LESSON 1 SOLVING NONLINEAR INEQUALITIES

In this lesson, we will make use of the Axiom of Trichotomy given below.

Axiom of Trichotomy A real number can only be one of the following: positive, negative, or zero.

NOTE: When you substitute a real number in for the variable in a nonlinear expression, you will either get another real number (which is either positive, negative, or zero) or something that is undefined as a real number.

For example, when we replace the x in the nonlinear expression $\dfrac{7 - 3x}{x^2 + 5x - 24}$ by 4, we get the real number $-\dfrac{5}{12}$ obtained by $\dfrac{7 - 12}{16 + 20 - 24} = \dfrac{-5}{12}$. The resulting real number is negative. If we replace the x by -9 in the expression, we get the real number $\dfrac{17}{6}$ obtained by $\dfrac{7 + 27}{81 - 45 - 24} = \dfrac{34}{12} = \dfrac{17}{6}$. This number is positive. If you replace the x by $\dfrac{7}{3}$ in the expression, you will get the real number zero obtained by $\dfrac{7 - 7}{\frac{49}{9} + \frac{35}{3} - 24} = \dfrac{0}{\frac{49}{9} + \frac{105}{9} - \frac{216}{9}} = \dfrac{0}{-\frac{62}{9}} = 0$. Finally, if we replace the x by 3, we get an undefined real number since we get division by zero obtained by $\dfrac{7 - 9}{9 + 15 - 24} = \dfrac{-2}{0}$.

Thus, to solve a nonlinear inequality, we will find all the real numbers that make a nonlinear expression equal to zero. We will also have to find all the numbers that make the nonlinear expression undefined. Thus, all the remaining real numbers, when substituted for the variable in the nonlinear expression, would make the resulting real number either be positive or negative. Thus, a nonlinear expression has the ability to change signs at the real numbers where the expression is either zero or undefined.

We will determine when a nonlinear expression is positive and negative using the following three steps:

Step 1 Find all the real numbers that make the nonlinear expression equal zero and all the real numbers that make the expression undefined.

Step 2 Plot all the numbers found in Step 1 on the real number line.

Step 3 Using the real number line in Step 2, identify the open intervals determined by the plotted numbers. For each open interval, pick a real number that is in the interval. We will call this number the "test value" for the interval. Substitute the test value for the variable in the nonlinear expression. Whatever sign the expression has for this test value, the expression will have the same sign for any number in the open interval.

<u>Examples</u> Solve the following nonlinear inequalities.

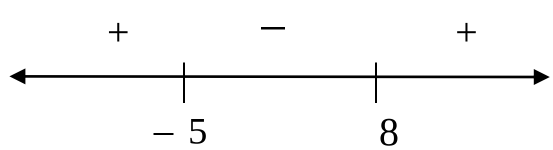
1. $x^2 - 3x - 40 < 0$

Step 1: Find when the nonlinear expression $x^2 - 3x - 40$ is equal to zero. That is, solve the equation $x^2 - 3x - 40 = 0$.

$x^2 - 3x - 40 = 0 \Rightarrow (x+5)(x-8) = 0 \Rightarrow x = -5, \; x = 8$

Find when the nonlinear expression $x^2 - 3x - 40$ is undefined. The expression $x^2 - 3x - 40$ is defined for all real numbers *x*.

Step 2: Plot all the numbers found in Step 1 on the real number line.



Sign of $x^2 - 3x - 40$

Step 3: Use the real number line to identify the open intervals determined by the plotted numbers. Pick a test value for each open interval.

Interval	Test Value	Sign of $x^2 - 3x - 40 = (x+5)(x-8)$
$(-\infty, -5)$	-6	$(-6+5)(-6-8) = (-)(-) = +$
$(-5, 8)$	0	$(0+5)(0-8) = (+)(-) = -$
$(8, \infty)$	9	$(9+5)(9-8) = (+)(+) = +$

Answer: $(-5, 8)$

2. $\dfrac{6-x}{3x-14} \leq 0$

NOTE: This is a **two** part problem. One part of the problem is to solve the nonlinear inequality $\dfrac{6-x}{3x-14} < 0$. The other part of the problem is to solve the equation $\dfrac{6-x}{3x-14} = 0$.

We will use the three step method to solve the nonlinear inequality $\dfrac{6-x}{3x-14} < 0$:

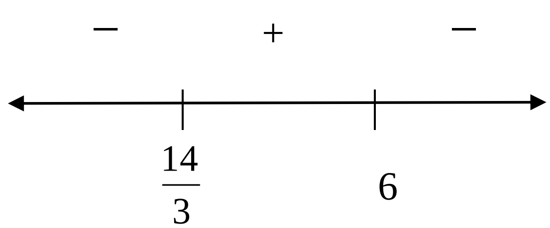
Step 1: Find when the nonlinear expression $\dfrac{6-x}{3x-14}$ is equal to zero. That is, solve the equation $\dfrac{6-x}{3x-14} = 0$. The fraction is equal to zero if and only if the numerator of the fraction is equal to zero.

That is, $\dfrac{6-x}{3x-14} = 0 \Rightarrow 6 - x = 0 \Rightarrow x = 6$

Find when the nonlinear expression $\frac{6-x}{3x-14}$ is undefined. The fraction is undefined if and only if the denominator of the fraction is equal to zero.

That is, $\frac{6-x}{3x-14}$ undefined $\Rightarrow$ $3x - 14 = 0 \Rightarrow$ $x = \frac{14}{3}$

Step 2: Plot all the numbers found in Step 1 on the real number line.



Sign of $\frac{6-x}{3x-14}$

Step 3: Use the real number line to identify the open intervals determined by the plotted numbers. Pick a test value for each open interval.

Interval	Test Value	Sign of $\frac{6-x}{3x-14}$
$\left(-\infty, \frac{14}{3}\right)$	0	$\frac{6-0}{0-14} = \frac{(+)}{(-)} = -$
$\left(\frac{14}{3}, 6\right)$	5	$\frac{6-5}{15-14} = \frac{(+)}{(+)} = +$
$(6, \infty)$	7	$\frac{6-7}{21-14} = \frac{(-)}{(+)} = -$

Thus, the solution for the nonlinear inequality $\frac{6-x}{3x-14} < 0$ is the set of real numbers given by $\left(-\infty,\ \frac{14}{3}\right) \cup (6, \infty)$. The solution for $\frac{6-x}{3x-14} = 0$ was found in Step 1 above. Thus, the solution for $\frac{6-x}{3x-14} = 0$ is the set $\{6\}$. Putting these two solutions together, we have that the solution for $\frac{6-x}{3x-14} \le 0$ is the set of real numbers $\left(-\infty,\ \frac{14}{3}\right) \cup [6, \infty)$.

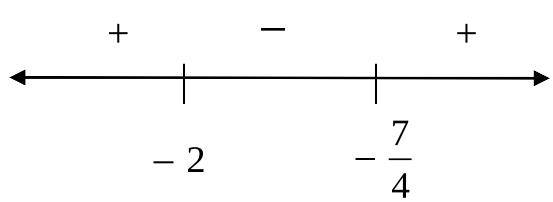
Answer: $\left(-\infty,\ \frac{14}{3}\right) \cup [6, \infty)$

3. $4t^2 + 15t + 14 > 0$

Step 1: $4t^2 + 15t + 14 = 0 \Rightarrow (t+2)(4t+7) = 0 \Rightarrow t = -2,\ t = -\frac{7}{4}$

The expression $4t^2 + 15t + 14$ is defined for all real numbers *t*.

Step 2:



Sign of $4t^2 + 15t + 14$

Step 3:

Interval	Test Value	Sign of $(t+2)(4t+7)$
$(-\infty, -2)$	-3	$(-3+2)(-12+7) = (-)(-) = +$
$\left(-2, -\frac{7}{4}\right)$	-1.8	$(-1.8+2)(-7.2+7) = (+)(-) = -$
$\left(-\frac{7}{4}, \infty\right)$	0	$(0+2)(0+7) = (+)(+) = +$

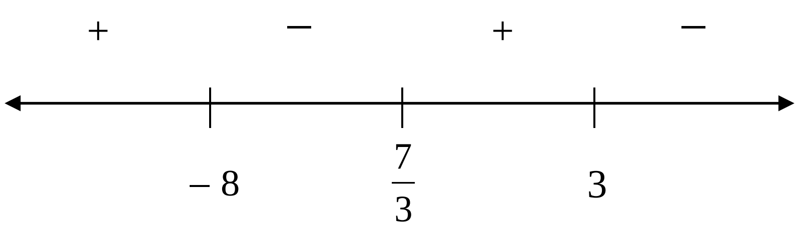
Answer: $(-\infty, -2) \cup \left(-\frac{7}{4}, \infty\right)$

4. $\dfrac{7 - 3x}{x^2 + 5x - 24} < 0$

Step 1: $\dfrac{7 - 3x}{x^2 + 5x - 24} = 0 \Rightarrow 7 - 3x = 0 \Rightarrow x = \dfrac{7}{3}$

$\dfrac{7 - 3x}{x^2 + 5x - 24}$ undefined $\Rightarrow x^2 + 5x - 24 = 0 \Rightarrow$
$(x + 8)(x - 3) = 0 \Rightarrow x = -8, \; x = 3$

Step 2:



Sign of $\dfrac{7 - 3x}{x^2 + 5x - 24}$

Step 3:

Interval	Test Value	Sign of $\dfrac{7 - 3x}{(x + 8)(x - 3)}$
$(-\infty, -8)$	-9	$\dfrac{(+)}{(-)(-)} = \dfrac{(+)}{(+)} = +$
$\left(-8, \frac{7}{3}\right)$	0	$\dfrac{(+)}{(+)(-)} = \dfrac{(+)}{(-)} = -$
$\left(\frac{7}{3}, 3\right)$	2.4	$\dfrac{(-)}{(+)(-)} = \dfrac{(-)}{(-)} = +$
$(3, \infty)$	4	$\dfrac{(-)}{(+)(+)} = \dfrac{(-)}{(+)} = -$

Answer: $\left(-8, \frac{7}{3}\right) \cup (3, \infty)$

5. $16 - 5y^2 \geq 0$

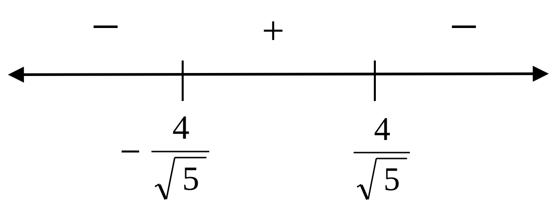
NOTE: This is a **two** part problem. One part of the problem is to solve the nonlinear inequality $16 - 5y^2 > 0$. The other part of the problem is to solve the equation $16 - 5y^2 = 0$.

We will use the three step method to solve the nonlinear inequality $16 - 5y^2 > 0$:

Step 1: $16 - 5y^2 = 0 \Rightarrow 16 = 5y^2 \Rightarrow y^2 = \frac{16}{5} \Rightarrow y = \pm \frac{4}{\sqrt{5}}$

The expression $16 - 5y^2$ is defined for all real numbers *y*.

Step 2:



Sign of $16 - 5y^2$

Step 3:

Interval	Test Value	Sign of $16 - 5y^2$
$\left(-\infty, -\frac{4}{\sqrt{5}}\right)$	-2	$16 - 20 = -$
$\left(-\frac{4}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$	0	$16 - 0 = +$
$\left(\frac{4}{\sqrt{5}}, \infty\right)$	2	$16 - 20 = -$

Thus, the solution for the nonlinear inequality $16 - 5y^2 > 0$ is the set of real numbers given by $\left(-\frac{4}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right)$. The solution for $16 - 5y^2 = 0$ was found in Step 1 above. Thus, the solution for $\frac{6 - x}{3x - 14} = 0$ is the set $\left\{ -\frac{4}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right\}$. Putting these two solutions together, we have that the solution for $16 - 5y^2 \geq 0$ is the set of real numbers $\left[-\frac{4}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right]$.

Answer: $\left[-\frac{4}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right]$

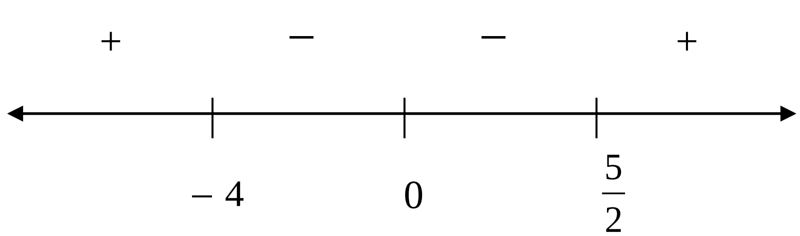
6. $\frac{5w^3 + 20w^2}{2w - 5} > 0$

NOTE: Since $5w^3 + 20w^2 = 5w^2(w + 4)$, then $\frac{5w^3 + 20w^2}{2w - 5} = \frac{5w^2(w + 4)}{2w - 5}$

Step 1: $\frac{5w^2(w + 4)}{2w - 5} = 0 \Rightarrow w = 0, w = -4$

$\frac{5w^2(w + 4)}{2w - 5}$ undefined $\Rightarrow w = \frac{5}{2}$

Step 2:



Sign of $\frac{5w^2(w + 4)}{2w - 5}$

Step 3:

Interval	Test Value	Sign of $\dfrac{5w^2(w+4)}{2w-5}$
$(-\infty, -4)$	-5	$\dfrac{(+)(+)(-)}{(-)} = \dfrac{(-)}{(-)} = +$
$(-4, 0)$	-1	$\dfrac{(+)(+)(+)}{(-)} = \dfrac{(+)}{(-)} = -$
$\left(0, \dfrac{5}{2}\right)$	1	$\dfrac{(+)(+)(+)}{(-)} = \dfrac{(+)}{(-)} = -$
$\left(\dfrac{5}{2}, \infty\right)$	3	$\dfrac{(+)(+)(+)}{(+)} = \dfrac{(+)}{(+)} = +$

Answer: $(-\infty, -4) \cup \left(\dfrac{5}{2}, \infty\right)$

7. $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5} \geq 0$

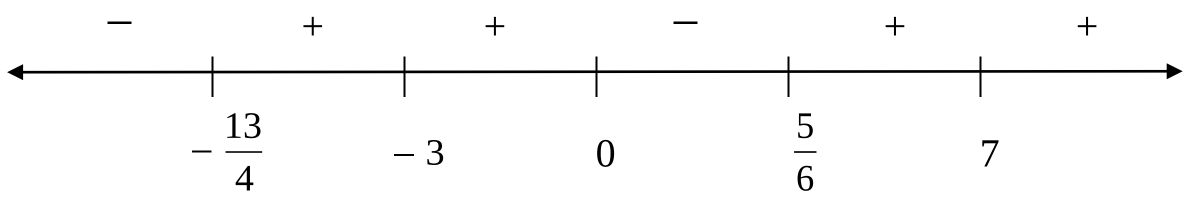
NOTE: This is a **two** part problem. One part of the problem is to solve the nonlinear inequality $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5} > 0$. The other part of the problem is to solve the equation $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5} = 0$.

We will use the three step method to solve the nonlinear inequality $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5} > 0$:

Step 1: $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5} = 0 \Rightarrow x = 0,\ x = -3,\ x = \dfrac{5}{6}$

$$\frac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5} \text{ undefined } \Rightarrow x=7,\ x=-\frac{13}{4}$$

Step 2: Sign of $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5}$:



Step 3:

Interval	Test Value	Sign of $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5}$
$\left(-\infty, -\frac{13}{4}\right)$	-4	$\frac{(-)(+)(-)}{(+)(-)} = \frac{(+)}{(-)} = -$
$\left(-\frac{13}{4}, -3\right)$	-3.1	$\frac{(-)(+)(-)}{(+)(+)} = \frac{(+)}{(+)} = +$
$(-3, 0)$	-1	$\frac{(-)(+)(-)}{(+)(+)} = \frac{(+)}{(+)} = +$
$\left(0, \frac{5}{6}\right)$	0.1	$\frac{(+)(+)(-)}{(+)(+)} = \frac{(-)}{(+)} = -$
$\left(\frac{5}{6}, 7\right)$	1	$\frac{(+)(+)(+)}{(+)(+)} = \frac{(+)}{(+)} = +$
$(7, \infty)$	8	$\frac{(+)(+)(+)}{(+)(+)} = \frac{(+)}{(+)} = +$

Thus, the solution for the nonlinear inequality $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5} > 0$ is the set of real numbers $\left(-\dfrac{13}{4}, -3\right) \cup (-3, 0) \cup \left(\dfrac{5}{6}, 7\right) \cup (7, \infty)$.

The solution for $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5} = 0$ was found in Step 1 above. Thus, the solution for $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5} = 0$ is the set $\left\{-3, 0, \dfrac{5}{6}\right\}$. Putting these two solutions together, we have that the solution for $\dfrac{x^3(x+3)^2(6x-5)}{(7-x)^4(4x+13)^5} \geq 0$ is the set of real numbers $\left(-\dfrac{13}{4}, 0\right] \cup \left[\dfrac{5}{6}, 7\right) \cup (7, \infty)$.

Answer: $\left(-\dfrac{13}{4}, 0\right] \cup \left[\dfrac{5}{6}, 7\right) \cup (7, \infty)$