LESSON 10 LOGARITHMIC FUNCTIONS

Definition The logarithmic function with base b is the function defined by $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.

Recall that $y = \log_b x$ if and only if $b^y = x$

Recall the following information about logarithmic functions:

1. The domain of $f(x) = \log_b x$ is the set of positive real numbers. That is, the domain of $f(x) = \log_b x$ is $(0, \infty)$.

2. The range of $f(x) = \log_b x$ is the set of real numbers. That is, the range of $f(x) = \log_b x$ is $(-\infty, \infty)$.

3. The logarithmic function $f(x) = \log_b x$ and the exponential function $g(x) = b^x$ are inverses of one another:

   $(f \circ g)(x) = f(g(x)) = \log_b g(x) = \log_b b^x = x \log_b b = x(1) = x$, for all x in the domain of g, which is the set of all real numbers.

   $(g \circ f)(x) = g(f(x)) = g(\log_b x) = b^{\log_b x} = x$, for all x in the domain of *f*, which is the set of real numbers in the interval $(0, \infty)$.

Definition The natural logarithmic function is the logarithmic function whose base is the irrational number *e*. Thus, the natural logarithmic function is the function defined by $f(x) = \log_e x$, where $e = 2.718281828\ldots$. Recall that $\log_e x = \ln x$.

Definition The common logarithmic function is the logarithmic function whose base is the number 10. Thus, the common logarithmic function is the function defined by $f(x) = \log_{10} x$. Recall that $\log_{10} x = \log x$.

Theorem (Properties of Logarithms)

1. $\log_b u^r = r \log_b u$

2. $\log_b uv = \log_b u + \log_b v$

3. $\log_b \frac{u}{v} = \log_b u - \log_b v$

4. $\log_b b = 1$

5. $\log_b 1 = 0$

6. $b^{\log_b u} = u$

7. $\log_b b^u = u$

8. Change of Bases Formula: $\log_b u = \dfrac{\log_a u}{\log_a b}$

Proof

1. Let $y = \log_b u$. Then by the definition of logarithms, $b^y = u$. Thus, $u^r = (b^y)^r = b^{yr} = b^{ry}$. Writing the exponential equation $u^r = b^{ry}$ in terms of a logarithmic equation, we have that $\log_b u^r = ry$. Since $y = \log_b u$, then we have that $\log_b u^r = r \log_b u$.

2. Let $y = \log_b u$ and $w = \log_b v$. Then by the definition of logarithms, $b^y = u$ and $b^w = v$. Thus, $uv = b^y b^w = b^{y+w}$. Writing the exponential equation $uv = b^{y+w}$ in terms of a logarithmic equation, we have that $\log_b uv = y + w$. Since $y = \log_b u$ and $w = \log_b v$, then $\log_b uv = \log_b u + \log_b v$.

3. Let $y = \log_b u$ and $w = \log_b v$. Then by the definition of logarithms, $b^y = u$ and $b^w = v$. Thus, $\frac{u}{v} = \frac{b^y}{b^w} = b^{y-w}$. Writing the exponential equation $\frac{u}{v} = b^{y-w}$ in terms of a logarithmic equation, we have that $\log_b \frac{u}{v} = y - w$. Since $y = \log_b u$ and $w = \log_b v$, then $\log_b \frac{u}{v} = \log_b u - \log_b v$.

Alternate proof: Since $\frac{u}{v} = uv^{-1}$, we have that $\log_b \frac{u}{v} = \log_b uv^{-1}$. Now, applying Property 2, we have that $\log_b uv^{-1} = \log_b u + \log_b v^{-1}$. Now, applying Property 1, we have that $\log_b v^{-1} = -\log_b v$. Thus, we have that $\log_b \frac{u}{v} = \log_b uv^{-1} = \log_b u + \log_b v^{-1} = \log_b u - \log_b v$.

6. Let $y = \log_b u$. Then by the definition of logarithms, $b^y = u$. Since $y = \log_b u$, then $b^{\log_b u} = u$.

7. Follows from applying Property 1 and then Property 4.

8. Let $y = \log_b u$, $w = \log_a u$, and $z = \log_a b$. Then by the definition of logarithms, we have that $b^y = u$, $a^w = u$, and $a^z = b$. Since $a^z = b$, then $b^y = (a^z)^y = a^{yz}$. Since $b^y = u$ and $b^y = a^{yz}$, then $a^{yz} = u$. Since $a^w = u$, then $a^{yz} = a^w$. Thus, $yz = w$. Since $y = \log_b u$, $z = \log_a b$, and $w = \log_a u$, then $(\log_b u)(\log_a b) = \log_a u$. Since b is the base of a logarithm, then $b \neq 1$. Since $\log_a b = 0$ if and only if $b = 1$, then $\log_a b \neq 0$. So, we can solve for $\log_b u$ by dividing both sides of the equation $(\log_b u)(\log_a b) = \log_a u$ by $\log_a b$. Thus, we obtain that $\log_b u = \frac{\log_a u}{\log_a b}$.

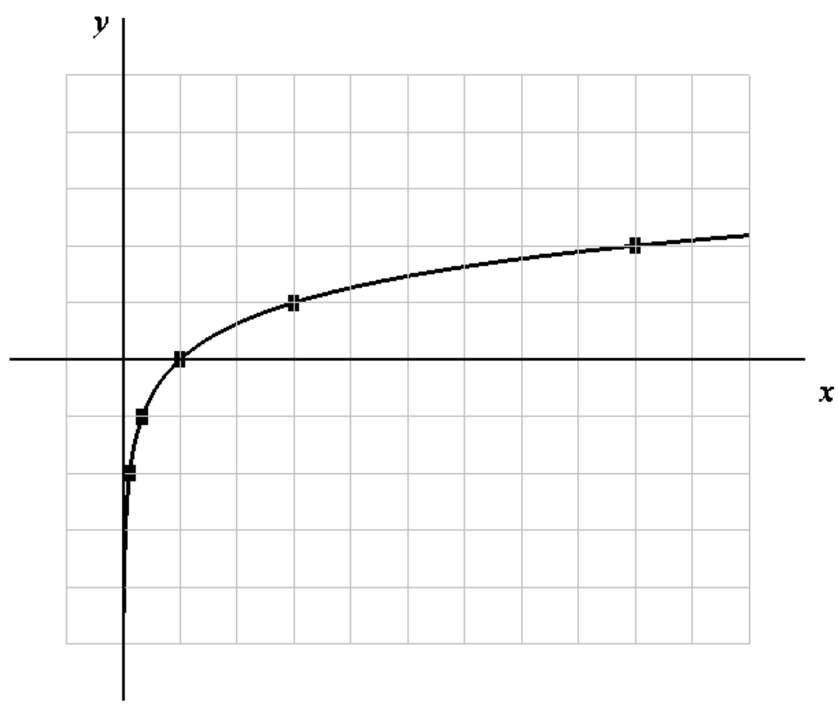
Alternate proof: Let $y = \log_b u$. Then by the definition of logarithms, $b^y = u$. Taking the logarithm base a of both sides of this equation, we obtain that $\log_a b^y = \log_a u$. By Property 1, we have that $\log_a b^y = y \log_a b$. Thus, $\log_a b^y = \log_a u \Rightarrow y \log_a b = \log_a u$. Since b is the base of a logarithm, then $b \neq 1$. Since $\log_a b = 0$ if and only if $b = 1$, then $\log_a b \neq 0$. Solving for y, we obtain that $y = \dfrac{\log_a u}{\log_a b}$. Since $y = \log_b u$, then $\log_b u = \dfrac{\log_a u}{\log_a b}$.

Examples Graph the following logarithmic functions.

1. $g(x) = \log_3 x$

Note that the domain of the logarithmic function g is $(0, \infty)$. In order to graph the function g given by $g(x) = \log_3 x$, we set $g(x) = y$ and graph the equation $y = \log_3 x$. By the definition of logarithm, $y = \log_3 x$ if and only if $x = 3^y$.

x	y
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2



The Drawing of this Graph

The x-intercept of the graph of the function is the point $(1, 0)$.

Note that as $x \to 0$ from the right, $y = \log_3 x \to -\infty$. Thus, the vertical line of $x = 0$, which is the y-axis, is a vertical asymptote of the graph of the function.

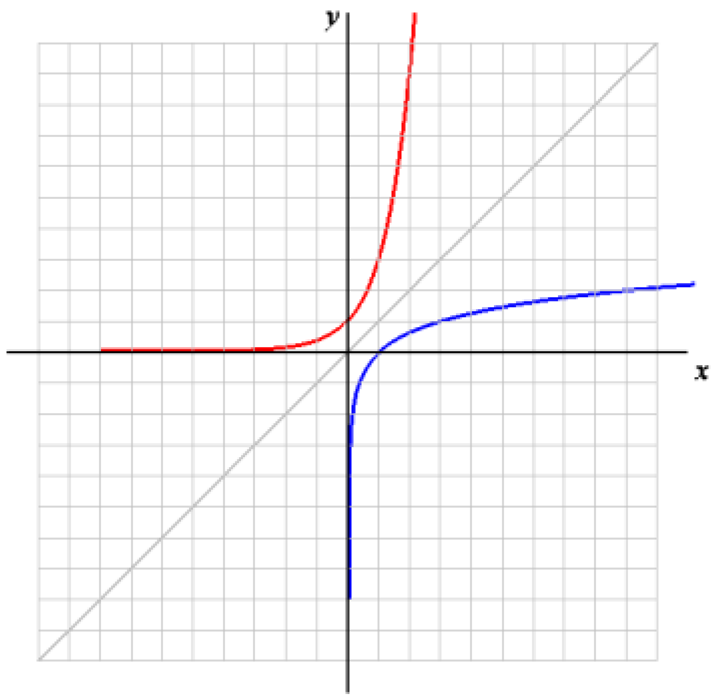
The functions $y = 3^x$ and $y = \log_3 x$ are inverse functions of one another:

$$y = 3^x \Rightarrow x = \log_3 y$$

$$y = \log_3 x \Rightarrow x = 3^y$$

We graphed the function $f(x) = 3^x$ in Lesson 9.

The graph of $y = 3^x$ is red and the graph of $y = \log_3 x$ is blue:



The Drawing of these Graphs

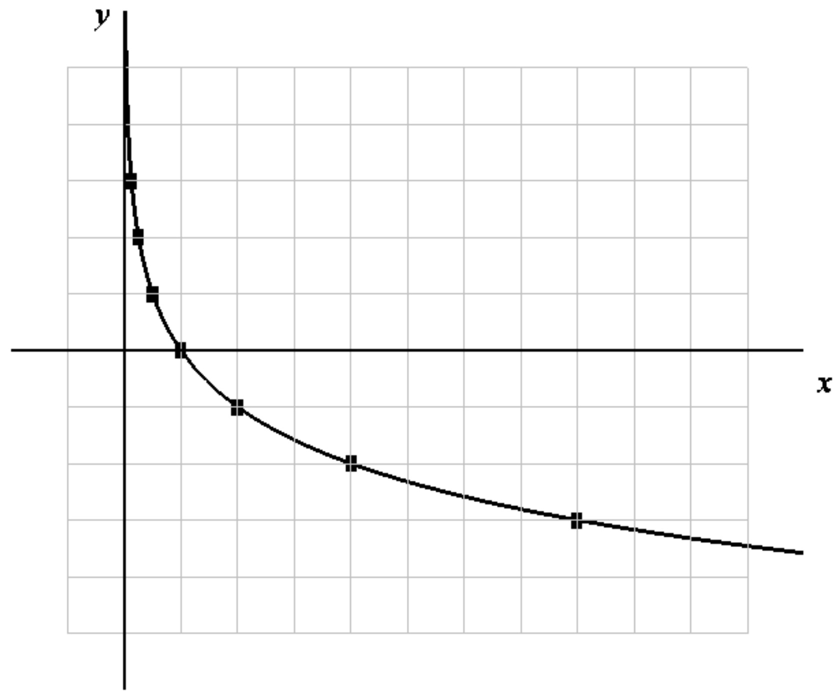
Each graph is a reflection of the other through the line $y = x$, which is gray.

2. $f(x) = \log_{1/2} x$

Note that the domain of the logarithmic function f is $(0, \infty)$. In order to graph the function f given by $f(x) = \log_{1/2} x$, we set $f(x) = y$ and graph the equation $y = \log_{1/2} x$. By the definition of logarithm,

$y = \log_{1/2} x$ if and only if $x = \left(\frac{1}{2}\right)^y$.

x	y
8	-3
4	-2
2	-1
1	0
$\frac{1}{2}$	1
$\frac{1}{4}$	2
$\frac{1}{8}$	3



The Drawing of this Graph

The x-intercept of the graph of the function is the point $(1, 0)$.

Note that as $x \to 0$ from the right, $y = \log_{1/2} x \to \infty$. Thus, the vertical line of $x = 0$, which is the y-axis, is a vertical asymptote of the graph of the function.

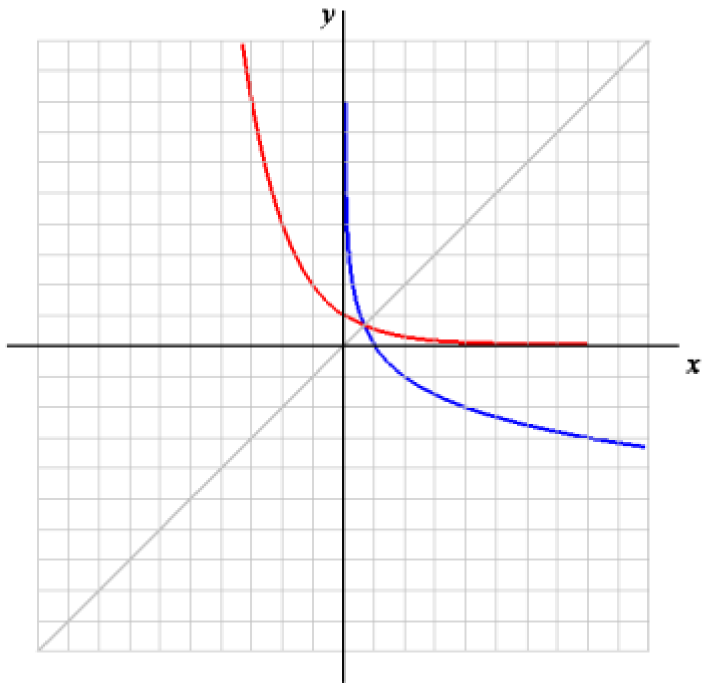
The functions $y = \log_{1/2} x$ and $y = \left(\frac{1}{2}\right)^x$ are inverse functions of one another:

$$y = \left(\frac{1}{2}\right)^x \Rightarrow x = \log_{1/2} y$$

$$y = \log_{1/2} x \;\Rightarrow\; x = \left(\frac{1}{2}\right)^y$$

We graphed the function $g(x) = \left(\frac{1}{2}\right)^x$ in Lesson 9.

The graph of $y = \left(\frac{1}{2}\right)^x$ is red and the graph of $y = \log_{1/2} x$ is blue:



The Drawing of these Graphs

Each graph is a reflection of the other through the line $y = x$, which is gray.

3. $h(x) = \log_4(-x)$

Note that the domain of the logarithmic function h is $(-\infty, 0)$. In order to graph the function h given by $h(x) = \log_4(-x)$, we set $h(x) = y$ and graph the equation $y = \log_4(-x)$. By the definition of logarithm, $y = \log_4(-x)$ if and only if $-x = 4^y \Rightarrow x = -4^y$.

x	y
$-\frac{1}{16}$	-2
$-\frac{1}{4}$	-1
-1	0
-4	1
-16	2

The Drawing of this Graph

The *x*-intercept of the graph of the function is the point $(-1, 0)$.

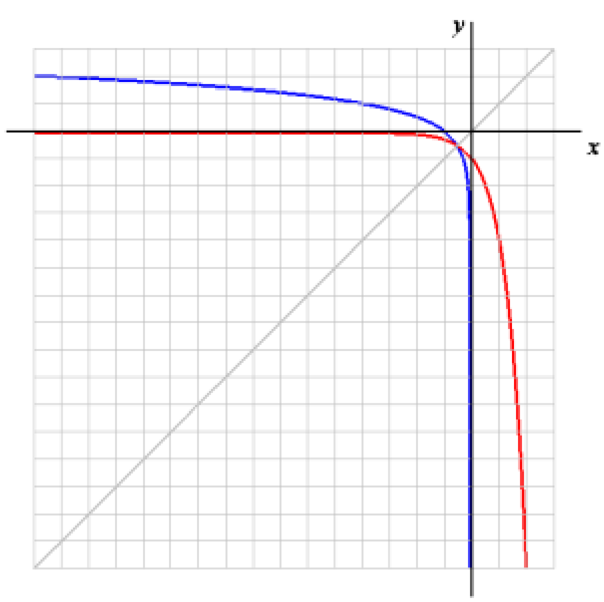
Note that as $x \to 0$ from the left, $y = \log_4(-x) \to -\infty$. Thus, the vertical line of $x = 0$, which is the *y*-axis, is a vertical asymptote of the graph of the function.

The functions $y = \log_4(-x)$ and $y = -4^x$ are inverse functions of one another:

$$y = \log_4(-x) \Rightarrow -x = 4^y \Rightarrow x = -4^y$$

$$y = -4^x \Rightarrow -y = 4^x \Rightarrow x = \log_4(-y)$$

The graph of $y = -4^x$ is red and the graph of $y = \log_4(-x)$ is blue:



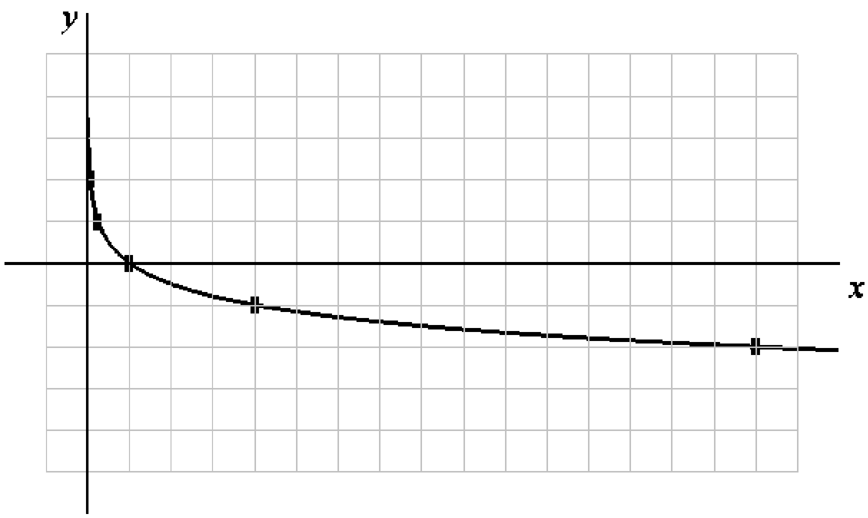
The Drawing of these Graphs

Each graph is a reflection of the other through the line $y = x$, which is gray.

4. $k(x) = -\log_4 x$

Note that the domain of the logarithmic function k is $(0, \infty)$. In order to graph the function k given by $k(x) = -\log_4 x$, we set $h(x) = y$ and graph the equation $y = -\log_4 x$. Since $y = -\log_4 x \Rightarrow -y = \log_4 x$, then by the definition of logarithm, $-y = \log_4 x$ if and only if $x = 4^{-y}$.

x	y
16	-2
4	-1
1	0
$\frac{1}{4}$	1
$\frac{1}{16}$	2



The Drawing of this Graph

The x-intercept of the graph of the function is the point $(1, 0)$.

Note that as $x \to 0$ from the right, $y = -\log_4 x \to \infty$. Thus, the vertical line of $x = 0$, which is the y-axis, is a vertical asymptote of the graph of the function.

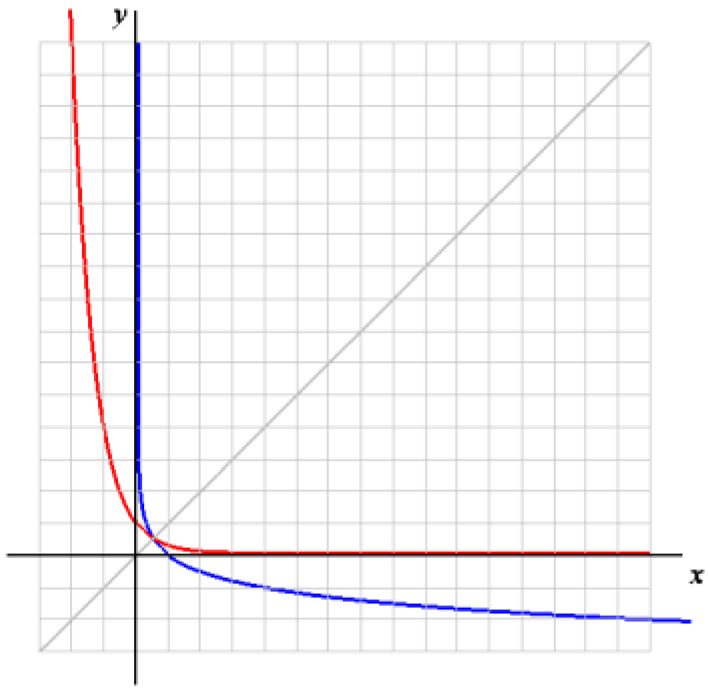
The functions $k(x) = -\log_4 x$ and $h(x) = 4^{-x}$ are inverse functions of one another:

$$y = 4^{-x} \Rightarrow -x = \log_4 y \Rightarrow x = -\log_4 y$$

$$y = -\log_4 x \Rightarrow -y = \log_4 x \Rightarrow x = 4^{-y}$$

We graphed the function $h(x) = 4^{-x}$ in Lesson 9.

The graph of $y = 4^{-x}$ is red and the graph of $y = -\log_4 x$ is blue:



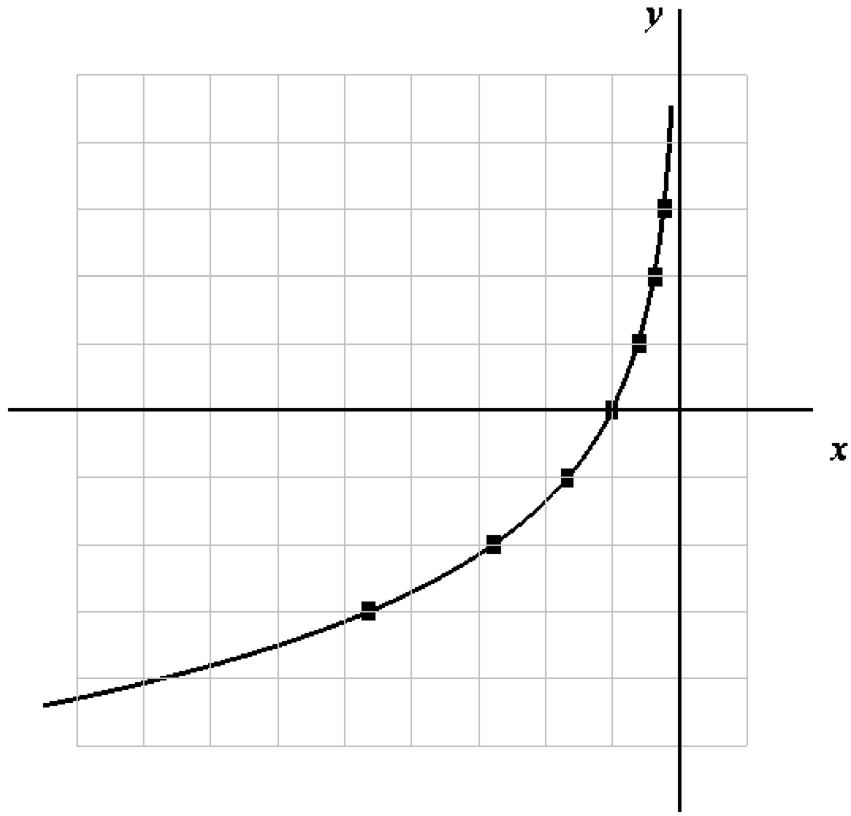
The Drawing of these Graphs

Each graph is a reflection of the other through the line $y = x$, which is gray.

5. $y = \log_{3/5}(-x)$

Note that the domain of the logarithmic function is $(-\infty, 0)$. By the definition of logarithm, $y = \log_{3/5}(-x)$ if and only if $-x = \left(\frac{3}{5}\right)^y \Rightarrow x = -\left(\frac{3}{5}\right)^y$.

x	y
$-\frac{125}{27}$	-3
$-\frac{25}{9}$	-2
$-\frac{5}{3}$	-1
-1	0
$-\frac{3}{5}$	1
$-\frac{9}{25}$	2
$-\frac{27}{125}$	3



The Drawing of this Graph

The x-intercept of the graph of the function is the point $(-1, 0)$.

Note that as $x \to 0$ from the left, $y = \log_{3/5}(-x) \to \infty$. Thus, the vertical line of $x = 0$, which is the y-axis, is a vertical asymptote of the graph of the function.

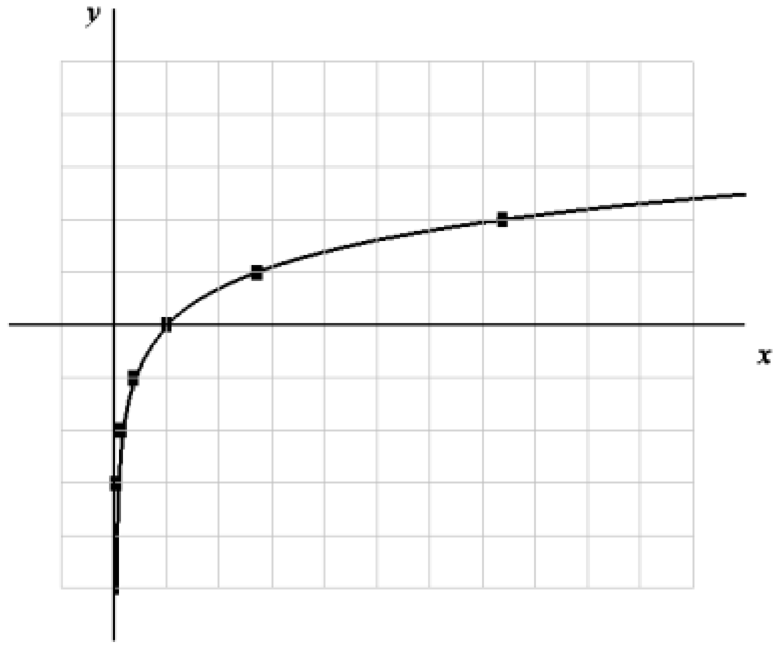
The functions $y = \log_{3/5}(-x)$ and $y = -\left(\frac{3}{5}\right)^x$ are inverse functions of one another.

6. $f(x) = \ln x$

Recall: $\ln x = \log_e x$, where $e = 2.718281828....$

Note that the domain of the logarithmic function f is $(0, \infty)$. In order to graph the function f given by $f(x) = \ln x$, we set $f(x) = y$ and graph the equation $y = \ln x$. By the definition of logarithm, $y = \ln x$ if and only if $x = e^{y}$.

x	y
$e^{-3} \approx 0.04979$	-3
$e^{-2} \approx 0.13534$	-2
$e^{-1} \approx 0.36788$	-1
1	0
$e \approx 2.71828$	1
$e^{2} \approx 7.38906$	2
$e^{3} \approx 20.08554$	3



The Drawing of this Graph

The x-intercept of the graph of the function is the point $(1, 0)$.

Note that as $x \to 0$ from the right, $y = \ln x \to -\infty$. Thus, the vertical line of $x = 0$, which is the y-axis, is a vertical asymptote of the graph of the function.

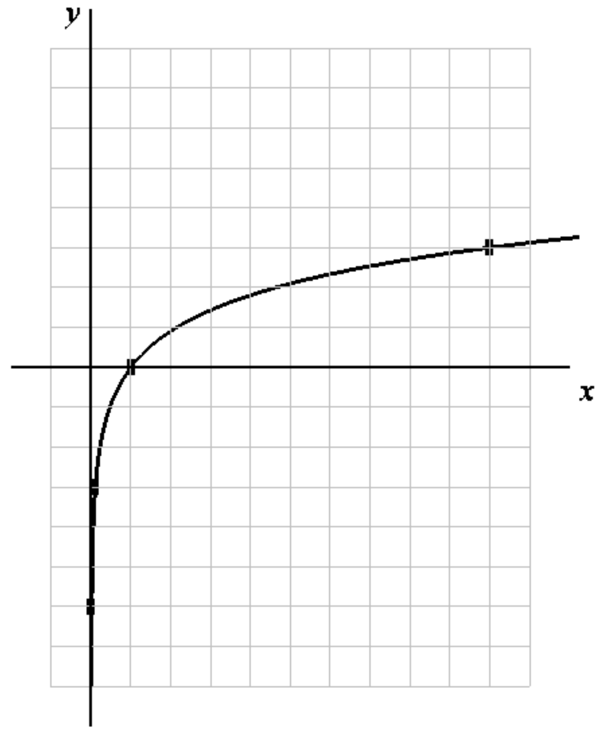
The functions $y = \ln x$ and $y = e^{x}$ are inverse functions of one another.

7. $g(x) = 3 \log x$

Recall: $\log x = \log_{10} x$

Note that the domain of the logarithmic function g is $(-\infty, 0)$. In order to graph the function g given by $g(x) = 3 \log x$, we set $g(x) = y$ and graph the equation $y = 3 \log x$. Since $y = 3 \log x \Rightarrow \dfrac{y}{3} = \log x$, then by the definition of logarithm, $\frac{y}{3} = \log x$ if and only if $x = 10^{y/3}$.

x	y
$\frac{1}{100}$	-6
$\frac{1}{10}$	-3
1	0
10	3
100	6



The Drawing of this Graph

The x-intercept of the graph of the function is the point $(1, 0)$.

Note that as $x \to 0$ from the right, $y = \log x \to -\infty$. Thus, the vertical line of $x = 0$, which is the y-axis, is a vertical asymptote of the graph of the function.

The functions $y = 3 \log x$ and $y = 10^{x/3}$ are inverse functions of one another.

8. $h(x) = -2 \log_{1/3}(-x)$

Note that the domain of the logarithmic function h is $(-\infty, 0)$. In order to graph the function h given by $h(x) = -2 \log_{1/3}(-x)$, we set $h(x) = y$ and graph the equation $y = -2 \log_{1/3}(-x)$. Since $y = -2 \log_{1/3}(-x)$

$\Rightarrow\ -\dfrac{y}{2} = \log_{1/3}(-x)$, then by the definition of logarithm, $-\dfrac{y}{2} = \log_{1/3}(-x)$ if and only if $-x = \left(\dfrac{1}{3}\right)^{-y/2} \Rightarrow -x = 3^{y/2} \Rightarrow x = -3^{y/2}$.

NOTE: $h\left(-\dfrac{1}{9}\right) = -2\log_{1/3}\dfrac{1}{9} = -2(2) = -4$

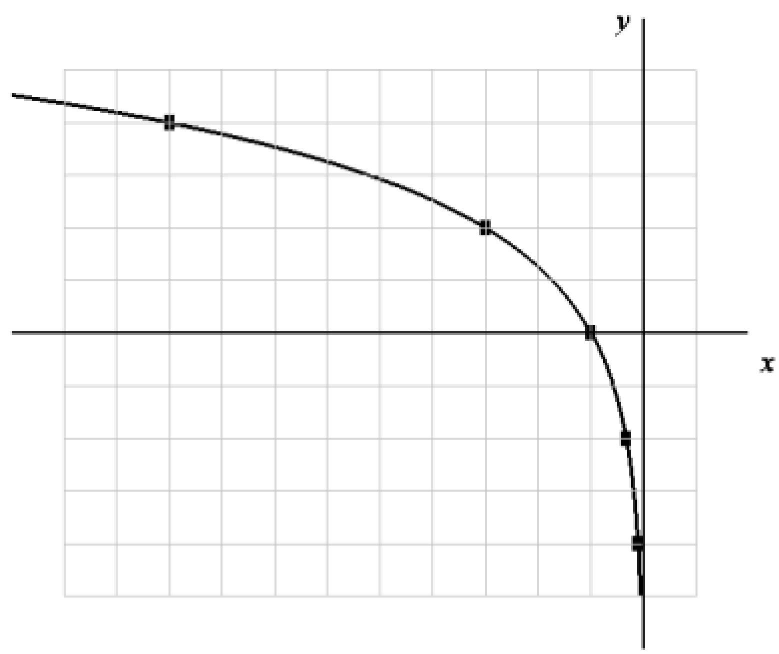
$h\left(-\dfrac{1}{3}\right) = -2\log_{1/3}\dfrac{1}{3} = -2(1) = -2$

$h(-1) = -2\log_{1/3}1 = -2(0) = 0$

$h(-3) = -2\log_{1/3}3 = -2(-1) = 2$

$h(-9) = -2\log_{1/3}9 = -2(-2) = 4$

x	y
$-\frac{1}{9}$	-4
$-\frac{1}{3}$	-2
-1	0
-3	2
-9	4



The Drawing of this Graph

The x-intercept of the graph of the function is the point $(-1, 0)$.

Note that as $x \to 0$ from the right, $y = -2 \log_{1/3}(-x) \to -\infty$. Thus, the vertical line of $x = 0$, which is the *y*-axis, is a vertical asymptote of the graph of the function.

The functions $y = -2 \log_{1/3}(-x)$ and $y = -3^{x/2}$ are inverse functions of one another.

9. $f(t) = -\frac{3}{4} \log_2 t$

Note that the domain of the logarithmic function f is $(0, \infty)$. In order to graph the function f given by $f(t) = -\frac{3}{4} \log_2 t$, we set $f(t) = y$ and graph the equation $y = -\frac{3}{4} \log_2 t$. Since $y = -\frac{3}{4} \log_2 t \Rightarrow -\frac{4y}{3} = \log_2 t$, then by the definition of logarithm, $-\frac{4y}{3} = \log_2 t$ if and only if $t = 2^{-4y/3}$.

NOTE:

$$f\left(\frac{1}{8}\right) = -\frac{3}{4} \log_2 \frac{1}{8} = -\frac{3}{4}(-3) = \frac{9}{4}$$

$$f\left(\frac{1}{4}\right) = -\frac{3}{4} \log_2 \frac{1}{4} = -\frac{3}{4}(-2) = \frac{3}{2}$$

$$f\left(\frac{1}{2}\right) = -\frac{3}{4} \log_2 \frac{1}{2} = -\frac{3}{4}(-1) = \frac{3}{4}$$

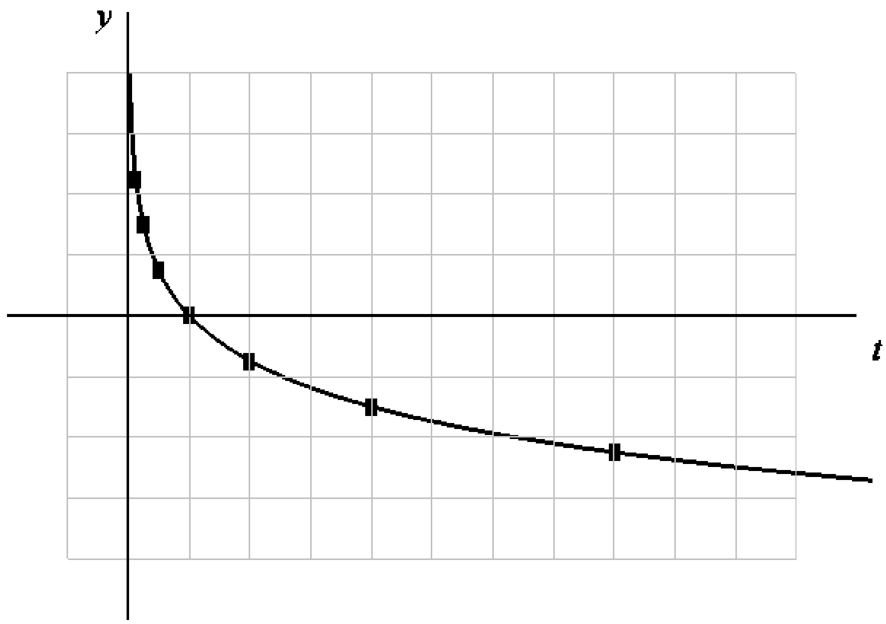
$$f(1) = -\frac{3}{4} \log_2 1 = -\frac{3}{4}(0) = 0$$

$$f(2) = -\frac{3}{4} \log_2 2 = -\frac{3}{4}(1) = -\frac{3}{4}$$

$$f(4) = -\frac{3}{4}\log_2 4 = -\frac{3}{4}(2) = -\frac{3}{2}$$

$$f(8) = -\frac{3}{4}\log_2 8 = -\frac{3}{4}(3) = -\frac{9}{4}$$

t	y
$\frac{1}{8}$	$\frac{9}{4}$
$\frac{1}{4}$	$\frac{3}{2}$
$\frac{1}{2}$	$\frac{3}{4}$
1	0
2	$-\frac{3}{4}$
4	$-\frac{3}{2}$
8	$-\frac{9}{4}$



The Drawing of this Graph

The t-intercept of the graph of the function is the point $(1, 0)$.

Note that as $t \to 0$ from the right, $y = -\frac{3}{4}\log_2 t \to \infty$. Thus, the vertical line of $t = 0$, which is the y-axis, is a vertical asymptote of the graph of the function.

The functions $y = -\frac{3}{4}\log_2 t$ and $y = 2^{-4t/3}$ are inverse functions of one another.

10. $g(t) = \frac{1}{2}\log_{3/4}(-t)$

Note that the domain of the logarithmic function g is $(-\infty, 0)$. In order to graph the function g given by $g(t) = \frac{1}{2}\log_{3/4}(-t)$, we set $g(t) = y$ and graph the equation $y = \frac{1}{2}\log_{3/4}(-t)$. Since $y = \frac{1}{2}\log_{3/4}(-t) \Rightarrow$ $2y = \log_{3/4}(-t)$, then by the definition of logarithm, $2y = \log_{3/4}(-t)$ if and only if $-t = \left(\frac{3}{4}\right)^{2y} \Rightarrow t = -\left(\frac{3}{4}\right)^{2y}$.

NOTE: Since $\left(\frac{3}{4}\right)^{2y} = \left[\left(\frac{3}{4}\right)^{2}\right]^{y} = \left(\frac{9}{16}\right)^{y}$, then $t = -\left(\frac{3}{4}\right)^{2y} =$ $-\left(\frac{9}{16}\right)^{y}$

NOTE: $g\left(-\frac{64}{27}\right) = \frac{1}{2}\log_{3/4}\frac{64}{27} = \frac{1}{2}(-3) = -\frac{3}{2}$

$$g\left(-\frac{16}{9}\right) = \frac{1}{2}\log_{3/4}\frac{16}{9} = \frac{1}{2}(-2) = -1$$

$$g\left(-\frac{4}{3}\right) = \frac{1}{2}\log_{3/4}\frac{4}{3} = \frac{1}{2}(-1) = -\frac{1}{2}$$

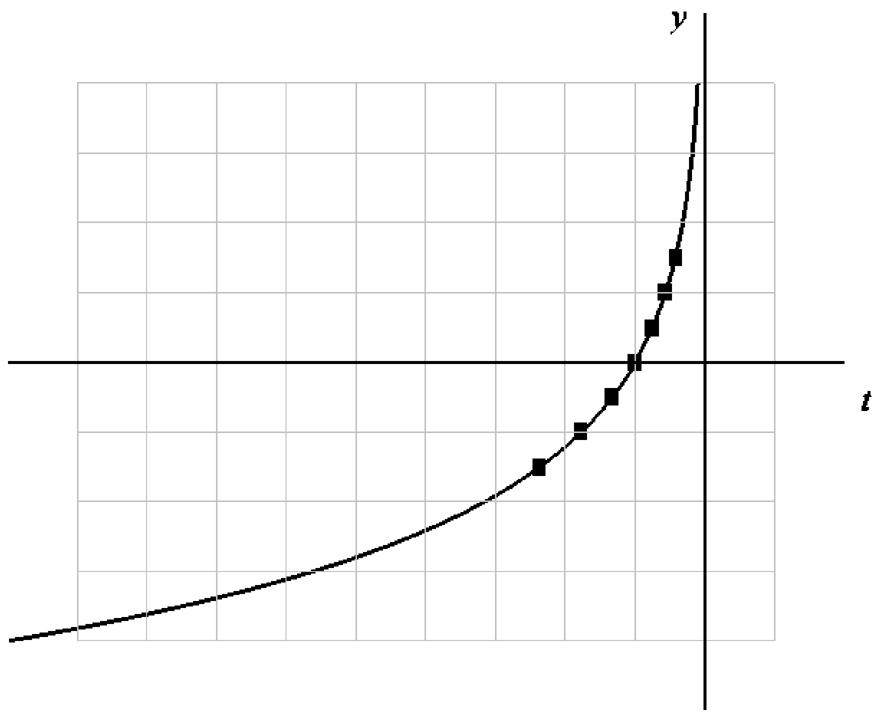
$$g(-1) = \frac{1}{2}\log_{3/4}1 = \frac{1}{2}(0) = 0$$

$$g\left(-\frac{3}{4}\right) = \frac{1}{2}\log_{3/4}\frac{3}{4} = \frac{1}{2}(1) = \frac{1}{2}$$

$$g\left(-\frac{9}{16}\right) = \frac{1}{2}\log_{3/4}\frac{9}{16} = \frac{1}{2}(2) = 1$$

$$g\left(-\frac{27}{64}\right) = \frac{1}{2}\log_{3/4}\frac{27}{64} = \frac{1}{2}(3) = \frac{3}{2}$$

t	y
$-\frac{64}{27}$	$-\frac{3}{2}$
$-\frac{16}{9}$	-1
$-\frac{4}{3}$	$-\frac{1}{2}$
-1	0
$-\frac{3}{4}$	$\frac{1}{2}$
$-\frac{9}{16}$	1
$-\frac{27}{64}$	$\frac{3}{2}$



The Drawing of this Graph

The t-intercept of the graph of the function is the point $(-1, 0)$.

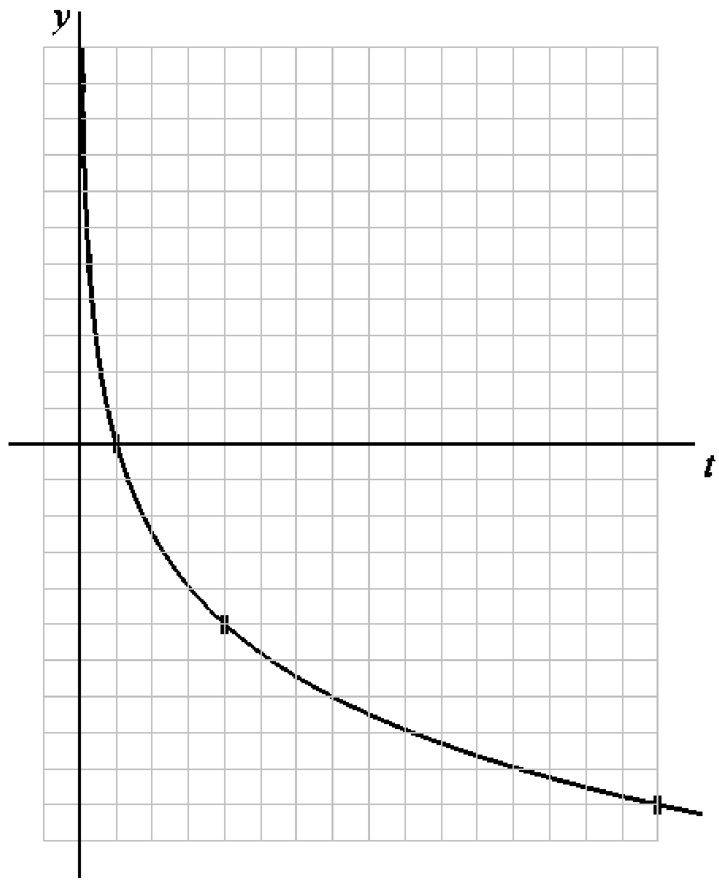
Note that as $t \to 0$ from the left, $y = \frac{1}{2}\log_{3/4}(-t) \to \infty$. Thus, the vertical line of $t = 0$, which is the y-axis, is a vertical asymptote of the graph of the function.

The functions $y = \frac{1}{2}\log_{3/4}(-t)$ and $y = -\left(\frac{3}{4}\right)^{2t} = -\left(\frac{9}{16}\right)^{t}$ are inverse functions of one another.

11. $h(t) = 5 \log_{1/4} t$

Note that the domain of the logarithmic function h is $(0, \infty)$. In order to graph the function h given by $h(t) = 5 \log_{1/4} t$, we set $h(t) = y$ and graph the equation $y = 5 \log_{1/4} t$. Since $y = 5 \log_{1/4} t \Rightarrow$ $\frac{y}{5} = \log_{1/4} t$, then by the definition of logarithm, $\frac{y}{5} = \log_{1/4} t$ if and only if $t = \left(\frac{1}{4}\right)^{y/5} = 4^{-y/5}$.

t	y
$\frac{1}{16}$	10
$\frac{1}{4}$	5
1	0
4	-5
16	-10



The Drawing of this Graph

The t-intercept of the graph of the function is the point $(1, 0)$.

Note that as $t \to 0$ from the right, $y = 5 \log_{1/4} t \to \infty$. Thus, the vertical line of $t = 0$, which is the y-axis, is a vertical asymptote of the graph of the function.

The functions $y = 5\log_{1/4} t$ and $y = \left(\frac{1}{4}\right)^{t/5} = 4^{-t/5}$ are inverse functions of one another.

12. $f(x) = \log_3 2x$

13. $g(t) = \log_4\left(-\frac{t}{3}\right)$

14. $h(x) = 5\log_{1/2} 4x$

Examples Sketch the graph of the following functions. State the domain of the function and use the sketch to state the range of the function.

1. $f(x) = \log_5(x - 3)$

2. $g(x) = 3\log x - 4$

3. $h(x) = \log_{2/3}(x + 5) + 8$

4. $f(x) = \ln(-x) + 2$

5. $g(t) = -2\log_{3/4}(t - 1) + 6$

6. $h(x) = -\ln(x + 4) - 3$

7. $f(x) = \sqrt{3}\log_{12/19}(4x + 8)$

8. $g(x) = \log_{\pi}(6 - x) - 1$

9. $h(t) = \frac{1}{3}\log_{1/2}(-3t - 5) - 12$

10. $f(x) = -4\log(8 - x) + 15$