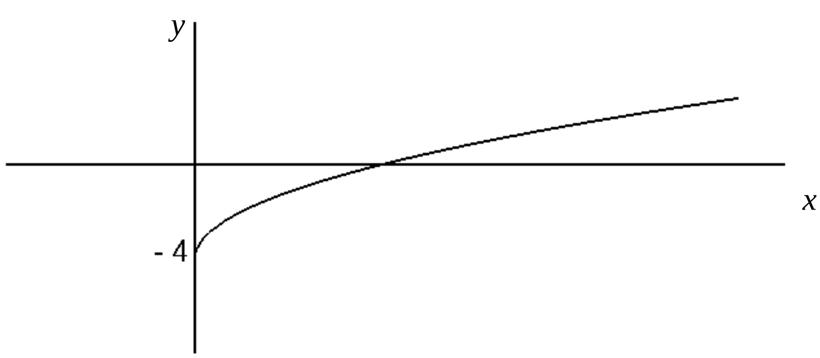
5. $g(x) = 2\sqrt{x} - 4$

The domain of g is $[0, \infty)$.

To graph the function g, we set $g(x) = y$ and graph the equation $y = 2\sqrt{x} - 4$.

$$y = 2\sqrt{x} - 4 \Rightarrow y + 4 = 2\sqrt{x}$$

The graph of $y + 4 = 2\sqrt{x}$ is the graph of $y = 2\sqrt{x}$ shifted 4 units downward. There is no horizontal shift.



The range of g is $[-4, \infty)$.

NOTE: The vertical shift of 4 units downward is determined from the expression $y + 4$ in the equation $y + 4 = 2\sqrt{x}$.

NOTE: To find the y-coordinate of the y-intercept of the graph of the equation $y = 2\sqrt{x} - 4$, we would set $x = 0$ obtaining that $y = 2\sqrt{0} - 4$. Thus, $y = 2(0) - 4 = -4$. Thus, the y-intercept is the point $(0, -4)$. Of course, we could have obtain this from the sketch of the graph of the equation $y = 2\sqrt{x} - 4$.

To find the x-coordinate of the x-intercept of the graph of the equation $y + 4 = 2\sqrt{x}$, we would set $y = 0$ obtaining that $4 = 2\sqrt{x}$. Solving for x, we have that $4 = 2\sqrt{x} \Rightarrow 2 = \sqrt{x} \Rightarrow x = 4$. Thus, the x-intercept is the point $(4, 0)$.

6. $h(t) = \sqrt{-5t - 30} - 11$

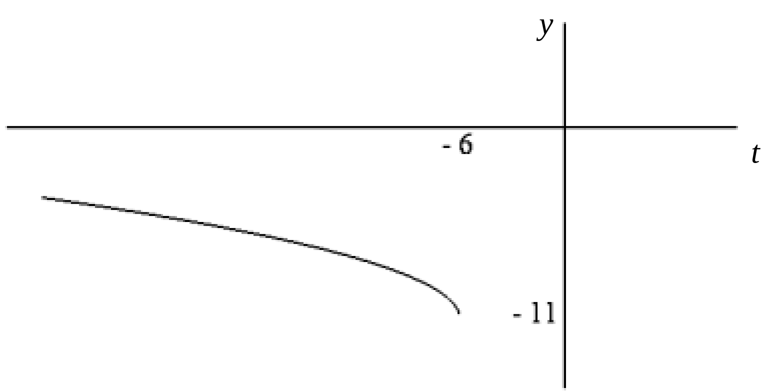
To graph the function h, we set $h(t) = y$ and graph the equation $y = \sqrt{-5t - 30} - 11$.

NOTE: The coefficient of the t variable in the expression $-5t - 30$ is **not** one. We will need to factor out the coefficient of -5 in order to identify the amount and the direction of the horizontal shift.

$$y = \sqrt{-5t - 30} - 11 \Rightarrow y + 11 = \sqrt{-5(t + 6)}$$

For the domain of h, we need that $-5(t + 6) \geq 0$. Thus, $t + 6 \leq 0 \Rightarrow t \leq -6$. Thus, the domain of h is $(-\infty, -6]$.

The graph of $y + 11 = \sqrt{-5(t + 6)}$ is the graph of $y = \sqrt{-5t}$ shifted 6 units to the left and 11 units downward.



The range of h is $[-11, \infty)$.

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NOTE: The horizontal shift of 6 units to the left is determined from the expression $t + 6$ in the equation $y + 11 = \sqrt{-5(t + 6)}$ and the vertical shift of 11 units downward is determined from the expression $y + 11$ in the equation.

NOTE: To find the *y*-coordinate of the *y*-intercept of the graph of the equation $y = \sqrt{-5t - 30} - 11$, we would set $t = 0$ obtaining that $y = \sqrt{-30} - 11$. Thus, $y = -11 + i\sqrt{30}$, a complex number. Thus, the equation $y = \sqrt{-5t - 30} - 11$ does not have a *y*-intercept. Of course, we could have obtain this from the sketch of the graph of the equation $y = \sqrt{-5t - 30} - 11$.

To find the *t*-coordinate of the *t*-intercept of the graph of the equation $y + 11 = \sqrt{-5(t + 6)}$, we would set $y = 0$ obtaining that $11 = \sqrt{-5(t + 6)}$. Solving for *t*, we have that $11 = \sqrt{-5(t + 6)} \Rightarrow$ $121 = -5(t + 6) \Rightarrow 121 = -5t - 30 \Rightarrow -5t = 151 \Rightarrow t = -\dfrac{151}{5}$.

Thus, the *t*-intercept is the point $\left(-\dfrac{151}{5}, 0\right)$.

7. $y = -\dfrac{11}{6}\sqrt{3x - 7} + 2$

NOTE: Functional notation is not being used here. However, we still have a function. The variable y is a function of the variable x. If we wanted, we could introduce functional notation by writing $y(x) = -\dfrac{11}{6}\sqrt{3x - 7} + 2$. The name of the function would become y when you do this.

NOTE: The coefficient of the *x* variable in the expression $3x - 7$ is **not** one. We will need to factor out the coefficient of 3 in order to identify the amount of the horizontal shift.

$$y = -\frac{11}{6}\sqrt{3x - 7} + 2 \Rightarrow y - 2 = -\frac{11}{6}\sqrt{3\left(x - \frac{7}{3}\right)}$$

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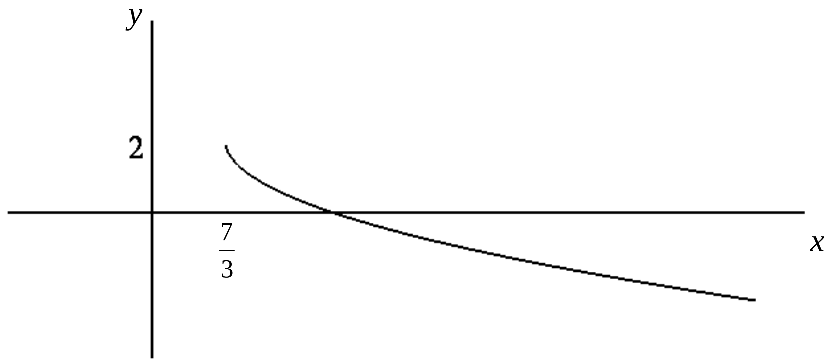
For the domain of the function, we need that $3\left(x - \frac{7}{3} \right) \geq 0$. Thus, $x - \frac{7}{3} \geq 0 \Rightarrow x \geq \frac{7}{3}$. Thus, the domain of the function is $\left[\frac{7}{3}, \infty \right)$.

The graph of $y - 2 = -\frac{11}{6}\sqrt{3\left(x - \frac{7}{3} \right)}$ is the graph of $y = -\frac{11}{6}\sqrt{3x}$ shifted $\frac{7}{3}$ units to the right and 2 units upward.



The range of the function is $(-\infty, 2]$.

NOTE: The horizontal shift of $\frac{7}{3}$ units to the right is determined from the expression $x + \frac{7}{3}$ in the equation $y - 2 = -\frac{11}{6}\sqrt{3\left(x - \frac{7}{3} \right)}$ and the vertical shift of 2 units upward is determined from the expression $y - 2$ in the equation.

NOTE: To find the *y*-coordinate of the *y*-intercept of the graph of the equation $y = -\frac{11}{6}\sqrt{3x - 7} + 2$, we would set $x = 0$ obtaining that $y = -\frac{11}{6}\sqrt{-7} + 2$. Since $\sqrt{-7} = i\sqrt{7}$, then *y* is a complex number. Thus, the equation $y = -\frac{11}{6}\sqrt{3x - 7} + 2$ does not have a *y*-intercept. Of course, we could have obtain this from the sketch of the graph of the equation $y = -\frac{11}{6}\sqrt{3x - 7} + 2$.

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To find the x-coordinate of the x-intercept of the graph of the equation $y - 2 = -\frac{11}{6}\sqrt{3x - 7}$, we would set $y = 0$ obtaining that $-2 = -\frac{11}{6}\sqrt{3x - 7}$. Solving for x, we have that $-2 = -\frac{11}{6}\sqrt{3x - 7} \Rightarrow$ $\frac{12}{11} = \sqrt{3x - 7} \Rightarrow 3x - 7 = \frac{144}{121} \Rightarrow 363x - 847 = 144 \Rightarrow$

$363x = 991 \Rightarrow x = \frac{991}{363}$. Thus, the x-intercept is the point $\left(\frac{991}{363}, 0\right)$.

8. $f(x) = \sqrt[3]{5 - x} - 4$

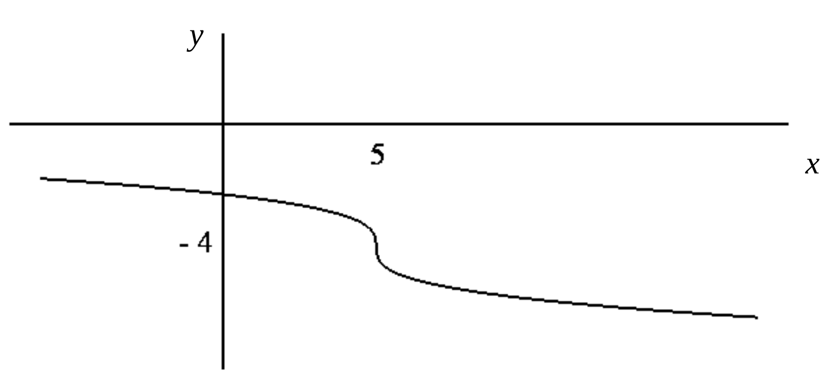
The domain of f is the set of all real numbers.

To graph the function f, we set $f(x) = y$ and graph the equation $y = \sqrt[3]{5 - x} - 4$.

NOTE: The coefficient of the x variable in the expression $5 - x$ is **not** one. We will need to factor out the coefficient of -1 in order to identify the amount and the direction of the horizontal shift.

$$y = \sqrt[3]{5 - x} - 4 \Rightarrow y + 4 = \sqrt[3]{-(x - 5)} \Rightarrow y + 4 = -\sqrt[3]{x - 5}$$

The graph of $y + 4 = -\sqrt[3]{x - 5}$ is the graph of $y = -\sqrt[3]{x}$ shifted 5 units to the right and 4 units downward.



The range of f is the set of all real numbers.

NOTE: The horizontal shift of 5 units to the right is determined from the expression $x - 5$ in the equation $y + 4 = -\sqrt[3]{x - 5}$ and the vertical shift of 4 units downward is determined from the expression $y + 4$ in the equation.

NOTE: To find the y-coordinate of the y-intercept of the graph of the equation $y = \sqrt[3]{5 - x} - 4$, we would set $x = 0$ obtaining that $y = \sqrt[3]{5} - 4$. Thus, the y-intercept is the point $(0, \sqrt[3]{5} - 4)$.

To find the x-coordinate of the x-intercept of the graph of the equation $y + 4 = \sqrt[3]{5 - x}$, we would set $y = 0$ obtaining that $4 = \sqrt[3]{5 - x}$. Solving for x, we have that $4 = \sqrt[3]{5 - x} \Rightarrow 64 = 5 - x \Rightarrow x = -59$. Thus, the x-intercept is the point $(-59, 0)$.

9. $g(x) = |6x - 18|$

The domain of g is the set of all real numbers.

To graph the function g, we set $g(x) = y$ and graph the equation $y = |6x - 18|$.

NOTE: The coefficient of the x variable in the expression $6x - 18$ is **not** one. We will need to factor out the coefficient of 6 in order to identify the amount of the horizontal shift.

$$y = |6x - 18| \Rightarrow y = |6(x - 3)| \Rightarrow y = 6|x - 3|$$

The graph of $y = 6|x - 3|$ is the graph of $y = 6|x|$ shifted 3 units to the right. There is no vertical shift.

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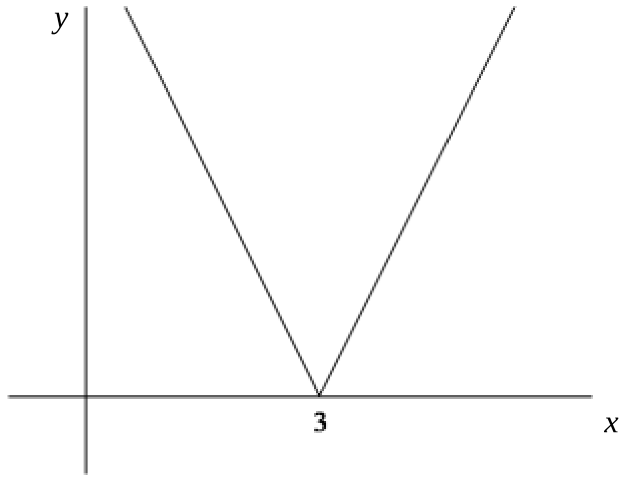

The range of g is $[0, \infty)$.

NOTE: The horizontal shift of 3 units to the right is determined from the expression $x - 3$ in the equation $y = 6|x - 3|$.

NOTE: To find the *y*-coordinate of the *y*-intercept of the graph of the equation $y = 6|x - 3|$, we would set $x = 0$ obtaining that $y = 6|-3|$. Thus, $y = 6|-3| = 18$ Thus, the *y*-intercept is the point $(0, 18)$.

To find the *x*-coordinate of the *x*-intercept of the graph of the equation $y = 6|x - 3|$, we would set $y = 0$ obtaining that $0 = 6|x - 3|$. Solving for *x*, we have that $0 = 6|x - 3| \Rightarrow 0 = |x - 3| \Rightarrow$ $0 = x - 3 \Rightarrow x = 3$. Thus, the *x*-intercept is the point $(3, 0)$. Of course, we could have obtain this from the sketch of the graph of the equation $y = 6|x - 3|$.

10. $h(x) = -|4x + 9| + 7$

The domain of h is the set of all real numbers.

To graph the function h, we set $h(x) = y$ and graph the equation $y = -|4x + 9| + 7$.

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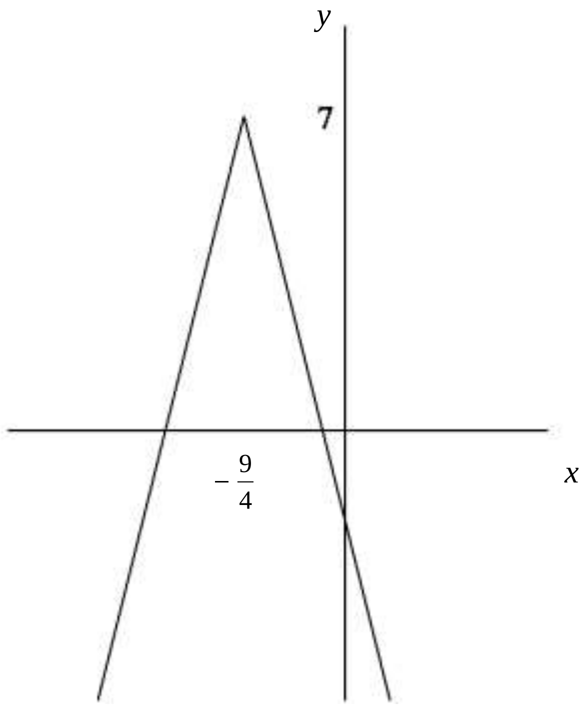
NOTE: The coefficient of the x variable in the expression $4x + 9$ is **not** one. We will need to factor out the coefficient of 4 in order to identify the amount of the horizontal shift.

$$y - 7 = -|4x + 9| \Rightarrow y - 7 = -\left|4\left(x + \frac{9}{4}\right)\right| \Rightarrow y - 7 = -4\left|x + \frac{9}{4}\right|$$

The graph of $y - 7 = -4\left|x + \frac{9}{4}\right|$ is the graph of $y = -4|x|$ shifted $\frac{9}{4}$ units to the left and 7 units upward.



The range of h is $(-\infty, 7]$.

NOTE: The horizontal shift of $\frac{9}{4}$ units to the left is determined from the expression $x + \frac{9}{4}$ in the equation $y - 7 = -4\left|x + \frac{9}{4}\right|$ and the vertical shift of 7 units upward is determined from the expression $y - 7$ in the equation.

NOTE: To find the y-coordinate of the y-intercept of the graph of the equation $y = -|4x + 9| + 7$, we would set $x = 0$ obtaining that

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$y = -|9| + 7$. Thus, $y = -|9| + 7 = -9 + 7 = -2$ Thus, the y-intercept is the point $(0, -2)$.

To find the x-coordinate of the x-intercepts of the graph of the equation $y - 7 = -|4x + 9|$, we would set $y = 0$ obtaining that $-7 = -|4x + 9|$. Solving for x, we have that $-7 = -|4x + 9| \Rightarrow$ $7 = |4x + 9| \Rightarrow 4x + 9 = \pm 7 \Rightarrow 4x = -9 \pm 7 \Rightarrow x = \dfrac{-9 \pm 7}{4}$. $x = \dfrac{-9 + 7}{4} = \dfrac{-2}{4} = -\dfrac{1}{2}$ or $x = \dfrac{-9 - 7}{4} = \dfrac{-16}{4} = -4$ Thus, the x-intercepts are the points $(-4, 0)$ and $\left(-\frac{1}{2}, 0\right)$.

11. $f(x) = \dfrac{4}{3x^2} + 8$

The domain of f is the set of all real numbers x such that $x \neq 0$.

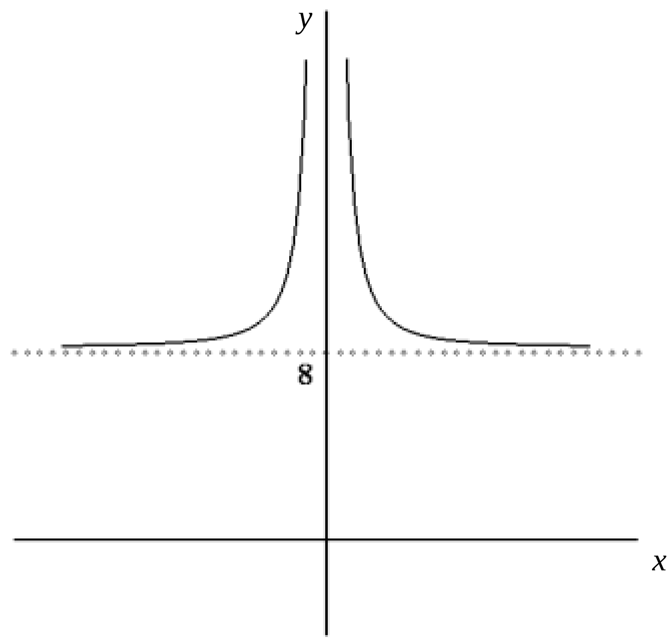
To graph the function f, we set $f(x) = y$ and graph the equation $y = \dfrac{4}{3x^2} + 8$.

$$y = \frac{4}{3x^2} + 8 \Rightarrow y - 8 = \frac{4}{3x^2}$$

The graph of $y - 8 = \dfrac{4}{3x^2}$ is the graph of $y = \dfrac{4}{3x^2}$ shifted 8 units upward. There is no horizontal shift.

NOTE: We know that the graph of $y = \dfrac{4}{3x^2}$ has the x-axis as a horizontal asymptote. The vertical shift of 8 units upward will shift this horizontal asymptote 8 units upward. Thus, the graph of $y - 8 = \dfrac{4}{3x^2}$ will have the line $y = 8$ as a horizontal asymptote. We also know that the graph of

$y = \dfrac{4}{3x^2}$ has the *y*-axis as a vertical asymptote. The vertical shift will not affect this vertical line. Thus, the graph of $y - 8 = \dfrac{4}{3x^2}$ will have the *y*-axis as a vertical asymptote.



The range of f is $(8, \infty)$.

NOTE: The vertical shift of 8 units upward is determined from the expression $y - 8$ in the equation $y - 8 = \dfrac{4}{3x^2}$.

NOTE: From the sketch of the graph of the equation $y - 8 = \dfrac{4}{3x^2}$, we can see that the graph does not have any *x*-intercepts nor a *y*-intercept.

12. $g(x) = -\dfrac{14}{2x + 5} - 6$

The domain of g is the set of all real numbers x such that $x \neq -\dfrac{5}{2}$.

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To graph the function g, we set $g(x) = y$ and graph the equation $y = -\dfrac{14}{2x + 5} - 6$.

NOTE: The coefficient of the x variable in the expression $2x + 5$ is **not** one. We will need to factor out the coefficient of 2 in order to identify the amount of the horizontal shift.

$$y = -\frac{14}{2x + 5} - 6 \Rightarrow y + 6 = -\frac{14}{2\left(x + \frac{5}{2}\right)} \Rightarrow y + 6 = -\frac{7}{x + \frac{5}{2}}$$

The graph of $y + 6 = -\dfrac{7}{x + \frac{5}{2}}$ is the graph of $y = -\dfrac{7}{x}$ shifted $\dfrac{5}{2}$ units to left and 6 units downward.

NOTE: We know that the graph of $y = -\dfrac{7}{x}$ has the *x*-axis as a horizontal asymptote. The vertical shift of 6 units downward will shift this horizontal asymptote 6 units downward. The horizontal shift will not affect this horizontal line. Thus, the graph of $y + 6 = -\dfrac{7}{x + \frac{5}{2}}$ will have the line $y = -6$ as a horizontal asymptote. We also know that the graph of $y = -\dfrac{7}{x}$ has the *y*-axis as a vertical asymptote. The horizontal shift of $\dfrac{5}{2}$ units to the left will shift this vertical asymptote $\dfrac{5}{2}$ units to the left. The vertical shift will not affect this vertical line. Thus, the graph of $y + 6 = -\dfrac{7}{x + \frac{5}{2}}$ will have the line $x = -\dfrac{5}{2}$ as a vertical asymptote.

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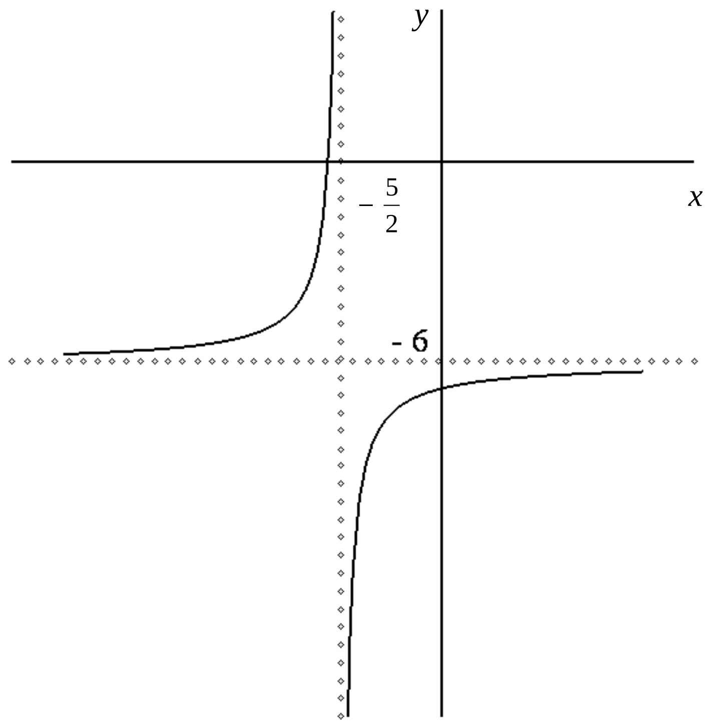

The range of g is $(-\infty, -6) \cup (-6, \infty)$.

NOTE: The horizontal shift of $\frac{5}{2}$ units to the left is determined from the expression $x + \frac{5}{2}$ in the equation $y + 6 = -\frac{7}{x + \frac{5}{2}}$ and the vertical shift of 6 units downward is determined from the expression $y + 6$ in the equation.

NOTE: To find the *y*-coordinate of the *y*-intercept of the graph of the equation $y = -\frac{14}{2x + 5} - 6$, we would set $x = 0$ obtaining that $y = -\frac{14}{5} - 6$. Thus, $y = -\frac{14}{5} - 6 = -\frac{14}{5} - \frac{30}{5} = -\frac{44}{5}$ Thus, the *y*-intercept is the point $\left(0, -\frac{44}{5}\right)$.

To find the *x*-coordinate of the *x*-intercept of the graph of the equation $y + 6 = -\frac{14}{2x + 5}$, we would set $y = 0$ obtaining that

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$6 = -\frac{14}{2x + 5}$. Solving for *x*, we have that $6 = -\frac{14}{2x + 5} \Rightarrow$

$6(2x + 5) = -14 \Rightarrow 3(2x + 5) = -7 \Rightarrow 6x + 15 = -7 \Rightarrow$

$6x = -22 \Rightarrow x = -\frac{11}{3}$. Thus, the *x*-intercept is the points $\left(-\frac{11}{3}, 0\right)$.

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