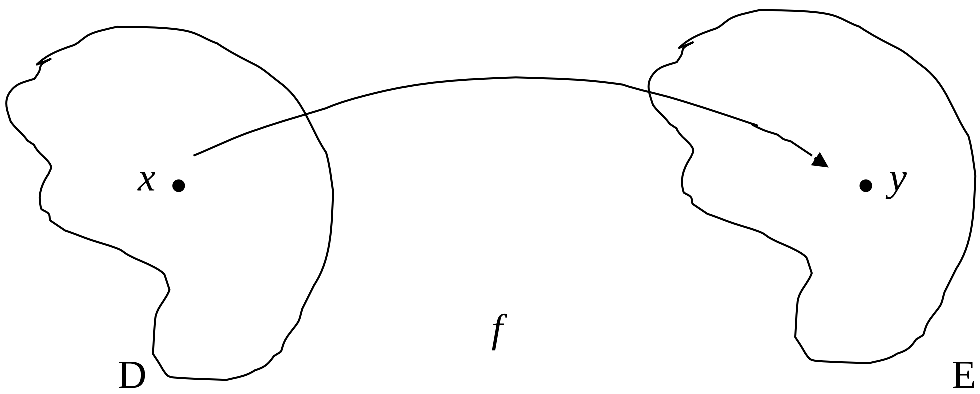
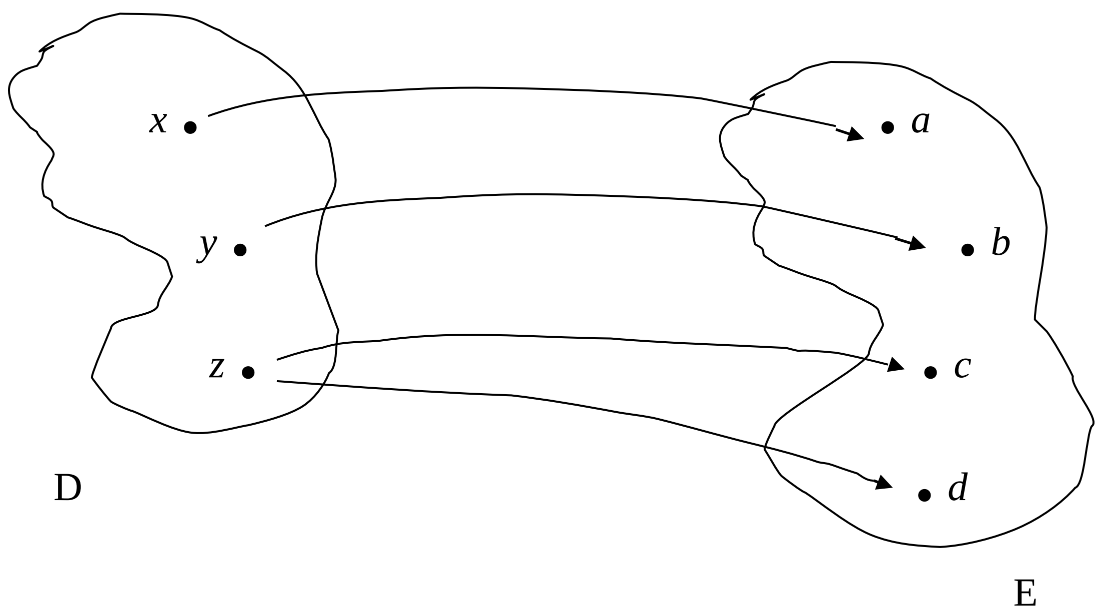
LESSON 2 FUNCTIONS

<u>Definiton</u> A function f from a set D to a set E is a correspondence that assigns to each element x of D a unique element y of E .

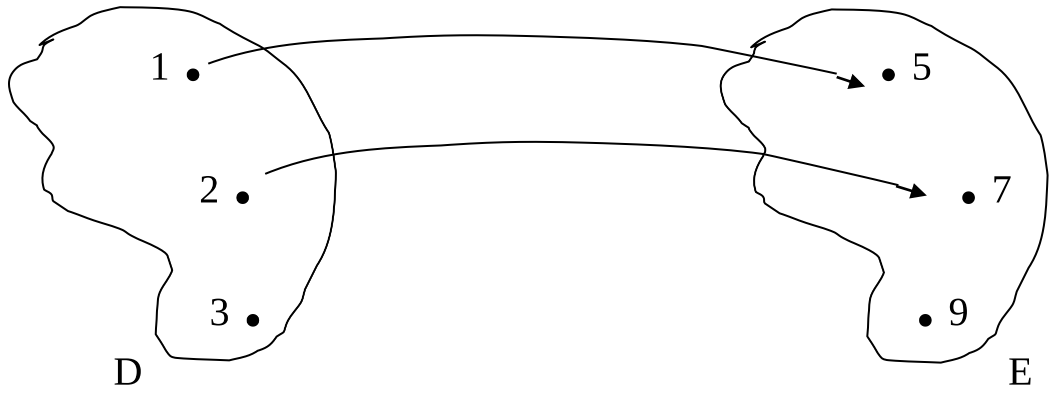


<u>Example</u> Determine if the following is a function or not.



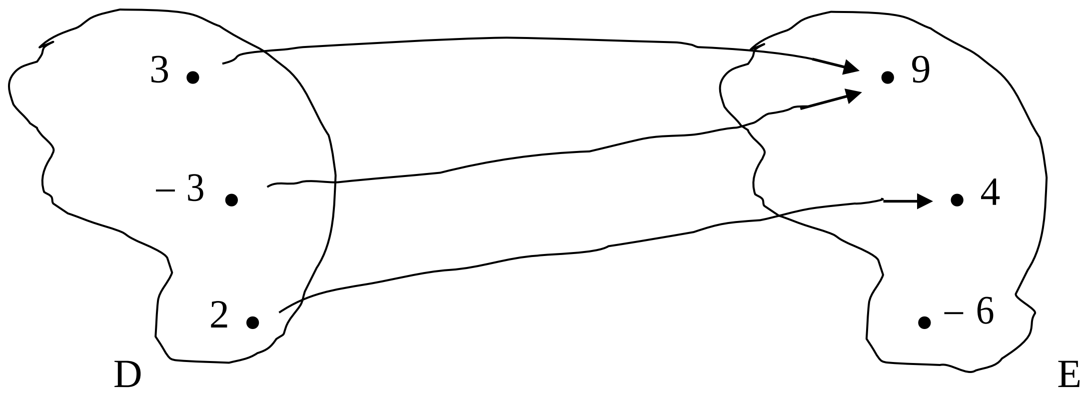
All the elements in the set D have been corresponded with an element in the set E. Now, we need to see if it has been done uniquely. x has been uniquely corresponded with a. y has been uniquely corresponded with b. z has **not** been uniquely corresponded. z has been corresponded with c and d. Thus, this is not a function.

Example Determine if the following is a function or not.



Not all the elements in the set D have been corresponded with an element in the set E. 3 has **not** been corresponded with anything in E. Thus, this is not a function.

Example Determine if the following is a function or not.

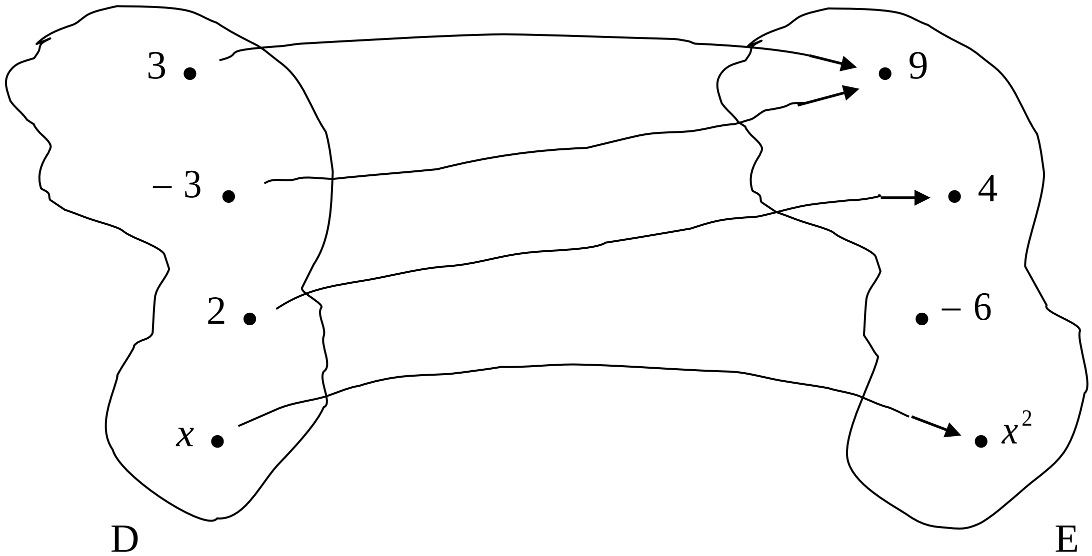


All the elements in the set D have been corresponded with an element in the set E. Now, we need to see if it has been done uniquely. 3 has been uniquely corresponded with 9. -3 has been uniquely corresponded with 9. 2 has been uniquely corresponded with 4. Thus, this is a function.

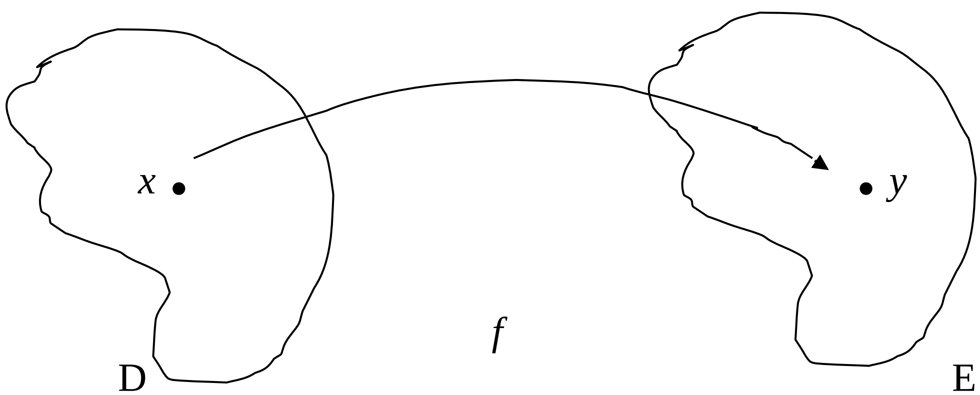
NOTE: This function is the function that corresponds each real number with its square.

Since any real number can be squared, we could not show the picture for all real numbers.

However, if we let x represent any real number, we could do the following:

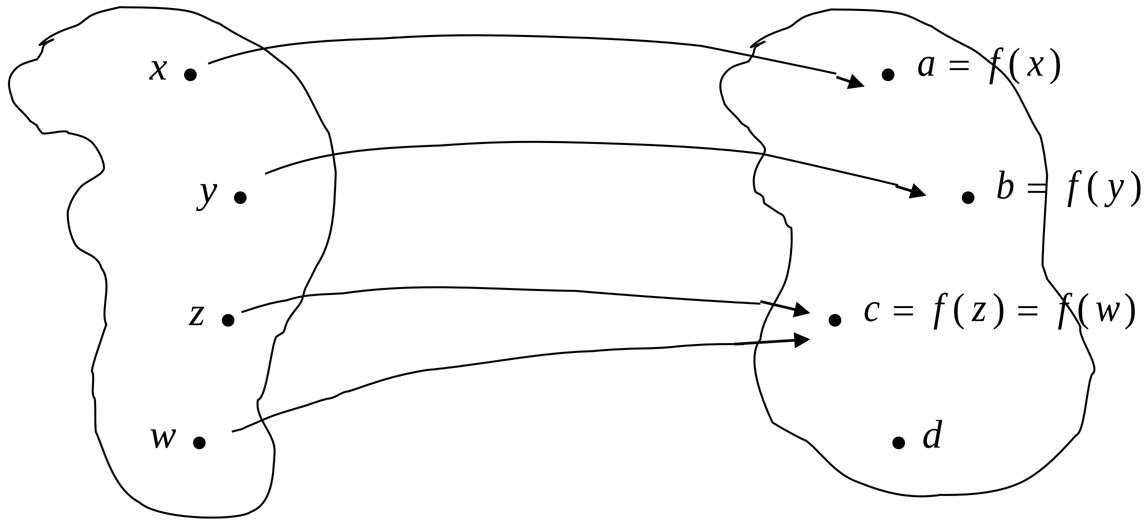


Let's develop some notation and terminology using the following picture for the function f.



The element y in the set E is called the value of the function f at x and is denoted by $f(x)$, read "f of x." A common mistake, which is made by students, is saying that $f(x)$ is the function. The name of the function is f and $f(x)$ is a functional value, namely the value of the function at x. The set D is called the domain of the function. The range of the function f is the set consisting of all possible functional values $f(x)$, where x is in the domain D.

Example



Domain of f = { x, y, z, w }

Range of f = { a, b, c }

Example If $f(x) = 6x^2 - 17x - 14$, then find $f(1)$, $f(-2)$, $f(0)$, and $f(x+h)$.

To find $f(1)$, replace **all** the x's in $f(x) = 6x^2 - 17x - 14$ by 1. Simplifying the exponential expression and the multiplication as we go, we have that

$$f(1) = 6 - 17 - 14 = -25$$

NOTE: This function f corresponds 1 with -25. **Answer:** -25

To find $f(-2)$, replace **all** the x's in $f(x) = 6x^2 - 17x - 14$ by -2. Simplifying the exponential expression and the multiplication as we go, we have that

$$f(-2) = 24 + 34 - 14 = 44$$

NOTE: This function f corresponds -2 with 44. **Answer:** 44

Clearly, $f(0) = 0 - 0 - 14 = -14$

NOTE: This function f corresponds 0 with -14. **Answer:** -14

Before we find $f(x+h)$, I want to show you another way to find $f(1)$ and $f(-2)$. Sometimes, we might have to factor the expression $6x^2 - 17x - 14$ in the statement $f(x) = 6x^2 - 17x - 14$ in order to answer another question like finding the *x*-intercept(s) of the graph of the function *f*. If we have to do the factorization, then we can use it in order to find functional values of *f*. It is also possible that the function *f* is given to us in factored form. In this case, we do not have to do any work to get the factorization. Since $6x^2 - 17x - 14 = (2x-7)(3x+2)$, then $f(x) = (2x-7)(3x+2)$.

Thus, $f(1) = -5(5) = -25$ and $f(-2) = -11(-4) = 44$

Now, let's find $f(x+h)$. To find $f(x+h)$, first replace **all** the *x*'s in $f(x) = 6x^2 - 17x - 14$ by $x + h$:

$$f(x+h) = 6(x+h)^2 - 17(x+h) - 14$$

Now, simplify the algebraic expression of the right side:

$$f(x+h) = 6(x^2 + 2xh + h^2) - 17(x+h) - 14 =$$

$$6x^2 + 12xh + 6h^2 - 17x - 17h - 14$$

Answer: $6x^2 + 12xh + 6h^2 - 17x - 17h - 14$

__**Example**__ If $g(x) = \sqrt{9 - 4x}$, then find $g(8)$.

$$g(8) = \sqrt{9 - 32} = \sqrt{-23} = i\sqrt{23}$$

NOTE: $i\sqrt{23}$ is a **complex number**. We want the value of all our functions to be a **real number**. Thus, the value of the function g at 8 is undefined as a real number. Thus, 8 is not in the **domain** of the function g.

Answer: undefined

NOTE: For the functions, which we will work with in this class, the domain of the function is the set of real numbers for which the functional values are real numbers too. In other words, we want to be able to graph the function in the *xy*-plane.

This is the reason that 8 is not in the domain of the function g above.

<u>Examples</u> Find the domain of the following functions.

1. $g(x) = \sqrt{9 - 4x}$

When we replace *x* by a real number, the functional value of this real number is obtained by multiplying that number by 4. Then subtracting the resulting product from 9. Then taking the square root of this difference. We will obtain a real number as long as we do not take the square root of a negative number. The square root of a negative number is a complex number.

Thus, we want to find all the values of *x* that will make the expression $9 - 4x$ be greater than or equal not zero. Thus, we want to solve the inequality $9 - 4x \geq 0$. Solving this inequality, we have that

$$9 - 4x \geq 0 \;\Rightarrow\; 9 \geq 4x \;\Rightarrow\; x \leq \frac{9}{4}$$

Thus, if we replace *x* in the expression $9 - 4x$ by any number that is less than or equal to nine fourths, then the value of $9 - 4x$ will either be a positive number or will be zero. Thus, the square root of the positive number or zero will be a real number and will not be a complex number. Thus, the domain of the function g is the set of numbers given by the interval $\left(-\infty,\ \frac{9}{4}\right]$. Note that the number 8 is not in the domain of the function g as we saw in the example above.

Answer: $\left(-\infty,\ \frac{9}{4}\right]$

2. $f(x) = 5x^2 + 3x - 6$

If we replace x in the expression $5x^2 + 3x - 6$ by a real number, the functional value of this real number will real number. Since this is true for all real numbers that we would substitute for x, then the domain of f is the set of all real numbers.

NOTE: The function f is a polynomial function. The domain of any polynomial is the set of all real numbers.

Answer: All real numbers

3. $h(x) = \dfrac{x^2 - 9}{12x^2 - 80x + 48}$

If we replace x in the expressions $x^2 - 9$ and $12x^2 - 80x + 48$ by a real number, we will obtain a real number for each one. The functional value of the original real number is obtained by dividing the real number, that was obtained from the expression $x^2 - 9$, by the real number that was obtained from the expression $12x^2 - 80x + 48$. This functional value will be a real number if we don't divide by zero. Thus, we want (or need) that the denominator of $12x^2 - 80x + 48$ not equal zero. That is,

Want (Need): $12x^2 - 80x + 48 \neq 0$

$$12x^2 - 80x + 48 \neq 0 \Rightarrow 4(3x^2 - 20x + 12) \neq 0 \Rightarrow 4(x - 6)(3x - 2) \neq 0 \Rightarrow$$

$x - 6 \neq 0$ **and** $3x - 2 \neq 0 \Rightarrow x \neq 6$ **and** $x \neq \frac{2}{3}$

Thus, the domain of the function h is the set of all real numbers such that $x \neq 6$ and $x \neq \frac{2}{3}$.

Answer: $x \neq 6$ **and** $x \neq \frac{2}{3}$

or All real numbers such that $x \neq 6$ **and** $x \neq \frac{2}{3}$

or $\left(-\infty, \frac{2}{3} \right) \cup \left(\frac{2}{3}, 6 \right) \cup (6, \infty)$

4. $f(x) = \sqrt{2x^2 - 5x - 3}$

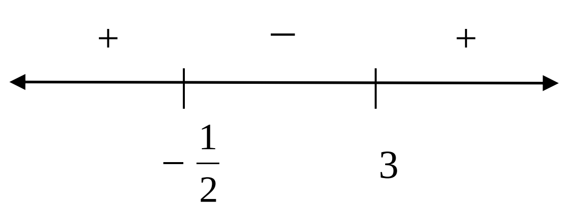
Want (Need): $2x^2 - 5x - 3 \geq 0$

We will need to solve this nonlinear inequality using the material from Lesson 1.

We will use the three step method to first solve the nonlinear inequality $2x^2 - 5x - 3 > 0$:

Step 1: $2x^2 - 5x - 3 = 0 \Rightarrow (x - 3)(2x + 1) = 0 \Rightarrow x = 3,\ x = -\frac{1}{2}$

Step 2:



Sign of $2x^2 - 5x - 3$

Step 3:

Interval	Test Value	Sign of $2x^2 - 5x - 3 = (x-3)(2x+1)$
$\left(-\infty, -\dfrac{1}{2}\right)$	-1	$(-)(-) = +$
$\left(-\dfrac{1}{2}, 3\right)$	0	$(-)(+) = -$
$(3, \infty)$	4	$(+)(+) = +$

Thus, the solution for the nonlinear inequality $2x^2 - 5x - 3 > 0$ is the set of real numbers given by $\left(-\infty, -\dfrac{1}{2}\right) \cup (3, \infty)$. The solution for $2x^2 - 5x - 3 = 0$ was found in Step 1 above. Thus, the solution for $2x^2 - 5x - 3 = 0$ is the set $\left\{-\dfrac{1}{2}, 3\right\}$. Putting these two solutions together, we have that the solution for $2x^2 - 5x - 3 \geq 0$ is the set of real numbers $\left(-\infty, -\dfrac{1}{2}\right] \cup [3, \infty)$.

Thus, the domain of the function f is the set $\left(-\infty, -\dfrac{1}{2}\right] \cup [3, \infty)$.

Answer: $\left(-\infty, -\dfrac{1}{2}\right] \cup [3, \infty)$

5. $g(x) = \sqrt[3]{2x^2 - 5x + 3}$

The index of the radical is three. This is an odd number (by the mathematical definition of odd). The cube root of a negative real number is also a negative real number. It is not a complex number. Of course, the cube root of a positive real number is also a positive real number and the cube root of zero is zero. Thus, we can take the cube root of any real number. Thus, the domain of the function g is the set of all real numbers.

In general, if the index of the radical is **odd**, the root of a negative real number is also a negative real number, the root of a positive real number is also a positive real number, and the root of zero is zero.

Answer: All real numbers.

6. $h(x) = \sqrt[4]{\dfrac{5x + 3}{x + 2}}$

The index of this radical is four, an even number. Thus, we want (or need) that the rational expression $\dfrac{5x + 3}{x + 2}$ not be negative. Thus,

Want (Need): $\dfrac{5x + 3}{x + 2} \geq 0$

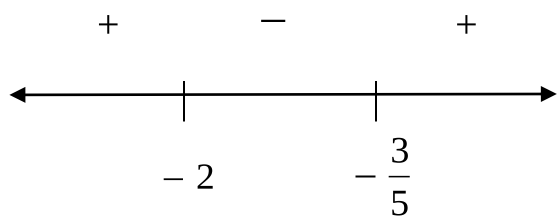
We will need to solve this nonlinear inequality using the material from Lesson 1.

We will use the three step method to first solve the nonlinear inequality $\dfrac{5x + 3}{x + 2} > 0$:

Step 1: $\dfrac{5x + 3}{x + 2} = 0 \Rightarrow 5x + 3 = 0 \Rightarrow x = -\dfrac{3}{5}$

$\dfrac{5x + 3}{x + 2}$ undefined $\Rightarrow x + 2 = 0 \Rightarrow x = -2$

Step 2:



Sign of $\dfrac{5x+3}{x+2}$

Step 3:

Interval	Test Value	Sign of $\dfrac{5x+3}{x+2}$
$(-\infty, -2)$	-3	$\dfrac{(-)}{(-)} = +$
$\left(-2, -\dfrac{3}{5}\right)$	-1	$\dfrac{(-)}{(+)} = -$
$\left(-\dfrac{3}{5}, \infty\right)$	0	$\dfrac{(+)}{(+)} = +$

Thus, the solution for the nonlinear inequality $\dfrac{5x+3}{x+2} > 0$ is the set of real numbers given by $(-\infty, -2) \cup \left(-\dfrac{3}{5}, \infty\right)$. The solution for $\dfrac{5x+3}{x+2} = 0$ was found in Step 1 above. Thus, the solution for $\dfrac{5x+3}{x+2} = 0$ is the set $\left\{-\dfrac{3}{5}\right\}$. Putting these two solutions together, we have that the solution for $\dfrac{5x+3}{x+2} \geq 0$ is the set of real numbers $(-\infty, -2) \cup \left[-\dfrac{3}{5}, \infty\right)$.

Thus, the domain of the function h is the set $(-\infty, -2) \cup \left[-\dfrac{3}{5}, \infty\right)$.

Answer: $(-\infty, -2) \cup \left[-\dfrac{3}{5}, \infty\right)$

7. $f(x) = \sqrt[5]{\dfrac{5x+3}{x+2}}$

The index of this radical is five, an odd number. The fifth root of a negative real number is a negative real number, the fifth root of a positive real number is a positive real number, and the fifth root of zero is zero. Thus, we only need to worry about division by zero.

Want (Need): $x + 2 \neq 0 \Rightarrow x \neq -2$

Thus, the domain of the function f is the set of all real numbers such that $x \neq -2$.

Answer: $x \neq -2$ **or** $(-\infty, -2) \cup (-2, \infty)$

8. $g(x) = \dfrac{\sqrt{45 - 9x^2}}{x^2 - 4x - 12}$

Want (Need): $45 - 9x^2 \geq 0$ **and** $x^2 - 4x - 12 \neq 0$

Using the material from Lesson 1, the solution for the nonlinear inequality $45 - 9x^2 \geq 0$ is the set of real numbers given by $\left[-\sqrt{5}, \sqrt{5}\right]$. Since $x^2 - 4x - 12 = (x+2)(x-6)$, then $x^2 - 4x - 12 \neq 0$ when $x \neq -2$ **and** $x \neq 6$. Since $-\sqrt{5} < -2 < \sqrt{5}$, then -2 is in the interval $\left[-\sqrt{5}, \sqrt{5}\right]$. Since $6 > \sqrt{5}$, then 6 is not in the interval $\left[-\sqrt{5}, \sqrt{5}\right]$. Thus, the real numbers x that satisfy the condition $45 - 9x^2 \geq 0$ **and** $x^2 - 4x - 12 \neq 0$ are the numbers in the interval $\left[-\sqrt{5}, -2\right) \cup \left(-2, \sqrt{5}\right]$. Thus, this is the domain of the function g.

Answer: $\left[-\sqrt{5}\,,\,-2\right) \cup \left(-2\,,\,\sqrt{5}\right]$

9. $h(x) = \dfrac{x^2 - 4x - 12}{\sqrt{45 - 9x^2}}$

Want (Need): $45 - 9x^2 > 0$

NOTE: The reason we want or need that $45 - 9x^2 > 0$ and not $45 - 9x^2 \geq 0$ is because if $45 - 9x^2 = 0$, then $\sqrt{45 - 9x^2} = \sqrt{0} = 0$ and we would have division by zero. Using the material from Lesson 1, the solution for the nonlinear inequality $45 - 9x^2 > 0$ is the set of real numbers given by $\left(-\sqrt{5}\,,\,\sqrt{5}\right)$. Thus, this is the domain of the function h.

Answer: $\left(-\sqrt{5}\,,\,\sqrt{5}\right)$

__**Example**__ If $g(x) = 6x^2 - 8x + 15$, then find $\dfrac{g(x+h) - g(x)}{h}$.

$g(x+h) = 6(x+h)^2 - 8(x+h) + 15 = 6(x^2 + 2xh + h^2) - 8x - 8h + 15 =$

$6x^2 + 12xh + 6h^2 - 8x - 8h + 15$

$g(x) = 6x^2 - 8x + 15$

NOTE: In the subtraction of $g(x)$ from $g(x+h)$, the $6x^2$ terms will cancel, the $-8x$ terms will cancel, and the 15's will cancel. Thus,

$g(x+h) - g(x) = 12xh + 6h^2 - 8h = h(12x + 6h - 8)$

NOTE: In the division of $g(x+h) - g(x)$ by h, the h's will cancel. Thus,

$$\frac{g(x+h)-g(x)}{h} = 12x + 6h - 8 \text{ provided that } h \neq 0.$$

Answer: $12x + 6h - 8$

<u>Example</u> If $f(x) = \dfrac{8}{3x+16}$, then find $\dfrac{f(-2+h) - f(-2)}{h}$.

$$f(-2+h) = f(h-2) = \frac{8}{3(h-2)+16} = \frac{8}{3h-6+16} = \frac{8}{3h+10}$$

$$f(-2) = \frac{8}{10} = \frac{4}{5}$$

$$f(-2+h) - f(-2) = \frac{8}{3h+10} - \frac{4}{5}$$

The least common denominator of 5 and $3h + 10$ is $5(3h + 10)$. Thus,

$$f(-2+h) - f(-2) = \frac{40}{5(3h+10)} - \frac{4(3h+10)}{5(3h+10)} = \frac{40 - 4(3h+10)}{5(3h+10)} =$$

$$\frac{40 - 12h - 40}{5(3h+10)} = \frac{-12h}{5(3h+10)}$$

NOTE: Division by h is the same as multiplying by $\dfrac{1}{h}$, provided that $h \neq 0$. When you multiply $f(-2+h) - f(-2)$, which is a fraction by $\dfrac{1}{h}$, the h's will cancel. Thus,

$$\frac{f(-2+h) - f(-2)}{h} = \frac{-12}{5(3h+10)} \text{, provided that } h \neq 0.$$

Answer: $\dfrac{-12}{5(3h+10)}$ **or** $-\dfrac{12}{5(3h+10)}$

Example If $h(x) = \begin{cases} -2x^2, & x \le 1 \\ 5 - x, & x > 1 \end{cases}$, then find $h(5)$, $h(-3)$, and $h(1)$. Then sketch the graph of the function h.

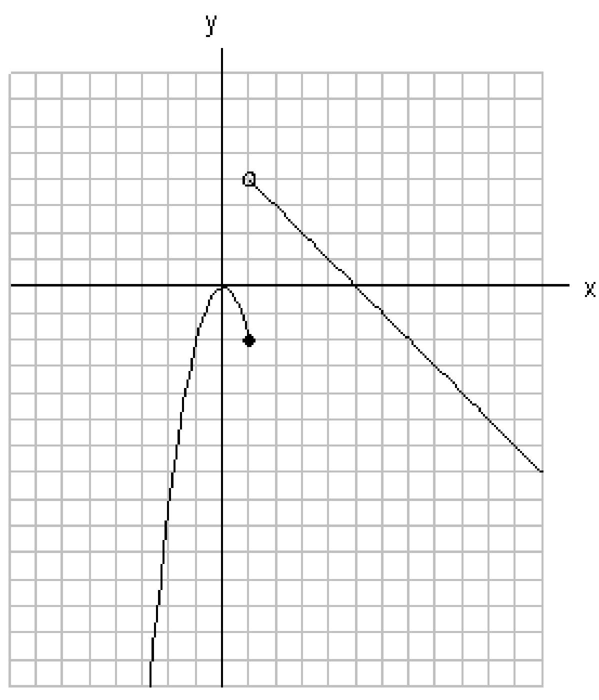
The function h is called a piecewise function. The number 1 is sometimes called a breakup point of the function h.

To find $h(5)$: Since $5 > 1$ and $h(x) = 5 - x$ when $x > 1$, then $h(5) = 0$.
Answer: 0

To find $h(-3)$: Since $-3 < 1$ and $h(x) = -2x^2$ when $x \le 1$, then $h(-3) = -18$.
Answer: -18

To find $h(1)$: Since $1 = 1$ and $h(x) = -2x^2$ when $x \le 1$, then $h(1) = -2$.
Answer: -2

Sketch of *h*:



This graph was created using Maple.

Example If $g(x) = \begin{cases} x^2 + 6x + 9\,, & x \leq -5 \\ \sqrt[3]{4x+9}\,, & -5 < x \leq -2 \\ 3x^2 + \dfrac{11}{2}x\,, & x > -2 \end{cases}$, then find $g(4)$, $g(-2)$, $g\left(-\dfrac{17}{4}\right)$, $g(-4)$ and $g(-8)$. Then sketch the graph of the function g.

The function g is a piecewise function. The numbers -5 and -2 are the breakup points of the function g.

To find $g(4)$: Since $4 > -2$ and $g(x) = 3x^2 + \dfrac{11}{2}x$ when $x > -2$, then $g(4) = 48 + 22 = 70$. NOTE: Since $3x^2 + \dfrac{11}{2}x = \dfrac{1}{2}x(6x + 11)$, then we could use this information in order to find $g(4)$. Thus, $g(4) = \dfrac{1}{2}(4)(35) = 70$.

Answer: 70

To find $g(-2)$: Since $-2 = -2$ and $g(x) = \sqrt[3]{4x+9}$ when $-5 < x \leq -2$, then $g(-2) = \sqrt[3]{-8+9} = \sqrt[3]{1} = 1$.

Answer: 1

To find $g\left(-\dfrac{17}{4}\right)$: Since $-5 < -\dfrac{17}{4} \leq -2$ and $g(x) = \sqrt[3]{4x+9}$ when $-5 < x \leq -2$, then $g\left(-\dfrac{17}{4}\right) = \sqrt[3]{-17+9} = \sqrt[3]{-8} = -2$.

Answer: -2

To find $g(-4)$: Since $-5 < -4 \leq -2$ and $g(x) = \sqrt[3]{4x+9}$ when $-5 < x \leq -2$, then $g(-4) = \sqrt[3]{-16+9} = \sqrt[3]{-7} = -\sqrt[3]{7}$.

Answer: $-\sqrt[3]{7}$

To find $g(-8)$: Since $-8 \leq -5$ and $g(x) = x^2 + 6x + 9$ when $x \leq -5$, then $g(-8) = 64 - 48 + 9 = 25$. NOTE: Since $x^2 + 6x + 9 = (x + 3)^2$, then we could use this information in order to find $g(-8)$. Thus, $g(-8) = (-5)^2 = 25$.
Answer: 25

Review of sketching the quadratic function $y = ax^2 + bx + c$, which is a parabola that either opens upward or downward.

The *x*-coordinate of the vertex of the parabola is given by $x = -\dfrac{b}{2a}$. The y-coordinate of the vertex of the parabola can be found by evaluating the quadratic function at $-\dfrac{b}{2a}$. That is, find $y\left(-\dfrac{b}{2a}\right)$. Thus, the vertex of the parabola is the point $\left(-\dfrac{b}{2a},\ y\left(-\dfrac{b}{2a}\right)\right)$.

The direction that the parabola opens is determined by the sign of *a*, the coefficient of x^2 in the quadratic function $y = ax^2 + bx + c$. If $a > 0$, then the parabola opens upward. If $a < 0$, then the parabola opens downward.

The parabola has a axis of symmetry, which is a vertical line passing through the vertex of the parabola, given by the equation $x = -\dfrac{b}{2a}$.

Sometimes, the parabola can be sketched using the *x*-intercept(s) of the quadratic function and/or the axis of symmetry to find the *x*-coordinate of the vertex of the parabola. If the quadratic function has one *x*-intercept, then this *x*-intercept is the vertex of the parabola. To sketch the graph of this parabola, you would plot the vertex, which is the one *x*-intercept, and make the parabola open upward if $a > 0$ or open downward if $a < 0$. If the quadratic function has two *x*-intercepts, say $(x_1, 0)$ and $(x_2, 0)$, then the *x*-coordinate of the vertex of the parabola is the midpoint of x_1 and x_2. Thus, the *x*-coordinate of the vertex of the parabola is

$\frac{x_1 + x_2}{2}$. To sketch this parabola, you would plot the *x*-intercepts and the vertex of the parabola.

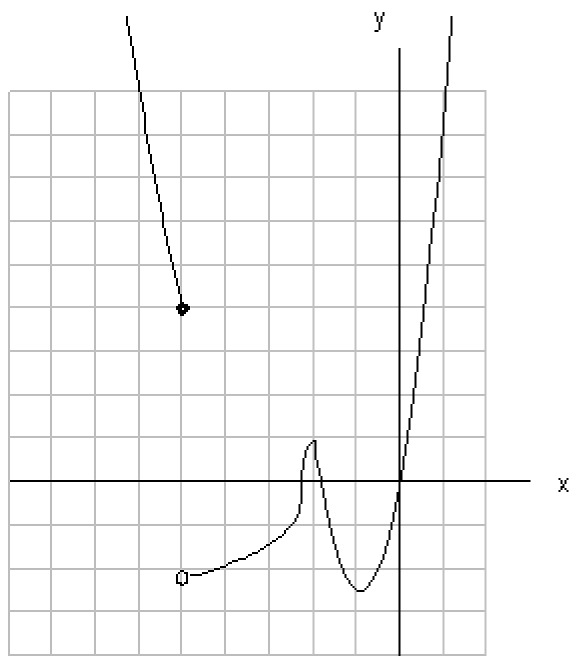
To sketch the graph of the quadratic function $y = x^2 + 6x + 9$:
Since $x^2 + 6x + 9 = (x + 3)^2$, then $y = (x + 3)^2$. To find the *x*-intercept(s), we set *y* equal to zero and solve the resulting equation. Thus, we want to solve the equation $(x + 3)^2 = 0$. There is only one solution to this equation. The solution is $x = -3$. Thus, the vertex of the parabola is $(-3, 0)$. Since the coefficient of x^2 is $1 > 0$, then the parabola opens upward. Since the piecewise function g is defined by $g(x) = x^2 + 6x + 9$ whenever $x \leq -5$, then we only use the sketch of the parabola $y = x^2 + 6x + 9$ for $x \leq -5$.

To sketch the graph of the quadratic function $y = 3x^2 + \frac{11}{2}x$: Since $3x^2 + \frac{11}{2}x$ = $\frac{1}{2}x(6x + 11)$, then $y = \frac{1}{2}x(6x + 11)$. To find the *x*-intercept(s), we set *y* equal to zero and solve the resulting equation. Thus, we want to solve the equation $\frac{1}{2}x(6x + 11) = 0$. There are two solutions to this equation. They are $x = 0$ and $x = -\frac{11}{6}$. Thus, the *x*-intercepts of the parabola are $(0, 0)$ and $\left(0, -\frac{11}{6}\right)$. Since the midpoint of 0 and $-\frac{11}{6}$ is $-\frac{11}{12}$, then the *x*-coordinate of the vertex of the parabola is $x = -\frac{11}{12}$. The *y*-coordinate of the vertex of the parabola can be obtained by evaluating the function $y = \frac{1}{2}x(6x + 11)$ at $-\frac{11}{12}$. Thus, $y\left(-\frac{11}{12}\right) = \frac{1}{2}\left(-\frac{11}{12}\right)\left(-\frac{11}{2} + 11\right) = \frac{1}{2}\left(-\frac{11}{12}\right)\left(-\frac{11}{2} + \frac{22}{2}\right) = \frac{1}{2}\left(-\frac{11}{12}\right)\left(\frac{11}{2}\right) =$

$-\dfrac{121}{48}$. Thus, the vertex of the parabola is $\left(-\dfrac{11}{12}, -\dfrac{121}{48}\right)$. Since the piecewise function g is defined by $g(x) = 3x^2 + \dfrac{11}{2}x$ whenever $x > -2$, then we only use the sketch of the parabola $y = 3x^2 + \dfrac{11}{2}x$ for $x > -2$.

To sketch the graph of the radical function $y = \sqrt[3]{4x+9}$: Since $y = \sqrt[3]{4\left(x + \dfrac{9}{4}\right)}$, then we shift the graph of $y = \sqrt[3]{4x}$ $\dfrac{9}{4}$ units to the left.

Sketch of ***g*****:**



This graph was created using Maple.

<u>Example</u> If $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$, then find $f(-4)$, $f(0)$, and $f(6)$. Then sketch the graph of the function f.

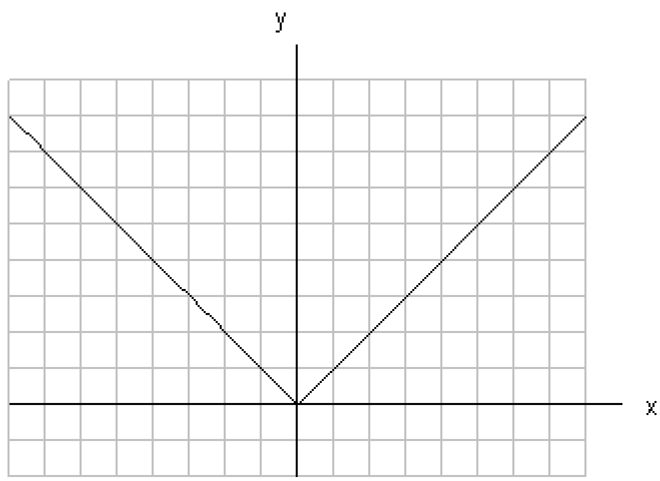
The number 0 is the breakup point of this piecewise function f.

To find $f(-4)$: Since $-4 < 0$ and $f(x) = -x$ when $x < 0$, then $f(-4) = -(-4) = 4$. **Answer:** 4

To find $f(0)$: Since $0 \geq 0$ and $f(x) = x$ when $x \geq 0$, then $f(0) = 0$.
Answer: 0

To find $f(6)$: Since $6 \geq 0$ and $f(x) = x$ when $x \geq 0$, then $f(6) = 6$.
Answer: 6

Sketch of *f*:



This graph was created using Maple.

NOTE: This piecewise function is the absolute value function.

We will have the need to find the absolute value of algebraic expressions. We will do this using the following definition.

<u>Definition</u> Let *a* be an algebraic expression. Then $|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$.

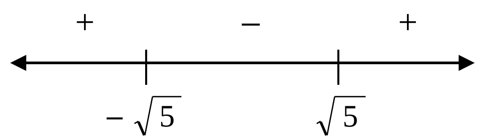
<u>Example</u> Find the absolute value of the following.

1. $|x^2 - 5|$

Using the definition of absolute value given above, we have that

$|x^2 - 5| = \begin{cases} x^2 - 5 \ , & x^2 - 5 \geq 0 \\ -(x^2 - 5)\ , & x^2 - 5 < 0 \end{cases}$. Now, we need to find when $x^2 - 5 \geq 0$ and when $x^2 - 5 < 0$. We can do this by finding the sign of $x^2 - 5$ using the three-step method from Lesson 1.

Sign of $x^2 - 5$:



Thus, $x^2 - 5 > 0$ when $x < -\sqrt{5}$ or $x > \sqrt{5}$. Since $x^2 - 5 = 0$ when $x = \pm\sqrt{5}$, then $x^2 - 5 \geq 0$ when $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$.

Also, $x^2 - 5 < 0$ when $-\sqrt{5} < x < \sqrt{5}$. Since

$|x^2 - 5| = \begin{cases} x^2 - 5 \ , & x^2 - 5 \geq 0 \\ -(x^2 - 5)\ , & x^2 - 5 < 0 \end{cases}$, then we have that

$|x^2 - 5| = \begin{cases} x^2 - 5 \ , & x \leq -\sqrt{5} \ \ or \ \ x \geq \sqrt{5} \\ -(x^2 - 5)\ , & -\sqrt{5} < x < \sqrt{5} \end{cases}$. Since $-(x^2 - 5) =$

$5 - x^2$, we may also write $|x^2 - 5| = \begin{cases} x^2 - 5\ , & x \leq -\sqrt{5} \ \ or \ \ x \geq \sqrt{5} \\ 5 - x^2\ , & -\sqrt{5} < x < \sqrt{5} \end{cases}$.

Also, since the expression $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$ is equivalent to the expression $|x| \geq \sqrt{5}$ and the expression $-\sqrt{5} < x < \sqrt{5}$ is equivalent to the expression $|x| < \sqrt{5}$, then we may also write

$$|x^2 - 5| = \begin{cases} x^2 - 5\ , & |x| \geq \sqrt{5} \\ 5 - x^2\ , & |x| < \sqrt{5} \end{cases}$$

Answer: $\left| x^2 - 5 \right| = \begin{cases} x^2 - 5\,, & x \le -\sqrt{5} \;\; or \;\; x \ge \sqrt{5} \\ 5 - x^2\,, & -\sqrt{5} < x < \sqrt{5} \end{cases}$ **or**

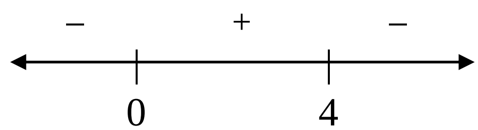
$$\left| x^2 - 5 \right| = \begin{cases} x^2 - 5\,, & |x| \ge \sqrt{5} \\ 5 - x^2\,, & |x| < \sqrt{5} \end{cases}$$

2. $\left| 4t - t^2 \right|$

Using the definition of absolute value given above, we have that

$$\left| 4t - t^2 \right| = \begin{cases} 4t - t^2\,, & 4t - t^2 \ge 0 \\ -(4t - t^2)\,, & 4t - t^2 < 0 \end{cases}.$$

Sign of $4t - t^2 = t(4 - t)$:



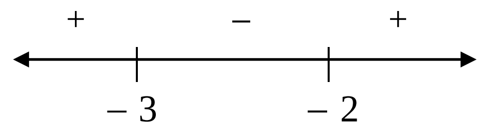
Answer: $\left| 4t - t^2 \right| = \begin{cases} 4t - t^2\,, & 0 \le t \le 4 \\ t^2 - 4t\,, & t < 0 \;\; or \;\; t > 4 \end{cases}$

3. $\left| w^2 + 5w + 6 \right|$

Using the definition of absolute value given above, we have that

$$\left| w^2 + 5w + 6 \right| = \begin{cases} w^2 + 5w + 6\,, & w^2 + 5w + 6 \ge 0 \\ -(w^2 + 5w + 6)\,, & w^2 + 5w + 6 < 0 \end{cases}.$$

Sign of $w^2 + 5w + 6 = (w + 2)(w + 3)$:



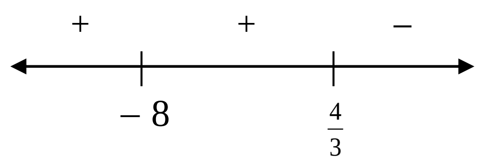
Answer: $\left|w^2 + 5w + 6\right| = \begin{cases} w^2 + 5w + 6 & , \ w \le -3 \ \mathit{or} \ w \ge -2 \\ -(w^2 + 5w + 6), & -3 < w < -2 \end{cases}$

4. $\left|(x + 8)^2 (4 - 3x)\right|$

Using the definition of absolute value given above, we have that

$$\left|(x + 8)^2 (4 - 3x)\right| = \begin{cases} (x + 8)^2 (4 - 3x) \ , & (x + 8)^2 (4 - 3x) \ge 0 \\ -(x + 8)^2 (4 - 3x) \ , & (x + 8)^2 (4 - 3x) < 0 \end{cases}.$$

Sign of $(x + 8)^2 (4 - 3x)$:



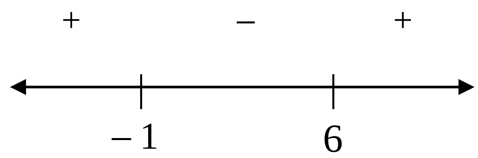
Answer: $\left|(x + 8)^2 (4 - 3x)\right| = \begin{cases} (x + 8)^2 (4 - 3x) \ , & x \le \dfrac{4}{3} \\ (x + 8)^2 (3x - 4) \ , & x > \dfrac{4}{3} \end{cases}$

5. $\left|\dfrac{t + 1}{t - 6}\right|$

Using the definition of absolute value given above, we have that

$$\left|\frac{t + 1}{t - 6}\right| = \begin{cases} \dfrac{t + 1}{t - 6} \ , & \dfrac{t + 1}{t - 6} \ge 0 \\ -\dfrac{t + 1}{t - 6} \ , & \dfrac{t + 1}{t - 6} < 0 \end{cases}.$$

Sign of $\dfrac{t+1}{t-6}$:



Answer: $\left|\dfrac{t+1}{t-6}\right| = \begin{cases} \dfrac{t+1}{t-6} & , \ t \le -1 \ or \ t > 6 \\[2ex] \dfrac{t+1}{6-t} & , \ -1 < t < 6 \end{cases}$

6. $\dfrac{3x - |x|}{2x+5}$

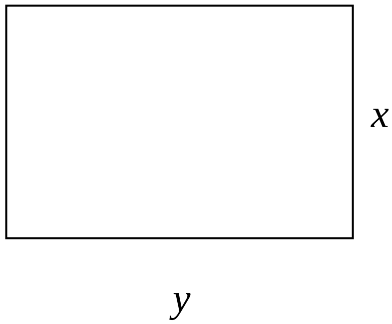
Since $|x| = \begin{cases} x \ , & x \ge 0 \\ -x \ , & x < 0 \end{cases}$, then $|x| = x$ when $x \ge 0$. Thus, $\dfrac{3x - |x|}{2x+5} = \dfrac{3x - x}{2x+5} = \dfrac{2x}{2x+5}$ when $x \ge 0$. Since $|x| = \begin{cases} x \ , & x \ge 0 \\ -x \ , & x < 0 \end{cases}$, then $|x| = -x$ when $x < 0$. Thus, $\dfrac{3x - |x|}{2x+5} = \dfrac{3x - (-x)}{2x+5} = \dfrac{3x + x}{2x+5} = \dfrac{4x}{2x+5}$ when $x < 0$.

Answer: $\dfrac{3x - |x|}{2x+5} = \begin{cases} \dfrac{2x}{2x+5} & , \ x \ge 0 \\[2ex] \dfrac{4x}{2x+5} & , \ x < 0 \end{cases}$

<u>**Examples**</u> Find a function of one variable for the following descriptions.

1. Betty wishes to fence a rectangular region of area 650 square yards. Express the amount F of fencing that is required as function of x, which is the length of the rectangle.

We will need to identify the width of the rectangle: Let y be the width of the rectangle.



NOTE: The amount of fencing, which is required to fence this rectangular region, is the perimeter of the rectangle. Thus, $F = 2x + 2y$.

NOTE: F is a function of two variables x and y. In order to get F as a function of one variable x, we will need to get a relationship between x and y. A relationship between x and y is an equation containing only the variables of x and y. We haven't used the information that the area of the rectangular enclosure is to 650 square yards. Since the area of a rectangle is given by the formula $A = lw$, then the area of our rectangular enclosure is $A = xy$. Thus, in order for the area of our enclosure to be 650 cubic feet, we need that $xy = 650$. This is our relationship between x and y.

Now, we can solve for y in terms of x. Thus, $xy = 650 \Rightarrow y = \dfrac{650}{x}$.

Since $F = 2x + 2y$ and $y = \dfrac{650}{x}$, then $F = 2x + 2y = 2x + 2\left(\dfrac{650}{x}\right)$

$= 2x + \dfrac{1300}{x}$.

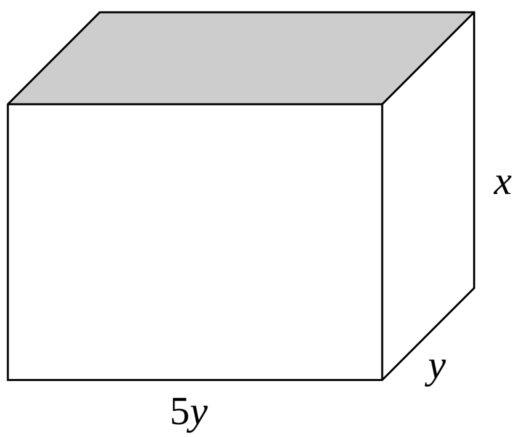
Answer: $F = 2x + \dfrac{1300}{x}$ (in yards)

NOTE: In Calculus I (MATH-1850), you will find the dimensions of the rectangle which require the least of amount of fencing in order to enclose 650 square yards.

2. A closed rectangular box is to be constructed having a volume of 85 ft^3. The width of the bottom of the box is y ft. The length of the bottom of the box is five times the width of the bottom. Express the amount M of material that is needed to make this box as a function of y.

NOTE: Closed box means that there will be a top to the box.

We will need to identify the height of the box: Let x be the height of the box.



The amount of material, which is needed to construct the base **or** the top of the box, is given by $5y^2$ square feet. (You only need to recognize this statement. You do not need to write it.)

Thus, the amount of material, which is needed to construct the base **and** the top of the box, is given by $2(5y^2)$ square feet. (You only need to recognize this statement. You do not need to write it.)

The amount of material, which is needed to construct the front **or** the back of the box, is given by $5xy$ square feet. (You only need to recognize this statement. You do not need to write it.)

Thus, the amount of material, which is needed to construct the front **and** the back of the box, is given by $2(5xy)$ square feet. (You only need to recognize this statement. You do not need to write it.)

The amount of material, which is needed to construct the left-side **or** the right-side of the box, is given by xy square feet. (You only need to recognize this statement. You do not need to write it.)

Thus, the amount of material, which is needed to construct the front **and** the back of the box, is given by $2xy$ square feet. (You only need to recognize this statement. You do not need to write it.)

Thus, $M = 2(5y^2) + 2(5xy) + 2xy \Rightarrow M = 10y^2 + 10xy + 2xy \Rightarrow$

$M = 10y^2 + 12xy$.

NOTE: M is a function of two variables x and y. In order to get M as a function of one variable y, we will need to get a relationship between x and y. A relationship between x and y is an equation containing only the variables of x and y. We haven't used the information that the volume of the box is to 85 cubic feet. Since the volume of a box is given by the formula $V = lwh$, then the volume of our box is $V = (5y)\,y\,x = 5xy^2$. Thus, in order for the volume of our box to be 85 cubic feet, we need that $5xy^2 = 85$. This is our relationship between x and y.

Now, we can solve for x in terms of y. Thus, $5xy^2 = 85 \Rightarrow x = \dfrac{85}{5y^2} \Rightarrow$

$x = \dfrac{17}{y^2}$. Since $M = 10y^2 + 12xy$ and $x = \dfrac{17}{y^2}$, then $M = 10y^2 + 12xy$

$= 10y^2 + 12\left(\dfrac{17}{y^2}\right)y = 10y^2 + \dfrac{204}{y}$.

NOTE: $17(12) = 17(10 + 2) = 170 + 34 = 204$

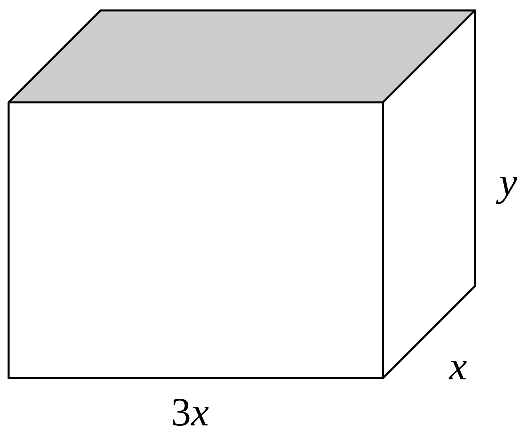
Answer: $M = 10y^2 + \dfrac{204}{y}$ (in ft^2)

NOTE: In Calculus I (MATH-1850), you will find the dimensions of the box which require the least of amount of material in order to have a volume of 85 cubic feet.

3. An open rectangular box is to be constructed having a volume of 288 in^3. The length of the bottom of the box is three times the width of the bottom.

The material for the bottom of the box costs 8 cents per square inch and the material for the four sides costs 5 cents per square inch. Express the cost C to make this box as a function of one variable.

NOTE: An open box means no top.



NOTE: The amount of material, which is needed to construct the bottom of the box, is given by $3x^2$ square inches. Thus, the cost to construct the bottom of the box is given by $8(3x^2)$ cents.

NOTE: The amount of material, which is needed to construct the front and the back of the box, is given by $2(3xy)$ square inches. Thus, the cost to construct the front and the back of the box is given by $5[2(3xy)]$ cents.

NOTE: The amount of material, which is needed to construct the left-side and the right-side of the box, is given by $2xy$ square inches. Thus, the cost to construct the front and the back of the box is given by $5(2xy)$ cents.

Thus, the cost, C, to construct this open box, is given by $C = 8(3x^2) + 5[2(3xy) + 2xy]$ in cents. Simplifying, we have that $C = 24x^2 + 5(6xy + 2xy) = 24x^2 + 5(8xy) = 24x^2 + 40xy$ in cents.

NOTE: C is a function of two variables x and y. In order to get C as a function of one variable in x or y, we will need to get a relationship between x and y. A relationship between x and y is an equation containing only the variables of x and y. We haven't used the information that the volume of the box is to be 288 cubic inches. Since the volume of a box is given by the formula $V = lwh$, then the volume of our box is $V = (3x)\,x\,y = 3x^2y$. Thus, in order for the volume of our box to be 288 cubic inches, we need that $3x^2y = 288$. This is our relationship between x and y.

Now, we can solve the equation $3x^2y = 288$ for x or y. It is easier to solve for y in terms of x. Thus, $3x^2y = 288 \;\Rightarrow\; y = \dfrac{288}{3x^2} \;\Rightarrow\; y = \dfrac{96}{x^2}$.

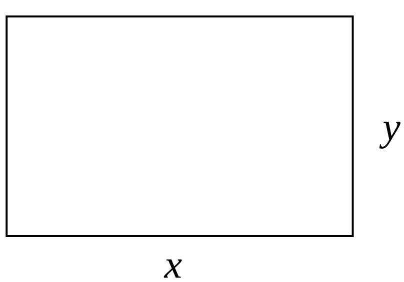
Since $C = 24x^2 + 40xy$ and $y = \dfrac{96}{x^2}$, then $C = 24x^2 + 40xy$ =

$$4\left[6x^2 + 10x\left(\frac{96}{x^2}\right)\right] = 24\left[x^2 + 10x\left(\frac{16}{x^2}\right)\right] = 24\left(x^2 + \frac{160}{x}\right).$$

Answer: $C = 24\left(x^2 + \dfrac{160}{x}\right)$ (in cents)

NOTE: In Calculus I (MATH-1850), you will find the dimensions of the box cheapest to construct and will have a volume of 28 cubic inches.

4. Bill can only afford to buy 100 yards of fencing. He uses the fencing to enclose his rectangular garden. Express the area A of the rectangular enclosure as a function of one variable.



Area of the enclosure: $A = xy$

Since the amount of the fencing is 100 yards, we have that $2x + 2y = 100 \;\Rightarrow\; x + y = 50$. Solving for y, we have that $y = 50 - x$.

Since $A = xy$ and $y = 50 - x$, then $A = x(50 - x) = 50x - x^2$.

Answer: $A = 50x - x^2$ (in square yards)