LESSON 11 EXPONENTIAL AND LOGARITHMIC EQUATIONS

Since any exponential function is one-to-one, then $b^u = b^v$ if and only if $u = v$.

<u>Examples</u> Solve the following exponential equations. Give exact answers. No decimal approximations.

1. $3^x = 81$

Using the one-to-one property: $3^x = 81 \Rightarrow 3^x = 3^4 \Rightarrow x = 4$

Using logarithms base 3: $3^x = 81 \Rightarrow \log_3 3^x = \log_3 81 \Rightarrow$

$x \log_3 3 = \log_3 81 \Rightarrow x = \log_3 81 = 4$

NOTE: $\log_3 3 = 1$

Using natural logarithms: $3^x = 81 \Rightarrow \ln 3^x = \ln 81 \Rightarrow x \ln 3 = \ln 81 \Rightarrow$

$x = \dfrac{\ln 81}{\ln 3} \Rightarrow x = 4$ (using a calculator)

Answer: 4

2. $5^{-t} = 25$

3. $2^{3x - 11} = 32$

Using the one-to-one property: $2^{3x - 11} = 32 \Rightarrow 2^{3x - 11} = 2^5 \Rightarrow$

$$3x - 11 = 5 \Rightarrow 3x = 16 \Rightarrow x = \frac{16}{3}$$

Using logarithms base 2: $2^{3x - 11} = 32 \Rightarrow \log_2 2^{3x - 11} = \log_2 32 \Rightarrow$

$$(3x - 11) \log_2 2 = \log_2 32 \Rightarrow 3x - 11 = 5 \Rightarrow 3x = 16 \Rightarrow x = \frac{16}{3}$$

NOTE: $\log_2 2 = 1$

Using natural logarithms: $2^{3x - 11} = 32 \Rightarrow \ln 2^{3x - 11} = \ln 32 \Rightarrow$

$$(3x - 11) \ln 2 = \ln 32 \Rightarrow 3x \ln 2 - 11 \ln 2 = \ln 32 \Rightarrow$$

$$3x \ln 2 = \ln 32 + 11 \ln 2 \Rightarrow x = \frac{\ln 32 + 11 \ln 2}{3 \ln 2} = \frac{\ln 32(2^{11})}{\ln 8} = \frac{16}{3}$$

(using a calculator)

Answer: $\frac{16}{3}$

4. $4^{x^3} = \frac{1}{16}$

5. $16^t = 64$

6. $9^{x + 4} = \frac{1}{27}$

7. $6^x = 12$

Using natural logarithms: $6^x = 12 \Rightarrow \ln 6^x = \ln 12 \Rightarrow$

$x \ln 6 = \ln 12 \Rightarrow x = \dfrac{\ln 12}{\ln 6}$

NOTE: $x = \dfrac{\ln 12}{\ln 6} \approx 1.38685$ and $6^{1.38685} \approx 11.99994$

Using logarithms base 6: $6^x = 12 \Rightarrow \log_6 6^x = \log_6 12 \Rightarrow$

$x \log_6 6 = \log_6 12 \Rightarrow x = \log_6 12$ NOTE: $\log_6 6 = 1$

Since your calculator does not have logarithm base 6 key, you would have to do a change of bases to obtain an approximation for $\log_6 12$. Since your calculator has a natural logarithm key $\boxed{\text{LN}}$, then we obtain that $\log_6 12 = \dfrac{\ln 12}{\ln 6}$ using the change of base formula that $\log_b u = \dfrac{\log_a u}{\log_a b}$, where $u = 12$, $b = 6$, and $a = e$. Or, Since your calculator has a common logarithm key $\boxed{\text{LOG}}$, then we obtain that $\log_6 12 = \dfrac{\log 12}{\log 6}$ using the change of base formula that $\log_b u = \dfrac{\log_a u}{\log_a b}$, where $u = 12$, $b = 6$, and $a = 10$.

Answer: $x = \dfrac{\ln 12}{\ln 6}$

8. $5^t = \dfrac{2}{3}$

9. $7^{-x} = \dfrac{3}{4}$

10. $8^{5 - 2x} = 65$

11. $3^{7x + 4} = 49$

Examples Solve the following logarithmic equations. Give exact answers. No decimal approximations.

1. $\log_x 16 = 2$

Using the definition of logarithm ($y = \log_b x$ if and only if $b^y = x$), we will write the logarithmic equation as an exponential equation:

$$\log_x 16 = 2 \Rightarrow x^2 = 16$$

Using square roots to solve the equation $x^2 = 16$, we have that

$$x^2 = 16 \Rightarrow \sqrt{x^2} = \sqrt{16} \Rightarrow |x| = 4 \Rightarrow x = \pm 4$$

Since the base of a logarithm can not be negative, then the solution of $x = -4$ can not be used. Thus, the only solution of the equation $\log_x 16 = 2$ is $x = 4$.

Answer: $x = 4$

2. $\log_x 5 = 3$

Using the definition of logarithm ($y = \log_b x$ if and only if $b^y = x$), we will write the logarithmic equation as an exponential equation:

$$\log_x 5 = 3 \Rightarrow x^3 = 5$$

Using cube roots to solve the equation $x^3 = 5$, we have that

$$x^3 = 5 \Rightarrow \sqrt[3]{x^3} = \sqrt[3]{5} \Rightarrow x = \sqrt[3]{5}$$

Answer: $x = \sqrt[3]{5}$

3. $\log_3 x = -2$

Using the definition of logarithm ($y = \log_b x$ if and only if $b^y = x$), we will write the logarithmic equation as an exponential equation:

$$\log_3 x = -2 \Rightarrow x = 3^{-2} \Rightarrow x = \frac{1}{9}$$

We need to check that the number $\frac{1}{9}$ makes the argument of the logarithm positive since the logarithm of a negative number or zero is undefined. The argument of $\log_3 x$ is x. Thus, x will be positive when $x = \frac{1}{9}$. Thus, $\frac{1}{9}$ is a solution of the equation $\log_3 x = -2$. Of course, it is the only solution.

Answer: $x = \frac{1}{9}$

4. $\log(4t - 9) = 2$

Recall that log is the notation for the common logarithm, and the base of the common logarithm is 10. Using the definition of logarithm ($y = \log_b x$ if and only if $b^y = x$), we will write the logarithmic equation as an exponential equation:

$$\log(4t - 9) = 2 \Rightarrow 4t - 9 = 10^2 \Rightarrow 4t - 9 = 100$$

Solving the equation $4t - 9 = 100$, we have that $t = \dfrac{109}{4}$.

We need to check that the number $\dfrac{109}{4}$ makes the argument of the logarithm positive since the logarithm of a negative number or zero is undefined. The argument of $\log(4t - 9)$ is $4t - 9$.

When $t = \dfrac{109}{4}$, we have that $4t - 9 = 4\left(\dfrac{109}{4}\right) - 9 = 109 - 9 > 0$.

Thus, $\dfrac{109}{4}$ is a solution of the equation $\log(4t - 9) = 2$. Of course, it is the only solution.

Answer: $t = \dfrac{109}{4}$

5. $\log_2 x + \log_2(x - 12) = 6$

First, we'll use the property of logarithms that $\log_b uv = \log_b u + \log_b v$ in order to write $\log_2 x + \log_2(x - 12)$ as $\log_2 x(x - 12)$. Thus,

$$\log_2 x + \log_2(x - 12) = 6 \Rightarrow \log_2 x(x - 12) = 6$$

Now, using the definition of logarithm ($y = \log_b x$ if and only if $b^y = x$), we will write the logarithmic equation $\log_2 x(x - 12) = 6$ as an exponential equation:

$$\log_2 x(x - 12) = 6 \Rightarrow x(x - 12) = 2^6 \Rightarrow x(x - 12) = 64$$

Solving the equation $x(x-12)=64$, we have that

$$x(x-12)=64 \Rightarrow x^2-12x=64 \Rightarrow x^2-12x-64=0 \Rightarrow$$

$$(x+4)(x-16)=0 \Rightarrow x=-4,\ x=16$$

We need to check that the numbers -4 and 16 make the argument of the logarithms positive since the logarithm of a negative number or zero is undefined. The argument of $\log_2 x$ is x and the argument of $\log_2(x-12)$ is $x-12$.

When $x=-4$, we have that $x=-4<0$. Thus, when $x=-4$, we have that $\log_2 x=\log_2(-4)$. However, $\log_2(-4)$ is undefined. Thus, -4 is a solution of the equation $x(x-12)=64$, but it is not a solution of the equation $\log_2 x+\log_2(x-12)=6$.

When $x=16$, we have that $x=16>0$ and $x-12=16-12>0$. Thus, 16 is a solution of the equation $\log_2 x+\log_2(x-12)=6$.

Answer: $x=16$

6. $\ln x=\ln(5-x)$

In order to solve this equation, we will use the fact that any logarithm function is one-to-one. Thus, $\log_b u=\log_b v$ if and only if $u=v$.

Thus, by the one-to-one property, we have that

$$\ln x=\ln(5-x) \Rightarrow x=5-x$$

Solving the equation $x=5-x$, we have that $x=\frac{5}{2}$.

We need to check that the number $\frac{5}{2}$ makes the argument of the logarithms positive since the logarithm of a negative number or zero is undefined. The argument of $\ln x$ is x and the argument of $\ln(5 - x)$ is $5 - x$.

When $x = \frac{5}{2}$, we have that $x = \frac{5}{2} > 0$ and $5 - x = 5 - \frac{5}{2} > 0$. Thus, $\frac{5}{2}$ is a solution of the equation $\ln x = \ln(5 - x)$.

Answer: $x = \frac{5}{2}$

7. $\log_8 t = \log_8(7t + 11)$

In order to solve this equation, we will use the fact that any logarithm function is one-to-one. Thus, $\log_b u = \log_b v$ if and only if $u = v$.

Thus, by the one-to-one property, we have that

$$\log_8 t = \log_8(7t + 11) \Rightarrow t = 7t + 11$$

Solving the equation $t = 7t + 11$, we have that $t = -\frac{11}{6}$.

We need to check that the number $-\frac{11}{6}$ makes the argument of the logarithms positive since the logarithm of a negative number or zero is undefined. The argument of $\log_8 t$ is t and the argument of $\log_8(7t + 11)$ is $7t + 11$.

When $t = -\frac{11}{6}$, we have that $t = -\frac{11}{6} < 0$. Thus, when $t = -\frac{11}{6}$, we have that $\log_8 t = \log_8\left(-\frac{11}{6}\right)$. However, $\log_8\left(-\frac{11}{6}\right)$ is undefined.

Thus, $-\frac{11}{6}$ is a solution of the equation $t = 7t + 11$, but it is not a solution of the equation $\log_8 t = \log_8 (7t + 11)$.

Answer: No solution